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ON THE ABILITY OF A CABLE-DRIVEN ROBOT TO GENERATE A PRESCRIBED SET OF WRENCHES

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Abstract

This paper presents a new geometry-based method to determine if a cable-driven robot operating in a d-degree-of-freedom workspace ($2 \leq d \leq 6$) with $n \geq d$ cables can generate a given set of wrenches in a given pose, considering acceptable minimum and maximum tensions in the cables. To this end, the fundamental nature of the *Available Wrench Set* is studied. The latter concept, defined here, is closely related to similar sets introduced in [23, 4]. It is shown that the Available Wrench Set can be represented mathematically by a zonotope, a special class of convex polytopes. Using the properties of zonotopes, two methods to construct the Available Wrench Set are discussed. From the representation of the Available Wrench Set, computationally-efficient and non-iterative tests are presented to verify if this set includes the *Task Wrench Set*, the set of wrenches needed for a given task.

INTRODUCTION AND PROBLEM DEFINITION

A cable-driven robot, or simply cable robot, is a parallel robot whose actuated limbs are cables. The length of the cables can be adjusted in a coordinated manner to control the pose (position and orientation) and/or wrench (force and torque) at the moving platform. Pioneer applications of such mechanisms are the NIST Robocrane [1], the Falcon high-speed manipulator [15] and the Skycam [7]. The fact that cables can only exert efforts in one direction impacts the capability of the mechanism to generate wrenches at the platform. Previous work already presented methods to test if a set of wrenches – ranging from one to all possible wrenches – could be generated by a cable robot in a given pose, considering that cables work only in tension. Some of the proposed methods focus on fully constrained cable robots while others apply to unconstrained robots. In all cases, minimum and/or maximum cable tensions is considered. A complete section of this paper is dedicated to the comparison of the proposed approach with previous methods.

A general geometric approach that addresses all possible cases without using an iterative algorithm is presented here. It will be shown that the results obtained with this approach are consistent with the ones previously presented in the literature [4, 5, 14, 17, 18, 22, 23, 24, 26]. This paper does not address the workspace of cable robots. The latter challenging problem was addressed in several papers over the recent years [10, 11, 12, 19, 25]. Before looking globally at the workspace, all proposed methods must go through the intermediate step of assessing the capability of a mechanism to generate a given set of wrenches. The approach proposed here is also compared with the intermediate steps of the papers on the workspace determination of cable robots.

The task that a robot has to achieve implies that it will have to be able to generate a given set of wrenches in a given pose $\mathbf{x}$. This *Task Wrench Set*, T, depends on the various applications of the considered robot, which can be for example to move a camera or other sensors [7, 6, 9, 3], manipulate payloads [15, 1] or simulate walking sensations to a user immersed in virtual reality [21], just to name a few. The *Available Wrench Set*, A, is the set of wrenches that the mechanism can generate. This set depends on the architecture of the robot, i.e., where the cables are attached on the platform and where the fixed winches are located. It also depends on the configuration pose as well as on the minimum and maximum acceptable tension in the cables. All the wrenches that are possibly needed to accomplish a task can

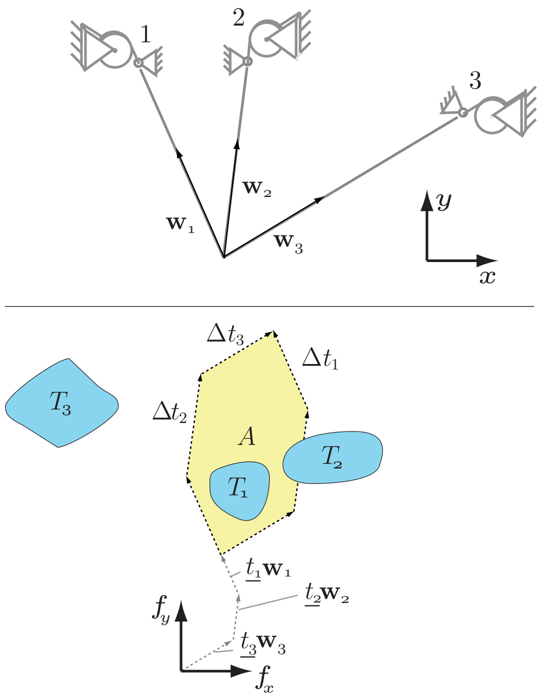

Figure 1 Planar mechanism $n = 3, d = 2$ with its associated A drawn (bottom) and example of three different T.

be generated if the following condition is met:

$$T \subset A. \tag{1}$$

If the Task Wrench Set is a subset of the Available Wrench Set, then all the wrenches in the Task Wrench Set can be generated by the robot. Take for example the simple $d = 2, n = 3$ planar mechanism presented in Figure 1. In the same figure, the Available Wrench Set is also shown, as well as three different T. The wrench that cable i can generate, in this case a force, is directed along cable i toward the ith winch. Its magnitude can vary from a minimum cable tension $\underline{t_i}$ to a maximum cable tension $\overline{t_i}$ and the corresponding range is defined as $\Delta t_i = \overline{t_i} - \underline{t_i}$. As shown in Figure 1, if the Task Wrench Set is completely inside the Available Wrench Set (e.g. T_1), then eq. (1) holds true and the task wrenches can be generated. This is not the case if T is partly in A (e.g. T_2) or if it is completely outside of it (e.g. T_3).

