ON THE ABEL SUMMABILITY OF MULTIPLE FOURIER SERIES BY SPHERICAL MEANS

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1. Introduction. Let $x=(x_1, x_2, \ldots, x_k)$ be a point in the k dimensional Euclidian space and $f(x)=f(x_1, x_2, \ldots, x_k)$ be a function of the Lebesgue class L having the period 2π in each variables; let

$$f(x) \sim \sum a_{n_1, n_2 \ldots n_k} e^{i(n_1 x_1 + n_2 x_2 + \ldots + n_k x_k)} \tag{1.1}$$

be its Fourier series, that is

$$a_{n_1 n_2 \ldots, n_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} f(x_1, x_2, \ldots, x_k) e^{-i(n_1 x_1 + \ldots + n_k x_k)} dx_1 \ldots dx_k . \tag{1.2}$$

We shall consider the spherical means of the series (1.1). This method was inaugurated by Prof. Bochner [1] and developed by other writers.

By the spherical sum of (1.1) we mean

$$S_R(x) = \sum_{\nu < R} a_{n_1, \ldots n_k} e^{i(n_1 x + \ldots + n_k x_k)} \tag{1.3}$$

$$n_1^2 + n_2^2 \ldots + n_k^2 = \nu$$

and write the spherical mean of the function $f(x)$ at a point x by

$$f_x(t) = \frac{\Gamma(k/2)}{2(\pi)^{k/2}} \int_\sigma f(x_1 + t\xi_1, x_2 + t\xi_2, \ldots, x_k + t\xi_k) d\sigma_\xi , \tag{1.4}$$

where σ denotes the unite sphere $\xi_1^2 + \xi_2^2 + \ldots + \xi_k^2 = 1$, and $d\sigma_\xi$ its $k-1$ dimensional volume element.

The general Abel mean of the series (1.3) is given by

$$\frac{1}{\varepsilon} \int_0^\infty \frac{(t/\varepsilon)^{k-1}}{[(t/\varepsilon)^{2(1+m)} + 1]^{(k+1)/2}} f_x(t) dt. \tag{1.5)'$$

If we put $m=0$, (1.5)' becomes ordinary Abel mean, which is established by Bochner [1]. The formula (1.5)' is reduced to

$$\frac{1}{\varepsilon} \int_0^1 \frac{(t/\varepsilon)^{k-1}}{[(t/\varepsilon)^{2(1+m)} + 1]^{(k+1)/2}} f_x(t) dt + o(1), \qquad \text{as } \varepsilon \to 0, \tag{1.5}$$

by the localization theorem, where $m > -\frac{1}{k+1}$, since

$$\left(\frac{1}{\varepsilon}\right) \int_1^\infty \frac{(t/\varepsilon)^{k-1}}{[(t/\varepsilon)^{2(1+m)} + 1]^{(k+1)/2}} f_x(t) dt$$

$$= \left| \left(\frac{1}{\varepsilon}\right) \varepsilon^{(1+m)(k+1)} \sum_{j=1}^{\infty} \int_{j}^{j+1} \frac{1}{[t^{2(1+m)} + \varepsilon^{2(1+m)}]^{(k+1)/2}} t^{k-1} f_x(t)\, dt \right|$$

$$\leqq \varepsilon^{mk+m+1} \sum_{j=0}^{\infty} \int_{j}^{j+1} \frac{1}{t^{(1+m)(k+1)}} |t^{k-1} f_x(t)|\, dt$$

$$\leqq \varepsilon^{mk+m+1} \sum_{j=1}^{\infty} \frac{1}{j^{(1+m)(k+1)}} \cdot O(j^{k-1}) \cdot \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(t_1, \ldots, t_k)|\, dt_1 \ldots dt_k$$

$$= \varepsilon^{mk+m+1} \cdot O\left(\sum_{j=1}^{\infty} \frac{1}{j^{mk+m+2}}\right) = o(1).$$

The general Abel mean of the function (1.4) is given by

$$\varepsilon \int_{0}^{1} \left(\frac{\varepsilon}{t}\right)^{\alpha} e^{-\left(\frac{\varepsilon}{t}\right)^{1+n}} \left(\frac{t}{\varepsilon}\right)^{k-1} f_x(t) \frac{dt}{t^2} \tag{1.6}$$

where $n > -1$.

