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ON THE A.E. CONVERGENCE OF WALSH-KACZMARZ-FOURIER SERIES

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ABSTRACT. It is shown that partial sums of Walsh-Kaczmarz-Fourier series of functions in the Orlicz class $L(\log^+ L)^2$ converge a.e. The proof utilizes an estimate of P. Sjölin on the partial sums of the usual Walsh-Fourier series.

The Walsh-Kaczmarz system is a reordering of the usual Walsh system within dyadic blocks of indices 2^N to 2^{N+1}, $N=0, 1, \cdots$. The a.e. convergence properties of Fourier series with respect to this system have been investigated by L. A. Balashov [1] and K. H. Moon [7]. Balashov showed that there exist functions in the Orlicz class $L(\log^+ L)^{1-\varepsilon}$, $0<\varepsilon<1$, whose Walsh-Kaczmarz-Fourier series diverge a.e. Moon established the a.e. convergence of Walsh-Kaczmarz-Fourier series of L^2 functions. In this note we prove, using a theorem of P. Sjölin [9] on the a.e. convergence of Walsh-Fourier series, the a.e. convergence result for functions in the class $L(\log^+ L)^2$.

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We recall the definition of the Walsh system $\{w_n\}$. Let r_n, where $r_n(x)=\operatorname{sgn}(\sin 2^{n+1}\pi x)$, be the nth Rademacher function. For any non-negative integer n, with dyadic expansion $n=\sum_{j=0}^{\infty} \varepsilon_j 2^j$, $w_n=\prod_{j=0}^{\infty} r_j^{\varepsilon_j}$.

The Walsh-Kaczmarz system $\{\phi_n\}$ is defined as follows: $\phi_0=1$, $\phi_1=r_0$, and for $N=1, 2, \cdots, 2^N \leqq n < 2^{N+1}$, with $n=\sum_{j=0}^{N} \varepsilon_j 2^j$, where $\varepsilon_j=0$ or 1 if $0\leqq j\leqq N-1$, and $\varepsilon_N=1$, $\phi_n=r_N \prod_{j=0}^{N-1} r_{N-j-1}^{\varepsilon_j}$. The system $\{\phi_n\}$ so defined is a rearrangement of $\{w_n\}$ within dyadic blocks of indices $2^N\leqq n<2^{N+1}$, $N=1, 2, \cdots$.

For $f\in L^1(0, 1)$, let $S_n f=\sum_{j=0}^{n-1} \phi_j \int_0^1 f\phi_j\, dt$ denote the nth partial sum of the Fourier series of f with respect to the Walsh-Kaczmarz system $\{\phi_n\}$.

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THEOREM. *If* $\int_0^1 |f|(\log^+|f|)^2\,dx<\infty$, *then* S_nf *converges to* f *a.e.*

We will show that there exist absolute constants C_1 and C_2 such that

$$m\left\{\sup_n |S_nf| > y\right\} \leqq y^{-1}\left(C_1\int_0^1 |f|\,(\log^+|f|)^2\,dx + C_2\right) \tag{1}$$

for all $y>0, f\in L(\log^+ L)^2$. The Theorem will follow from (1) by the usual density argument.

Before we proceed to prove (1) we need to make the following observation. Let τ be a permutation of the set of all nonnegative integers. An ordering $\{\theta_n\}$ of the Walsh functions is said to be the Paley ordering generated by $\{r_{\tau(n)}\}$ if for any nonnegative integer n with dyadic expansion $n=\sum_{j=0}^\infty \varepsilon_j 2^j$, $\theta_n=\prod_{j=0}^\infty r_{\tau(j)}^{\varepsilon_j}$. We will need several properties of the partial sums R_nf of the Fourier series of f with respect to $\{\theta_n\}$. These facts can easily be deduced from the corresponding ones of the Walsh-Fourier series since there is a 1-1 measure-preserving transformation E from (0, 1) onto itself such that $r_{\tau(N)}(x)=r_N(Ex)$ a.e., $N=0, 1, \cdots$, and hence $\theta_n(x)=w_n(Ex)$ a.e., $n=0, 1, \cdots$.