This idea was suggested in [23]. The reader familiar with the latter reference should note that the definitions of the sets A and T proposed here differ slightly from the ones used in [23]. Indeed, in [23], A is defined considering the minimum tension to be zero in all cables, while here the minimum tension can take any value, it can also differ from one cable to another and change with the pose. Also, set T is note used in [23] but rather a similar concept that the authors refer to as the *Desired Wrench Set*. The wrench needed to balance gravity is considered here as a subset of T while it is not included in the Desired Wrench Set used in [23]. Also, T here can be pose-dependent.

If a wrench $\mathbf{w}$ can be generated on the moving platform by the cables, there must exist a solution to the equation

$$\mathbf{W}\mathbf{t} = \mathbf{w} \tag{2}$$

that satisfies the condition

$$\underline{\mathbf{t}} \leq \mathbf{t} \leq \overline{\mathbf{t}}, \tag{3}$$

where

- $\mathbf{t}$ is a column vector whose ith component is the tension in the ith cable;
- $\underline{\mathbf{t}}$ and $\overline{\mathbf{t}}$ are vectors whose components are respectively the minimum and maximum acceptable tensions in the cables. The inequalities in eq. (3) must be interpreted component by component. For instance, $\underline{\mathbf{t}} < \mathbf{t}$ means that all components of $\underline{\mathbf{t}}$ are smaller the the corresponding component of $\mathbf{t}$, i.e., $\underline{t_1} < t_1$, $\underline{t_2} < t_2$, ... , $\underline{t_n} < t_n$;
- $\mathbf{W}$ is the $d \times n$ cable unit wrenches matrix. The ith column $\mathbf{w}_i$ of $\mathbf{W}$ is the unit wrench that cable i can exert on the platform. For a point-mass robot, $\mathbf{w}_i$ will be a unit vector in the direction of the cable, pointing toward the ith winch. For a six degree-of-freedom (DOF) robot, the first three components of $\mathbf{w}_i$ will be a unit force along the cable and the last three components will be the associated torque with respect to the reference frame attached to the moving platform.

Another way of formulating eq. (1) is that there must be a vector $\mathbf{t}$ that is a solution of eq. (2) for all $\mathbf{w} \in T$, and that respects the tension limit conditions of eq. (3). In the rest of this paper, a non-iterative method to verify this condition is presented. To do so, a geometry-based algorithm is developed that consists of two parts. The first step is to define A. The second is to determine if eq. (1) is valid.

It is noted that $\mathbf{W}$ and consequently A are pose-dependent. Depending on the application, T, $\underline{\mathbf{t}}$ and $\overline{\mathbf{t}}$ might also change with the pose.

NATURE OF THE AVAILABLE WRENCH SET

Cables can only pull. Hence, they can only apply a combination of positively weighted unit wrenches. In this work, the weights vary between a minimum and a maximum tension for each cable. Starting from this, eq. (2) can be used to define A:

$$A = \{\mathbf{w} \in \mathbb{R}^d \mid \mathbf{w} = \sum_{i=1}^{n} t_i \mathbf{w}_i,\ \underline{t_i} \leq t_i \leq \overline{t_i}\}. \tag{4}$$

With the change of variables

$$\beta_i = t_i - \underline{t_i}, \tag{5}$$

eq. (4) can be expressed as

$$A = \{\mathbf{w} \in \mathbb{R}^d \mid \mathbf{w} = \sum_{i=1}^{n} \beta_i \mathbf{w}_i + \mathbf{W}\underline{\mathbf{t}},\ 0 \leq \beta_i \leq (\overline{t_i} - \underline{t_i})\}. \tag{6}$$

With another change of variables

$$\alpha_i = \frac{\beta_i}{\Delta t_i}, \tag{7}$$

and recalling $\Delta t_i = \overline{t_i} - \underline{t_i}$, eq. (6) can now be written

$$A = \{\mathbf{w} \in \mathbb{R}^d \mid \mathbf{w} = \sum_{i=1}^{n} \alpha_i \Delta t_i \mathbf{w}_i + \mathbf{W}\underline{\mathbf{t}},\ 0 \leq \alpha_i \leq 1\}. \tag{8}$$