The formulas (1.5) and (1.6) were given by N. Levinson [3] for the case of one variable.

The object of this paper is to establish the relations between (1.5) and (1.6).

2. Tauberian lemmas. The following lemmas are essentially due to N. Levinson [3], and N. Wiener [7].

LEMMA 1. *Let*

$$\lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon}\right)^{k} \int_{0}^{1} N_1\left(\frac{t}{\varepsilon}\right) f(t)\, dt = 0 \qquad \textit{boundedly}, \ (k \geqq 1), \tag{2.0}$$

where

$$\int_{0}^{1} |f(t)|\, dt < \infty, \qquad f(t) = t^{k-1} f_x(t) \tag{2.1}$$

and

$$|N_1(t)| < A. \tag{2.2}$$

Let $R(t)$ be a function such that

$$\int_{0}^{\infty} t^{k-1} |R(t)|\, dt < \infty \tag{2.3}$$

and

$$\int_{0}^{\infty} |R(t)| \frac{dt}{t} < \infty. \tag{2.4}$$

Then, if

$$N_2(t)=\int_0^\infty R(y)N_1\left(\frac{t}{y}\right)\frac{dy}{y} \tag{2.5}$$

then

$$\lim_{\epsilon\to 0}\left(\frac{1}{\varepsilon}\right)^k\int_0^1 N_2\left(\frac{t}{\varepsilon}\right)f(t)\,dt=0. \tag{2.6}$$

PROOF. Since the following integral is absolutely convergent, we have

$$\left(\frac{1}{\varepsilon}\right)^k\int_0^\infty R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 N_1\left(\frac{t}{y}\right)f(t)\,dt \tag{2.7}$$

$$=\left(\frac{1}{\varepsilon}\right)^k\int_0^1 f(t)\,dt\int_0^\infty R\left(\frac{y}{\varepsilon}\right)N_1\left(\frac{t}{y}\right)\frac{dy}{y}=\left(\frac{1}{\varepsilon}\right)^k\int_0^1 N_2\left(\frac{t}{\varepsilon}\right)f(t)\,dt.$$

(2.0), (2.1) and (2.2) yields

$$\left|\left(\frac{1}{y}\right)^k\int_0^1 N_1\left(\frac{t}{y}\right)f(t)\,dt\right|<M,$$

and we get

$$\left|\left(\frac{1}{\varepsilon}\right)^k\int_0^\infty R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 N_1\left(\frac{t}{y}\right)f(t)dt\right|=\left|\left(\frac{1}{\varepsilon}\right)^k\int_0^\infty y^{k-1}R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y^k}\int_0^1 N_1\left(\frac{t}{y}\right)f(t)dt\right|$$

$$\leqq\left|\left(\frac{1}{\varepsilon}\right)^k\left\{\int_0^\delta+\int_\delta^\infty\right\}R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 N_1\left(\frac{t}{y}\right)f(t)dt\right|$$

$$\leqq\left|\left(\frac{1}{\varepsilon}\right)^k\int_0^\delta y^{k-1}R\left(\frac{y}{\varepsilon}\right)dy\left\{\frac{1}{y^k}\int_0^1 N_1\left(\frac{t}{y}\right)f(t)dt\right\}\right|+M\left|\left(\frac{1}{\varepsilon}\right)^k\int_\delta^\infty R\left(\frac{y}{\varepsilon}\right)y^{k-1}dy\right|.$$

Let we take small $\delta>0$, we have $\left|\left(\frac{1}{y}\right)^k\int_0^1 N_1\left(\frac{t}{y}\right)f(t)\,dt\right|<\eta$ for any small $\eta>0$, by (2.0). Therefore the last term is less than

$$\eta\left|\int_0^\infty R(u)u^{k-1}du\right|+M\int_{\delta/\epsilon}^\infty |u^{k-1}R(u)|\,du.$$

Therefore for sufficiently small η the first term is arbitrarily small independent of ε, and for small ε and fixed δ the second term is arbitrarily small. Combined this fact with (2.7) we have

$$\lim_{\epsilon\to 0}\left(\frac{1}{\varepsilon}\right)^k\int_0^1 N_2\left(\frac{t}{\varepsilon}\right)f(t)dt=0,$$

which proves the lemma.