First, if $n=\sum_{j=0}^\infty \varepsilon_j 2^j$, it follows from the definition that

$$R_nf = \theta_n \sum_{j=0}^{\infty} \varepsilon_j(R_{2^{j+1}}(\theta_nf) - R_{2^j}(\theta_nf)).$$

(See Paley [8].) Now, for any $g\in L^1$, $R_{2^j}(g)$ is the average of g over sets of the form $\{r_{\tau(0)}=c_0, \cdots, r_{\tau(j-1)}=c_{j-1}\}$ where $c_k=\pm 1$, $k=0, \cdots, j-1$, or, in terms of conditional expectation,

$$R_{2^j}(g) = E(g \mid r_{\tau(0)}, \cdots, r_{\tau(j-1)}), \qquad j = 1, 2, \cdots. \tag{2}$$

Thus we have

$$R_nf = \theta_n \sum_{j=0}^{\infty} \varepsilon_j(E(\theta_nf \mid r_{\tau(0)}, \cdots, r_{\tau(j)}) - E(\theta_nf \mid r_{\tau(0)}, \cdots, r_{\tau(j-1)})). \tag{3}$$

In the above equation, when $j=0$ the term $E(\theta_nf|r_{\tau(0)}, \cdots, r_{\tau(j-1)})$ is interpreted as the integral $\int_0^1 \theta_nf\,dt$.

Similarly, if $2^N\leqq n<2^{N+1}$, $n=\sum_{j=0}^N \varepsilon_j 2^j$,

$$\begin{aligned} R_nf - R_{2^N}f &= r_{\tau(N)}R_{n-2^N}(r_{\tau(N)}f) \\ &= \theta_n \sum_{j=0}^{N-1} \varepsilon_j(E(\theta_nf \mid r_{\tau(0)}, \cdots, r_{\tau(j)}) - E(\theta_nf \mid r_{\tau(0)}, \cdots, r_{\tau(j-1)})). \end{aligned} \tag{4}$$

Finally, from a theorem of Sjölin [9], we have

$$\int_0^1 \sup_n |R_nf|\,dx \leqq C_1\int_0^1 |f|\,(\log^+|f|)^2\,dx + C_2, \tag{5}$$

where C_1 and C_2 are absolute constants.

We now return to the proof of (1). We note that

$$\sup_n |S_n f| \leqq \sup_N \sup_{2^N \leqq n < 2^{N+1}} |S_n f - S_{2^N} f| + \sup_N |S_{2^N} f|.$$

Since $S_{2^N} f$ coincides with the 2^Nth partial sum of the Walsh-Fourier series, (2) and Doob's inequality [**10**, p. 91] give

$$(6)\quad m\left\{\sup_N |S_{2^N} f| > y\right\} = m\left\{\sup_N |E(f \mid r_0, \cdots, r_{N-1})| > y\right\} \leqq y^{-1}\int_0^1 |f|\, dx.$$

Hence it is sufficient to prove that for every positive integer N_0,

$$(7)\quad \begin{aligned} &m\left\{\sup_{N \leqq N_0} \sup_{2^N \leqq n < 2^{N+1}} |S_n f - S_{2^N} f| > y\right\} \\ &\qquad \leqq y^{-1}\left(C_1 \int_0^1 |f|\, (\log^+ |f|)^2\, dx + C_2\right). \end{aligned}$$

To this end we observe that for $2^N \leqq n < 2^{N+1}$, $N = 1, \cdots, N_0$, ϕ_n is equal to the nth term of the Paley ordering of the Walsh functions generated by the sequence $r_{N-1}, r_{N-2}, \cdots, r_0, r_N, r_{N+1}, \cdots$. Hence, it follows from (4) that for $2^N \leqq n < 2^{N+1}$, with $n = \sum_{j=0}^N \varepsilon_j(n) 2^j$,

$$(8)\quad \begin{aligned} S_n f - S_{2^N} f = \phi_n \sum_{j=0}^{N-1} \varepsilon_j(n)(&E(\phi_n f \mid r_{N-1}, \cdots, r_{N-j-1}) \\ &- E(\phi_n f \mid r_{N-1}, \cdots, r_{N-j})). \end{aligned}$$

Again, in the above equation, when $j=0$ the term $E(\phi_n f | r_{N-1}, \cdots, r_{N-j})$ is interpreted as the integral $\int_0^1 \phi_n f\, dt$.