The latter expression corresponds to the definition of a zonotope [13]: *A zonotope is the vector sum of a finite number of closed line segments in some Euclidean space*. The zonotope generated by the set of vectors $Y = \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\} \in \mathbb{R}^d$, denoted $zone(Y)$, is given by:

$$zone(Y) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i,\ 0 \leq \alpha_i \leq 1\}. \tag{9}$$

It is pointed out that some authors use different conventions for the range of α_i (i.e. $-1 \leq \alpha_i \leq 1$ in [16]). The line segments used to define a zonotope are called its *generators*. In our case, the generators are the $\Delta t_i \mathbf{w}_i$. Using the definition of a zonotope, eq. (8) can be written as

$$A = zone(G) \oplus \{\mathbf{W}\underline{\mathbf{t}}\}, \tag{10}$$

where $G = \{\Delta t_1 \mathbf{w}_1, \Delta t_2 \mathbf{w}_2, \cdots, \Delta t_n \mathbf{w}_n\}$ and where $\oplus$ stands for the Minkowski sum of two sets, which is obtained by adding each element of the second set to each element of the first set. Adding $\{\mathbf{W}\underline{\mathbf{t}}\}$, the wrench caused by the minimum acceptable tension in all the cables, produces a translation of $zone(G)$ that does not modify the shape of A. Indeed, the shape of the zonotope depends only on the directions of the unit wrenches ($\mathbf{W}$) as well as the difference between the maximum and minimum acceptable tensions ($\boldsymbol{\Delta}\mathbf{t}$). The translation depends also on the unit wrench matrix and on the minimum tension vector. The zonotope before the translation, $zone(G)$ is called for the rest of this paper the *base zonotope*. It does not depend on the minimum acceptable tensions. Its properties will be used to compare the proposed approach with the other approaches presented in the literature to define the Available Wrench Set.

A zonotope is a special class of convex polytope. A convex polytope is the generalization, in an arbitrary number of dimensions, of the concept of convex polygon in two dimensions or convex polyhedron in three dimensions. It is the convex hull of a finite set of points, which is the minimal convex set containing the set of points. The zonotope exhibits special features that will be discussed later.

Another way of defining a zonotope is by the Minkowski sum of n line segments [8]. As defined above, the Minkowski sum of two sets B and C is obtained by taking the sum of each element of B with each element of C:

$$B \oplus C = \{\mathbf{b} + \mathbf{c} \mid \mathbf{b} \in B, \mathbf{c} \in C\}. \tag{11}$$

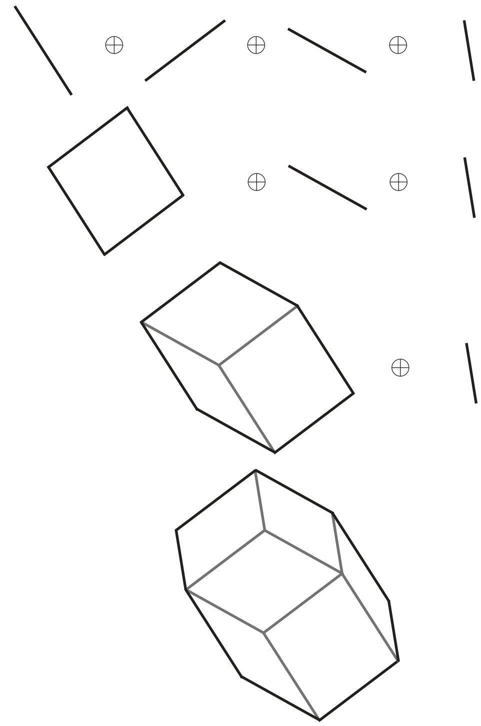

Figure 2 Minkowski sum of four line segments.

Using this definition, A can be defined as

$$A = S_1 \oplus S_2 \oplus \ldots \oplus S_n \oplus \{\mathbf{W}\underline{\mathbf{t}}\}, \tag{12}$$

where S_i is the set representing the line segment between the two end-points of vector $\Delta t_i \mathbf{w}_i$:

$$S_i = \{\mathbf{w} \in \mathbb{R}^d \mid \mathbf{w} = \alpha_i \Delta t_i \mathbf{w}_i,\ 0 \leq \alpha_i \leq 1\}. \tag{13}$$

Figure 2 shows how four line segments can be added in this manner to form a zonotope.

Physically, this definition as a combinatorial sum makes sense: To obtain the set of wrenches that the mechanism can generate, we add all wrenches that a cable can generate to all wrenches that all the other cables can generate. The result is thus the set of wrenches generated by all the acceptable tensions in the cables.