LEMMA 2. *Lemma* 1 *remains valid if* (2.4) *is replaced by*

$$|R(t)| < A, \qquad \text{for } t < \frac{1}{2},$$

and

$$\int_0^\infty |N_1(t)| \frac{dt}{t} < \infty.$$

PROOF. The proof is the same as that of N. Levinson [3].

LEMMA 3. *Let for some fixed* b, $t^{-b}N_1(t)$ *and* $t^{-b}N_2(t)$ *belong to* $L(0,\infty)$, *and let*

$$k_1(w) = \int_0^\infty N_1(t)t^{-w}dt,$$

$$k_2(w) = \int_0^\infty N_2(t)t^{-w}dt,$$

and

$$\gamma(w) = \frac{k_2(w)}{k_1(w)}.$$

If $\gamma(w)$ *is analytic, and*

$$\int_{-\infty}^\infty |\gamma(u+iv)|^2 dv < M < \infty$$

in the strip, $b-\delta \leqq u \leqq b+\delta$, *for some fixed* $\delta > 0$, *then*

$$R(t) = \underset{A\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i}\int_{-iA+u}^{iA+u} \gamma(w)t^{w-1}dw, \qquad (b-\delta \leqq u \leqq b+\delta), \tag{2.8}$$

is a solution of the equation (2.5) *and*

$$t^{-b}R(t) \tag{2.9}$$

belongs to the class $L(0,\infty)$.

PROOF. The proof is analogous to that of N. Levinson [3].

3. Summability theorems. Let $P(m)$ represent

$$\lim_{\epsilon\to 0} \frac{1}{\varepsilon}\int_0^1 \frac{(t/\varepsilon)^{k-1}}{[(t/\varepsilon)^{2(1+m)}+1]^{(k+1)/2}} f_x(t)\,dt = 0, \qquad m > -\frac{1}{k+1}, \tag{3.1}$$

and $E(n,\alpha)$ represent

$$\lim_{\epsilon\to 0}\varepsilon\int_0^1 \left(\frac{\varepsilon}{t}\right)^\alpha e^{-\left(\frac{\varepsilon}{t}\right)^{1+n}} \left(\frac{t}{\varepsilon}\right)^{k-1} f_x(t)\,\frac{dt}{t^2} = 0,\ n > -1,\ \alpha \geqq k-1. \tag{3.2}$$

THEOREM 1. $E(n,\alpha)$ *for* $n > m$, *implies* $P(m)$; *while* $P(m)$ *for* $m > n$, *implies* $E(n,\alpha)$.

PROOF. Let

$$N_1(t)=\frac{1}{[t^{2(1+m)}+1]^{(k+1)/2}}\quad\text{and}\quad N_2(t)=t^{-\alpha-2}e^{-\left(\frac{1}{t}\right)^{1+n}},$$

then

$$P(m)\ \text{ means }\ \lim_{\epsilon\to 0}\left(\frac{1}{\varepsilon}\right)^k\int_0^1 N_1\left(\frac{t}{\varepsilon}\right)t^{k-1}f_x(t)dt=0, \tag{3.3}$$

and

$$E(n,\alpha)\ \text{ means }\ \lim_{\epsilon\to 0}\left(\frac{1}{\varepsilon}\right)^k\int_0^1 N_2\left(\frac{t}{\varepsilon}\right)t^{k-1}f_x(t)dt=0. \tag{3.4}$$

In the sequel we shall use the notations of Lemmas 1,2 and 3. Then we have

$$|N_1(t)|<A\ \text{ and }\ |N_2(t)|<A \tag{2.2}$$

for $m>-\frac{1}{k+1}$, $n>-1$ and $t\in(0,\infty)$. Also, we have

$$\int_0^\infty |N_2(t)|\frac{dt}{t}<\infty.$$

By the Mellin transforms, we have

$$k_1(w)=k_1(u+iv)=\frac{1}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)}\Gamma\left(\frac{1-w}{2(1+m)}\right)\Gamma\left(\frac{k+1}{2}-\frac{1-w}{2(1+m)}\right) \tag{3.5}$$

for $1>u>-m(k+1)-k$, and

$$k_2(w)=k_2(u+iv)=\frac{1}{1+n}\Gamma\left(\frac{1+w+\alpha}{1+n}\right) \tag{3.6}$$

for $u>-(\alpha+1)$, (c.f. Titchmarsh [4] p.192), where $\Gamma(x)$ denotes Gamma function.