At this point we observe that for any L^1 function g, and any integers $n, m, l \geqq 0$,

$$(9)\quad E(g \mid r_n, \cdots, r_{n+m}) = E(E(g \mid r_n, \cdots, r_{n+m+l}) \mid r_0, \cdots, r_{n+m}).$$

To see this we first note that

$$E(g \mid r_n, \cdots, r_{n+m}) = E(E(g \mid r_n, \cdots, r_{n+m+l}) \mid r_n, \cdots, r_{n+m}).$$

The equality

$$\begin{aligned} &E(E(g \mid r_n, \cdots, r_{n+m+l}) \mid r_n, \cdots, r_{n+m}) \\ &\qquad = E(E(g \mid r_n, \cdots, r_{n+m+l}) \mid r_0, \cdots, r_{n+m}) \end{aligned}$$

is a consequence of the independence of the Rademacher functions $\{r_n\}$ and the following fact: (See, for example, [**3**, p. 285].)

Suppose $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ are three Borel fields such that $\mathscr{F}_1 \vee \mathscr{F}_2$, the Borel field generated by $\mathscr{F}_1 \cup \mathscr{F}_2$, is independent of $\mathscr{F}_3$. Then, for each integrable, $\mathscr{F}_1$-measurable function h, we have $E(h|\mathscr{F}_2) = E(h|\mathscr{F}_2 \vee \mathscr{F}_3)$.

Moreover, by the independence of the Rademacher functions, we have

$$\int_0^1 f\phi_n\,dt = E(E(f\phi_n \mid r_N, \cdots, r_{N_0}) \mid r_0, \cdots, r_{N-1}). \tag{10}$$

Substituting (9) and (10) into (8), we obtain

$$S_nf - S_{2^N}f = \phi_n E\Bigg(\sum_{j=0}^{N-1} \varepsilon_j(n)(E(f\phi_n \mid r_{N-j-1}, r_{N-j}, \cdots, r_{N_0}) - E(f\phi_n \mid r_{N-j}, r_{N-j+1}, \cdots, r_{N_0}))\Bigg| r_0, \cdots, r_{N-1}\Bigg)$$

Now we consider the Paley ordering $\{\psi_n\}$ of the Walsh functions generated by the sequence $r_{N_0}, r_{N_0-1}, \cdots, r_0, r_{N_0+1}, r_{N_0+2}, \cdots$. For each $2^N \leqq n < 2^{N+1}$, there corresponds an integer $m = m(n)$ such that

$$\phi_n = r_N \sum_{j=0}^{N-1} r_{N-j-1}^{\varepsilon_j(n)} = \psi_m.$$

In fact, we have $m = \sum_{j=0}^{N_0} \eta_j(m)2^j$, where

$$\eta_j(m) = \begin{cases} 0 & \text{if } j < N_0 - N, \\ 1 & \text{if } j = N_0 - N, \\ \varepsilon_{j-N_0+N-1}(n) & \text{if } j > N_0 - N. \end{cases}$$