Yet another way of defining a zonotope arising from n generators is as the image of the n-dimensional hypercube under an affine transformation [27]. To this end, eq. (8) can be expressed in matrix form as:

$$A = \{\mathbf{w} \in \mathbb{R}^d \mid \mathbf{w} = \mathbf{M}\boldsymbol{\alpha} + \mathbf{W}\underline{\mathbf{t}},\ 0 \leq \alpha_i \leq 1\}, \tag{14}$$

where

$$\begin{aligned} \mathbf{M} &= \mathbf{W} diag(\boldsymbol{\Delta}\mathbf{t}), && (15)\\ \boldsymbol{\Delta}\mathbf{t} &= [\Delta t_1, \Delta t_2, \ldots, \Delta t_n]^T, && (16)\\ \boldsymbol{\alpha} &= [\alpha_1, \alpha_2, \ldots, \alpha_n]^T. && (17) \end{aligned}$$

The following steps explain how A is constructed mathematically starting from a hypercube. The four steps are illustrated in Figure 3 for the case $d = 2, n = 3$.

1. Because the n α_i can have a value between 0 and 1, they represent a hypercube in the n-dimensional space.

2. Matrix $\mathbf{M}$ modifies the hypercube in two ways. First, it scales the length of the hypercube edge along axis i from 1 to Δt_i, transforming the hypercube into a hyper-rectangle. This is due to the square matrix $diag(\mathbf{\Delta t})$.

3. Second, the $d \times n$ unit wrench matrix $\mathbf{W}$ projects the n-dimensional hyper-rectangle onto the d-dimensional wrench-space. If $d = n$, the number of cables equals the number of DOF. In this case, the hyper-rectangle keeps its topology and is only deformed by $\mathbf{W}$. If $d < n$, there are more cables than DOF. In this case, the hyper-rectangle is projected onto a space of smaller dimension.

4. The last step consists in adding $\{\mathbf{W}\underline{\mathbf{t}}\}$, which translates the projected hyper-rectangle in the wrench-space so the wrench corresponding to $\boldsymbol{\alpha} = \mathbf{0}$ coincides with the wrench induced by the minimum tension in all the cables.

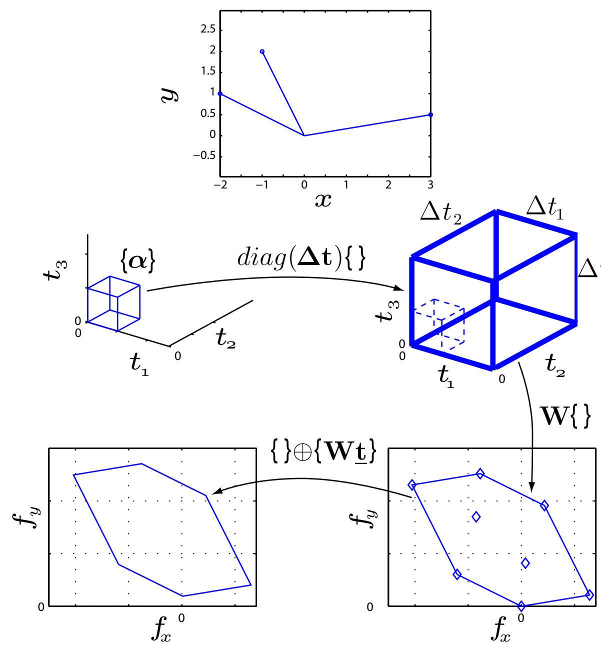

Figure 3 Illustration of the four steps in the mathematical construction of a zonotope for $n = 3$ and $d = 2$. The top image shows the mechanism.

The example of Figure 3 involves a small number of cables and DOFs in order to allow the visualization of construction of the zonotope. The mathematical expression remains the same for larger numbers of DOFs and cables. Other base zonotopes are illustrated in Figure 4 for $d = 3,\ n = 6$ and in Figure 5 for $d = 2,\ n = 5$.

To summarize, we can define the zonotope-shaped Available Wrench Set in three different ways:

1. The linear combinations of a set of vectors with minimum and maximum acceptable magnitude, eq. (8);

2. The Minkowski sum of line segments, eq. (12);

3. The affine transformation of a hypercube, eq. (14).

These three points of view provide different perspectives on the nature of the Available Wrench Set. Zonotopes have other interesting characteristics [27]. A zonotope is a type of polytope that is centrally symmetric. Each face of a zonotope has another face that is parallel to it. Faces of a zonotope are also themselves zonotopes and their edges correspond to generator segments. The faces which have a common generator form a belt zone that wraps around the surface (see Figure 6), hence the name zonotope. All these zones cover the whole surface. The number of these zones is the main measure of the complexity of a zonotope. For the Available Wrench Set of a cable robot, this number corresponds to the number of cables.