First we prove that $P(m)$ implies $E(n,\alpha)$ if $m>n$ and $\alpha\geqq k-1$. We use the Lemma 1 and Lemma 3. Since $t^{(k-1)}N_1(t)$ and $t^{(k-1)}N_2(t)$ belong to the class $L(0,\infty)$ under the hypothesis of the theorem, we have

$$\gamma(w)=\gamma(u+iv)=\frac{k_2(w)}{k_1(w)}=\frac{2(1+m)\Gamma\left(\frac{k+1}{2}\right)}{1+n}\cdot\frac{\Gamma\left(\frac{1+w+\alpha}{1+n}\right)}{\Gamma\left(\frac{1-w}{2(1+m)}\right)\Gamma\left(\frac{k+1}{2}-\frac{1-w}{2(1+m)}\right)}. \tag{3.7}$$

The well known formula of Γ-function

$$|\Gamma(u+iv)|\sim\sqrt{2\pi}\,|v|^{u-\frac{1}{2}}\exp\left\{-\frac{\pi}{2}|v|\right\},\ \text{ as }|v|\to\infty, \tag{3.8}$$

implies that

$$(3.9)' \qquad |\gamma(u+iv)| \sim C|v|^{\frac{1+u+\alpha}{1+n}-\frac{k}{2}} \exp\left\{-\frac{\pi}{2}\left[\left|\frac{v}{1+n}\right|-\left|\frac{v}{1+m}\right|\right]\right\}, \text{ as } |v|\to\infty,$$

in the strip $-(k-1)-\delta \leqq u \leqq -(k-1)+\delta$ for some fixed $\delta>0$, and some constant C. Since $\gamma(u+iv)$ is analytic in that strip, we have

$$(3.10) \qquad \int_{-\infty}^{\infty} |\gamma(u+iv)|^2 dv < M, \text{ in } -(k-1)-\delta \leqq u \leqq -(k-1)+\delta.$$

By Lemma 3, if we put $b=-(k-1)$, there exists a function $R(t)$ such that (2.3) and (2.5) of Lemma 1 are satisfied. Since $1/\Gamma(w)$ is an entire function, the regular property of the function $\gamma(u+iv)$ depends only on the behavior of numerator of the function. Consider the same way as above in the strip $-\delta \leqq u \leqq 1+\delta$, we have (2.9) in Lemma 3 for $b=-1$. (The details of this statement, see N. Levinson [3].) Thus we have (2.4).

Therefore all the conditions of Lemma 1 are satisfied, we have the theorem in this case.

Finally, we shall prove that $E(n, \alpha)$ implies $P(m)$ for $n>m$ and $\alpha \geqq k-1$. In Lemmas 1, 2 and 3, interchange N_1, N_2, k_1 and k_2 with N_2, N_1, k_2 and k_1 respectively. We have

$$(3.12) \quad \gamma(w)=\gamma(u+iv)=\frac{k_1(w)}{k_2(w)}=\frac{1+n}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)}\frac{\Gamma\left(\frac{1-w}{2(1+m)}\right)\Gamma\left(\frac{k+1}{2}-\frac{1-w}{2(1+m)}\right)}{\Gamma\left(\frac{1+w+\alpha}{1+n}\right)},$$

$$(3.13) \qquad |\gamma(u+iv)| \sim C|v|^{\frac{k}{2}-\frac{1+u+\alpha}{1+n}} \exp\left\{-\frac{\pi}{2}\left[\left|\frac{v}{1+m}\right|-\left|\frac{v}{1+n}\right|\right]\right\}, \text{ as } |v|\to\infty,$$

in the strip $-(k-1)-\delta \leqq u \leqq -(k-1)+\delta$ for some fixed $\delta>0$, and some constant C. $\gamma(w)$ is analytic in the strip $1>u>-m(k+1)-k$ and by (3.13), it belongs to L^2 on any ordinate of this strip. Thus as before, the conditions of Lemma 1 are satisfied except for (2.4). Again by Lemma 3, if we put $b=0$, we have an absolutely integrable

$$(3.14) \qquad R(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\gamma(w)t^{w-1}dw.$$