Therefore,

$$\begin{aligned} S_nf - S_{2^N}f &= \psi_m E\Bigg(\sum_{j=N_0-N+1}^{N_0} \eta_j(m)(E(f\psi_m \mid r_{N_0}, \cdots, r_{N_0-j}) - E(f\psi_m \mid r_{N_0}, \cdots, r_{N_0-j+1}))\Bigg| r_0, \cdots, r_{N-1}\Bigg) \\ &= \psi_m E\Bigg(\sum_{j=0}^{N_0} \eta_j(m)(E(f\psi_m \mid r_{N_0}, \cdots, r_{N_0-j}) - E(f\psi_m \mid r_{N_0}, \cdots, r_{N_0-j+1}))\Bigg| r_0, \cdots, r_{N-1}\Bigg) \\ &\quad - \psi_m E((E(f\psi_m \mid r_{N_0}, \cdots, r_N) - E(f\psi_m \mid r_{N_0}, \cdots, r_{N+1})) \mid r_0, \cdots, r_{N-1}). \end{aligned}$$

The last term vanishes since the independence of the Rademacher functions implies

$$\begin{aligned} E(E(f\psi_m \mid r_{N_0}, \cdots, r_N) \mid r_0, \cdots, r_{N-1}) &= \int_0^1 f\psi_m\,dt \\ &= E(E(f\psi_m \mid r_{N_0}, \cdots, r_{N+1}) \mid r_0, \cdots, r_{N-1}). \end{aligned}$$

Also, from (3), if $T_n f$ is the nth partial sum of the Fourier series of f with respect to $\{\psi_n\}$,

$$\sum_{j=0}^{N_0} \eta_j(m)(E(f\psi_m \mid r_{N_0}, \cdots, r_{N_0-j}) - E(f\psi_m \mid r_{N_0}, \cdots, r_{N_0-j+1})) = \psi_m(T_m f).$$

Hence, for $2^N \leqq n < 2^{N+1}$, $N \leqq N_0$,

$$S_n f - S_{2^N} f = \psi_m E(\psi_m T_m f \mid r_0, \cdots, r_{N-1}).$$

Consequently,

$$\begin{aligned}\sup_{2^N \leqq n < 2^{N+1}} |S_n f - S_{2^N} f| &\leqq \sup_{2^N \leqq n < 2^{N+1}} E(|T_{m(n)} f| \mid r_0, \cdots, r_{N-1}) \\ &\leqq E\left(\sup_k |T_k f| \mid r_0, \cdots, r_{N-1}\right)\end{aligned}$$

for all $N \leqq N_0$. Therefore

$$\begin{aligned} m\left\{\sup_{N \leqq N_0} \sup_{2^N \leqq n < 2^{N+1}} |S_n f - S_{2^N} f| > y\right\} & \\ \leqq m\left\{\sup_{N \leqq N_0} E\left(\sup_k |T_k f| \mid r_0, \cdots, r_{N-1}\right) > y\right\} & \\ \leqq y^{-1} \int_0^1 \sup_k |T_k f| \, dx & \\ \leqq y^{-1}\left(C_1 \int_0^1 |f| \, (\log^+ |f|)^2 \, dx + C_2\right). &\end{aligned}$$

Here we have made use of Doob's inequality (see (6)) and (5). This proves (7) and thus completes the proof of the Theorem.

REMARKS. For the usual Walsh-Fourier series, there is a gap between a.e. convergence results and a.e. divergence results. It is known that the Walsh-Fourier series converge a.e. for functions in the Orlicz class $L(\log^+ L)\log^+ \log^+ L$ (Sjölin [9]), and that there are functions in the class $L(\log^+ \log^+ L)^{1-\varepsilon}$ $(0<\varepsilon<1)$ whose Walsh-Fourier series diverge a.e. (Moon [7]). Such a gap also exists in the Walsh-Kaczmarz-Fourier series, where we have a.e. convergence for the class $L(\log^+ L)^2$ and a.e. divergence for the class $L(\log^+ L)^{1-\varepsilon}$ $(0<\varepsilon<1)$ (Balashov [1]).

Another proof involving modifications of the Carleson-Hunt technique [2], [5], [6] and estimates on maximal functions of the Hardy-Littlewood type yields a.e. convergence results for functions in the smaller class $L(\log^+ L)^2 \log^+ \log^+ L$. That proof, however, also works for more general rearrangements. It also gives various estimates on $\sup_n |S_n f|$. (See [11], [12] and [4].)

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