$\mathbf{t} \geq$	$\mathbf{t} \leq$	$n = d$	$n > d$
$\mathbf{0}$	∞	[19, 5]	[24, 18, 17, 25]
$\mathbf{0}$	$\bar{\mathbf{t}}$	[23]	[4]
$\underline{\mathbf{t}}$	$\bar{\mathbf{t}}$		[26, 14, 12, 22]

Table 1 Classification of the literature on the ability of a cable mechanism to generate wrenches.

Special cases

Several papers have already addressed the capability of a cable mechanism to generate a set of wrenches. Some of the authors addressed this issue with the ultimate goal of defining the workspace of this type of robot. In this section, it is shown that many of the previously proposed approaches are special cases of the zonotope approach for which the minimum tension is zero and/or the maximum tension is undefined. Table 1 classifies the principal methods found in the literature according to the type of tension limits considered and the relation between n and d. Note that the row $\mathbf{0} \leq \mathbf{t} \leq \bar{\mathbf{t}}$ is itself a special case of the last row. The next three subsections correspond to the three rows of the table. We discuss how the Available Wrench Set obtained using other approaches can be compared with the one obtained in this paper. This will show how the zonotope approach can encompass all the different cases.

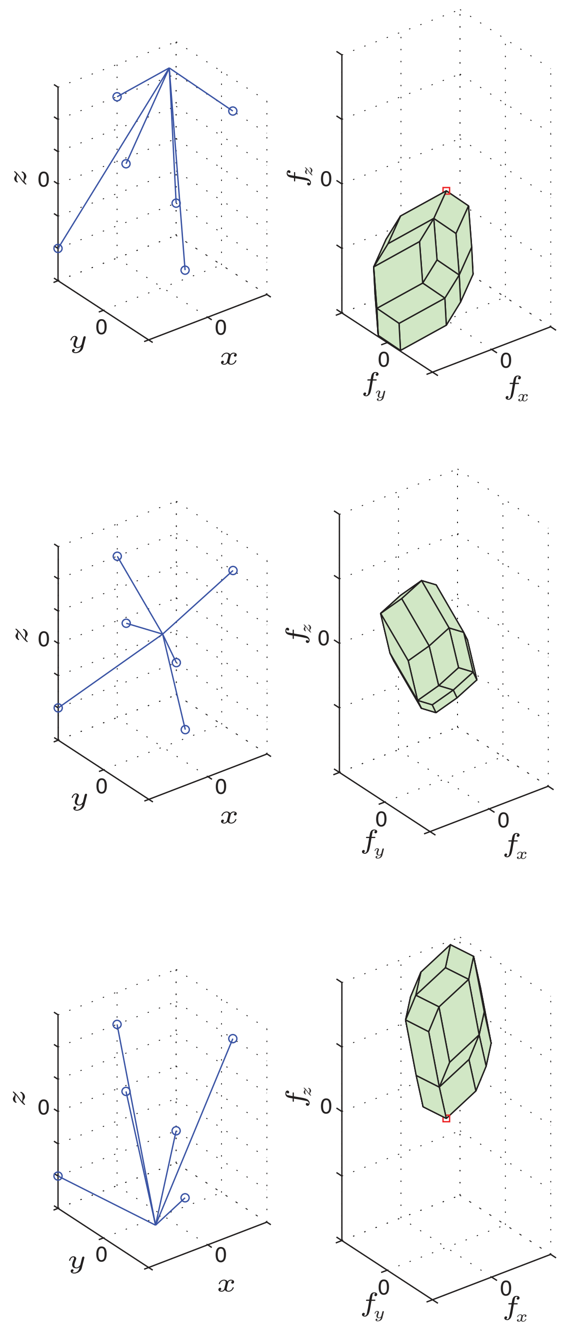

Figure 4 Mechanism $d = 3, n = 6$ in three different poses. The associated base zonotopes are shown on the right.