The integrand of (3.14) has poles at $w=1$ and $w=2m+3$, but it has no pole in the strip $1<u<2m+3$. We displace the path of integration to the right of $w=1$ and observe that $w=1$ is a pole, we have

$$(3.15) \quad R(t)=\frac{1+n}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)}\cdot\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1+1+\alpha}{1+n}\right)}$$

$$+\frac{1+n}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)}\frac{1}{2\pi i}\int_{-i\infty+2+2m}^{i\infty+2+2m}\frac{\Gamma\left(\frac{1-w}{2(1+m)}\right)\Gamma\left(\frac{k+1}{2}-\frac{1-w}{2(1+m)}\right)}{\Gamma\left(\frac{1+w+\alpha}{1+n}\right)}t^{w-1}dw,$$

so that

$$|R(t)| \leqq A + A't^{1+2m} \tag{3.16}$$

where A and A' are some positive constants. Since $m > -\frac{1}{2}$, $R(t)$ is bounded for finite t and the conditions of Lemma 2 are fulfilled. This proves the theorem of this case.

REMARK. The essential parts of our theorem are as follows; $E(n, \alpha)$ for $n > 0$, $\alpha \geqq k-1$ implies $P(0)$, while $P(0)$ implies $E(n, \alpha)$ for $0 > n > -1$ and $\alpha \geqq k-1$.

4. Absolute summability theorems. The Tauberian treatment of the absolute summability theorems was inaugurated by Prof. G. Sunouchi [4]. In his method, we get the following lemma.

LEMMA 4. *Under the hypothesis of Lemma* 1 *or Lemma* 2

$$\int_0^\infty \left| d_\varepsilon\left(\frac{1}{\varepsilon}\right) \int_0^1 N_1\left(\frac{t}{\varepsilon}\right) f(t)\, dt \right| < \infty \tag{4.0}$$

implies

$$\int_0^\infty \left| d_\varepsilon\left(\frac{1}{\varepsilon}\right)^k \int_0^1 N_2\left(\frac{t}{\varepsilon}\right) f(t)\, dt \right| < \infty. \tag{4.1}$$

PROOF. Put

$$S(x) = \int_0^x t^{k-1} R(t)\, dt, \tag{4.2}$$

$$F_1(y) = \left(\frac{1}{y}\right)^k \int_0^1 N_1\left(\frac{t}{y}\right) f(t)\, dt, \tag{4.3}$$

and

$$F_2(y) = \left(\frac{1}{y}\right)^k \int_0^1 N_2\left(\frac{t}{y}\right) f(t)\, dt \tag{4.4.}$$

then by (2.3), $S(0)$ and $S(\infty)$ exist. By (2.7), we get

$$\begin{aligned} F_2(\varepsilon) &= \left(\frac{1}{\varepsilon}\right)^k \int_0^\infty R\left(\frac{y}{\varepsilon}\right) \frac{dy}{y} \int_0^1 N_1\left(\frac{t}{y}\right) f(t)\, dt \\ &= \left(\frac{1}{\varepsilon}\right)^k \int_0^\infty y^{k-1} R\left(\frac{y}{\varepsilon}\right) \frac{dy}{y^k} \int_0^1 N_1\left(\frac{t}{y}\right) f(t)\, dt \\ &= \left(\frac{1}{\varepsilon}\right)^k \int_0^\infty y^{k-1} R\left(\frac{y}{\varepsilon}\right) F_1(y)\, dy \end{aligned} \tag{4.5}$$

$$= \frac{1}{\varepsilon}\int_0^\infty \left(\frac{y}{\varepsilon}\right)^{k-1} R\left(\frac{y}{\varepsilon}\right) F_1(y)dy.$$

Integrating by parts, the last term is

$$\left[S\left(\frac{y}{\varepsilon}\right) F_1(y) \right]_0^\infty - \int_0^\infty S\left(\frac{y}{\varepsilon}\right) dF_1(y). \tag{4.6}$$