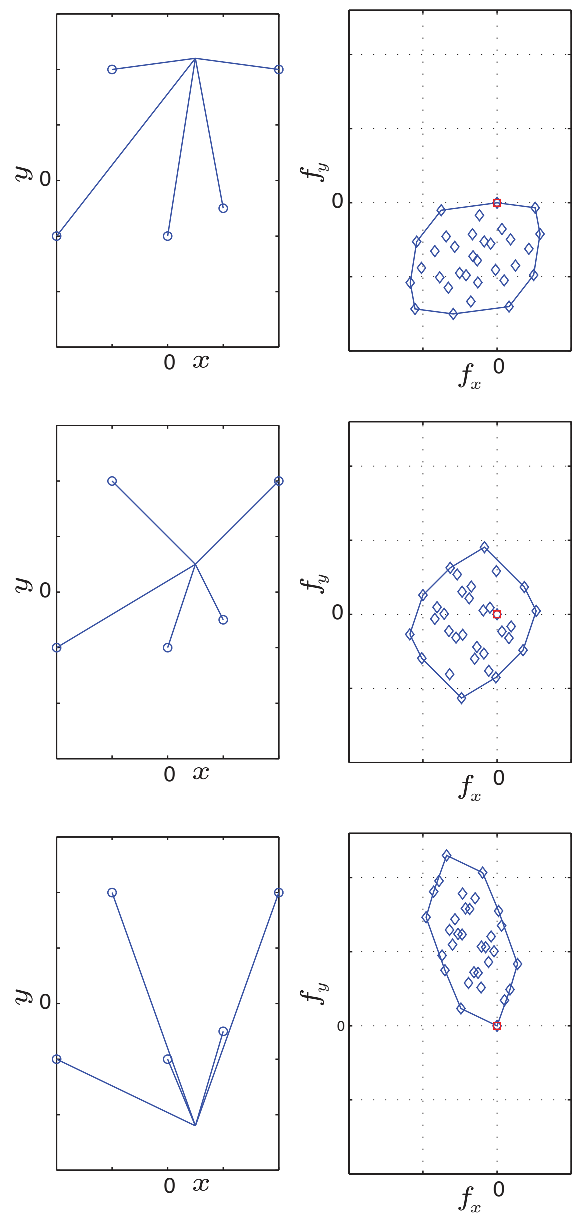

Figure 5 Mechanism $d = 2, n = 5$ in three different poses. The associated base zonotopes are shown on the right. The diamonds are the projected vertices of the hyper-rectangle.

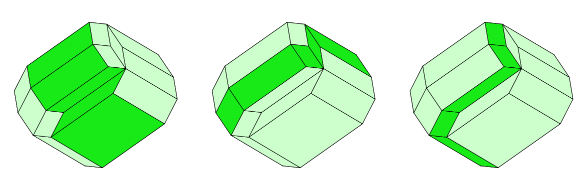

Figure 6 Three belt zones around a zonotope.

Zero Minimum Tension, Undefined Maximum Tensions In [24], the statics of cable mechanisms is studied using the properties of the unit wrench matrix $\mathbf{W}$. The authors suggest that the Available Wrench Set is limited by hyperplanes that can be defined from $d-1$ unit wrenches. This is an important result that is also used in [25] and later in this paper to build the zonotope. In fact, we will see that the faces of the zonotope are supported by such hyperplanes. Because the analysis presented in [24] does not consider maximum tensions, the convex hull of the hyperplanes limiting the Available Wrench Set is an open set corresponding to the open convex hull of the cable unit wrenches. For $n = d$, as in [5], the hull is a hyper-cone bounded by hyperplanes. As no upper tension limit is imposed, unit wrenches inside the open convex hull do not add any wrench-generating capability to the mechanism at a given pose. By imposing the maximum tension, the Available Wrench Set becomes a closed set and the zonotope shape appears.

In references [18, 17], the authors analyze a fully constrained planar mechanism. At a given pose, they determine if there exists a combination of positive tensions enabling the generation of the desired wrench. The solution corresponds to the minimum norm solution obtained using the pseudo-inverse of $\mathbf{W}$, added to $n-d$ vectors in the null space of $\mathbf{W}$. In our approach, if a wrench is feasible, it will be inside the zonotope. This point can be projected back into the hyper-rectangle using the pseudo-inverse of $\mathbf{W}$, which is equivalent to going from step 3 to step 2 in Figure 3. If $n > d$, there will also be other solutions, which will be this point, added to vectors in the null space of $\mathbf{W}$. In the particular example with $n = d+1$, the null-space would be a line, as it is suggested in other papers. The set of all possible tension combinations capable of generating a wrench is the null space of $\mathbf{W}$ added to the minimum norm solution using the pseudo-inverse, intersected with the hyper-rectangle.

In [10, 11], conditions for wrench-closure are stated. Wrench-closure implies that a mechanism can generate wrenches in all directions considering that it must have positive tensions in all cables but without considering maximum tension. Consider a mechanism in a wrench-closed pose. In this case, A is the complete wrench-space. If you start from infinity in the wrench-space and reduce the maximum acceptable tension in the cable, the zonotope shape will appear. If the mechanism is in a wrench-closed pose, even if the maximum tension limit becomes very low, the origin of the wrench space $\mathbf{w} = \mathbf{0}$ will still be included in the Available Wrench Set. To relate it to the zonotope approach, it means that if $\mathbf{w} = \mathbf{0}$ is strictly included inside the base zonotope, the mechanism is in a wrench-closed pose. Otherwise, the origin of the wrench space will be a vertex of the base zonotope. Examples are shown in Figures 4 and 5 where the middle pose in each figure corresponds to a wrench-closed pose while the other two poses do not.

In practice, the wrench-closure workspace has been used to optimize mechanisms (e.g. [21]). Figure 7 shows how the methods not considering an upper bound on the tension must be used with caution. In the figure, the mechanism is in a wrench-closed pose and hence it can generate wrenches in all directions. Accordingly, the origin of the wrench space is included in the base zonotope, but it is very close to the limit. Using the zonotope, one can see that the capability of the mechanism to generate a force in the $-f_y$ direction is very limited if an upper bound on the tension is taken into account.

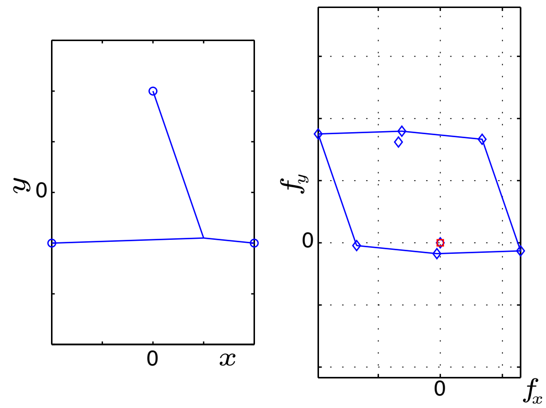

Figure 7 Example of a wrench-closed pose and its corresponding base zonotope.

Zero Minimum Tension, Given Maximum Tensions In [23], maximum tension limits are considered. The latter paper treats the case of suspended point-mass cable robots. The companion paper [4] treats the case of a mechanism with up to $n = 4$, $d = 3$. In both of these publications, a reader that is familiar with the nature of a zonotope will clearly recognize them in some of the figures illustrating the Available Wrench Set. Nevertheless, the authors did not identify this geometric entity and hence could not come up with a general method to obtain it for higher number of cables and DOF. They did notice, however, that the Available Wrench Set has parallel faces so they stated that it has the shape of a parallelepiped, which is almost true. In fact, a parallelepiped is a special class of zonotope for which $n = d$ and thus their statement will only be true in this condition. Because they consider a given maximum tension and a minimum tension of zero, the Available Wrench Set that they obtained corresponded to the base zonotope.

Non Zero Minimum Tension, Given Maximum Tensions In practice, it is relevant to impose a minimum tension larger than zero in order to ensure stiffness. In [26], the authors consider the homogeneous problem using an acceptable ratio between maximum and minimum tensions. They consider this ratio to be the same for all cables. The available tensions thus define a hypercube in the tension-space. They come up with conditions on the ratio to ensure that there exists a set of tensions inside this hypercube. If solutions exist, they are in a region which is the intersection of the translated null space of $\mathbf{W}$ and the hypercube. The authors also address the relevant issues of finding the optimal tension distribution and the continuity of solutions. The main difference between this approach and the one proposed here is that we consider absolute values of the tensions, not ratios. The analysis presented in [26] is performed in the tension domain while the one proposed here allows one to test the feasibility of a wrench directly in the wrench-space where the hyper-rectangle is projected.

Another approach based on the tension-space is also presented in [14]. In this case, the first objective is to minimize the norm of the tension vector. The authors use Dykstra's alternating projection algorithm to find the projection of a point on the acceptable tension convex set, which corresponds to the hyper-rectangle in this paper. With the proposed algorithm, the authors determine whether there is a common subset to the hyper-rectangle and the null space of $\mathbf{W}$ translated by the minimum norm solution. If solutions exist, they find the one that minimizes the norm of the tension vector. If there is no solution, the algorithm returns the minimum distance between the two sets, thereby providing insight on how the needed tensions are far from the hyper-rectangle of acceptable tensions. The algorithm exhibits good convergence but it is iterative and thus cannot be used in real time.

In [12], interval analysis is used to determine the wrench-feasible workspace considering minimum and maximum tension limits that are constant across the workspace. Since the authors are interested in the whole workspace, they do not study in detail the Available Wrench Set at a given pose.

The ability of fully constrained mechanisms to generate the efforts considering tension limits is approached from a different angle in [22]. Stating that the geometrical analysis becomes too complicated in higher dimensions, the authors reduce the problem to a set of inequalities in lower dimensions. Here again, the reader can clearly recognize zonotopes in the figures of the paper. However, the proposed numerical method does not cover the wrench-space homogeneously and thus the Available Wrench Set limits are not precise. Again, this type of calculation would not be suitable in the context of the real-time control of a robot.

As the empty field in Table 1 indicates, the authors are not aware of any paper that treats the case of upper and lower tension bounds for under constrained $n = d$ robots. The method presented here applies to this case as well.

CONSTRUCTING THE AVAILABLE WRENCH SET

Given the above results the shape of the Available Wrench Set is now known. As previously mentioned, a zonotope is a convex polytope. Two representations can be used to define such a polytope [13]: $\mathcal{V}$-representation (for vertices) and $\mathcal{H}$-representation (for hyperplanes). These representations define respectively the vertices of the polytope or the hyperplanes supporting its faces.