Since $S(0)$, $S(\infty)$, $F_1(0)$ and $F_2(\infty)$ exist by (2.3) and (4.0), we have

$$\int_0^\infty \left| d_\varepsilon \int_0^\infty S\left(\frac{y}{\varepsilon}\right) dF_1(y) \right| = \int_0^\infty |dF_2(\varepsilon)|. \tag{4.7}$$

Then, by (4.2) and Cameron-Martin's unsymmetric Fubini theorem [2],

$$\int_0^\infty \left| d_\varepsilon \int_0^\infty S\left(\frac{y}{\varepsilon}\right) dF_1(y) \right| \leqq \int_0^\infty |dF_1(y)| \int_0^\infty \left| d_\varepsilon S\left(\frac{y}{\varepsilon}\right) \right| \tag{4.8}$$

$$\leqq \int_0^\infty |dF_1(y)| \int_0^\infty \left| \frac{y}{\varepsilon^2}\left(\frac{y}{\varepsilon}\right)^{k-1} R\left(\frac{y}{\varepsilon}\right) \right| d\varepsilon$$

$$\leqq \int_0^\infty |dF_1(y)| \int_0^\infty |u^{k-1}R(u)|du < \infty.$$

This proves the Lemma.

If we denote by $|P(m)|$ the fact that

$$\frac{1}{\varepsilon}\int_0^\infty \frac{\left(\frac{t}{\varepsilon}\right)^{k-1}}{\left[\left(\frac{t}{\varepsilon}\right)^{2(1+m)} + 1\right]^{\frac{k+1}{2}}} f_x(t)dt, \qquad m \geqq 0, \tag{4.9}$$

is of bounded variation in $(0,\infty)$, and by $|E(n,\alpha)|$ the fact that

$$\varepsilon \int_0^1 \left(\frac{\varepsilon}{t}\right)^\alpha e^{-\left(\frac{\varepsilon}{t}\right)^{1+n}} f_x(t)\frac{dt}{t^2}, \qquad n > -1, \alpha \geqq k-1, \tag{4.10}$$

is of bounded variation in $(0,\infty)$. Then we have

THEOREM 2. *$|E(n, \alpha)|$ for $n > m$ and $\alpha \geqq k-1$ implies $|P(m)|$, while $|P(m)|$ implies $|E(n,\alpha)|$ for $m > n$, $\alpha \geqq k-1$.*

PROOF. To prove the localization property of absolute summability, we need to show that

$$\int_0^\infty \left| d_\varepsilon \frac{1}{\varepsilon}\int_1^\infty \frac{\left(\frac{t}{\varepsilon}\right)^{k-1}}{\left[\left(\frac{t}{\varepsilon}\right)^{2(1+m)} + 1\right]^{\frac{1}{2}(k+1)}} f_x(t)dt \right| < \infty, \quad m \geqq 0. \tag{4.11}$$

Since

$$\frac{d}{d\varepsilon}\left\{\left(\frac{1}{\varepsilon}\right)\int_1^\infty \frac{\left(\frac{t}{\varepsilon}\right)^{k-1}}{\left[\left(\frac{t}{\varepsilon}\right)^{2(1+m)}+1\right]^{\frac{1}{2}(k+1)}} f_x(t)\,dt\right\} \tag{4.12}$$

$$\leqq A\sum_{j=1}^\infty \frac{1}{\left[\left(\frac{j}{\varepsilon}\right)^{2(1+m)}+1\right]^{\frac{1}{2}(k+1)}}\left[\left(\frac{1}{\varepsilon}\right)^{k+1}+\left(\frac{1}{\varepsilon}\right)^{k+1}\frac{\left(\frac{j}{\varepsilon}\right)^{2(1+m)}}{\left(\frac{j}{\varepsilon}\right)^{(1+m)}+1}\right]$$

$$\cdot\int_j^{j+1}|t^{k-1}f_x(t)|\,dt$$

$$=\begin{cases} O(\varepsilon^{m(k+1)}) & \text{as } \varepsilon\to 0,\\ O(\varepsilon^{-(k+1)}) & \text{as } \varepsilon\to\infty,\end{cases}$$

where A is a constant, we have (4.11).

The existence of the solution $R(y)$ of (2.5) is the same as that of Theorem 1. Using Lemma 4 instead of Lemmas 1 and 2, we can show Theorem 2.

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