A hyperplane is a geometrical object that splits a space into two half-spaces. In one dimension (a line), a hyperplane is a point. In two dimensions, a hyperplane is a line, in three dimensions, it is a plane. For higher dimensions, it has no special name but the idea remains the same: it splits a space into two half-spaces.

We say that a hyperplane supports a set E if:

- E is completely included in one of the two half-spaces;
- E has at least one point that is also included in the hyperplane.

The support plane theorem states that *if E is a convex set and $\mathbf{x}$ is a point on the boundary of E, then there exists at least one support hyperplane that includes* $\mathbf{x}$.

The polytopes of interest in this work are convex. Hence, all their faces are supported by hyperplanes. The polytope is completely on one side of each of these support hyperplanes. For this reason, any convex polytope can be defined by the intersection of half-spaces limited by hyperplanes. In its $\mathcal{H}$-representation, the zonotope-shaped Available Wrench Set A is expressed as:

$$A = \{\mathbf{w} \in \mathbb{R}^d \mid \mathbf{N}\mathbf{w} \leq \mathbf{d}\}, \tag{18}$$

where $\mathbf{N}$ is a matrix whose ith line is $\mathbf{n}_i^T$, a unit vector normal to a hyperplane supporting a face, that points toward the exterior of the zonotope. The corresponding element of $\mathbf{d}$, d_i, can be obtained using a known point $\mathbf{w}_{i0}$ included in the hyperplane, such that

$$d_i = \mathbf{n}_i^T \mathbf{w}_{i0}. \tag{19}$$

For a zonotope, every line of $\mathbf{N}$ repeats itself with an opposite sign because each face has a parallel face. However, the two corresponding d_i differ as the two parallel hyperplanes include different points.

Using the $\mathcal{V}$-representation is the same as describing a mesh: A is defined using a table of points representing the vertices and another table that comprises the indices of the vertices that form the different faces. Both representations can be useful. The $\mathcal{V}$-representation is well suited for visualization. All the figures in this paper are based on it. On the other hand, the $\mathcal{H}$-representation is preferable for the second step of the algorithm that tests the relation between A and T. When all the support hyperplanes are defined, we can verify for all of them if T is included in all the half-spaces that intersect to form the polytope, similarly to what was done in [24, 25, 5]. The $\mathcal{H}$-representation expresses this problem as a set of inequalities.

We now explain two methods to construct the zonotope: the Convex Hull Method and the Hyperplane Shifting Method. The first one must use an iterative algorithm and can output both representations. The second one provides the $\mathcal{H}$-representation and does not use an iterative algorithm.

Convex Hull Method

Consider $zone(Y)$ from eq. (9) and a set H being given as

$$H = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i,\ \alpha_i \in \{0,1\}\}. \tag{20}$$

The set H is thus the set of points obtained from the combinations of the vectors $\mathbf{y}_i$ being multiplied by 0 or 1. Let $conv(H)$ be its convex hull, it is shown in [20] that

$$zone(Y) = conv(H). \tag{21}$$

Hence, a zonotope is the convex hull of the set of points in eq. (9) where all the α_i are either 0 or 1:

$$zone(Y) = conv\left\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i,\ \alpha_i = \{0, 1\}\right\}. \tag{22}$$

This implies that by finding the convex hull of all the wrenches produced when each cable has a tension equal to 0 or Δt_i, the base zonotope is obtained. In other words, the base zonotope is the convex hull of the vertices of the hyper-rectangle projected in the wrench-space. The diamonds in the base zonotope shown in Figures 3, 5 and 7 show all these projected vertices and their convex hulls for $d = 2$ planar mechanisms with different numbers of cables.

The vertices of the zonotope correspond to a change in the limiting condition. To generate a wrench at one of the zonotope's vertices, all cables must be at their minimum or maximum tension. Along the edges, one cable goes from minimum to maximum tension. The number of points from which the convex hull must be calculated is 2^n. It thus depends only on the number of cables and not on the number of DOFs.

It is possible to obtain quickly a matrix $\mathbf{C}$ whose columns represent all the possible combinations of extreme wrenches:

$$\mathbf{C} = \mathbf{M}\mathbf{A}_\alpha, \tag{23}$$

where $\mathbf{M}$ is defined in eq. (15). Each column of the permutation matrix $\mathbf{A}_\alpha$ is equivalent to a binary number that identifies uniquely a vertex of the hypercube according to the convention $\begin{bmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\end{bmatrix}^T$ where α_i is 0 or 1. This matrix has dimensions $n \times 2^n$. For example, for $n = 3$, it is:

$$\mathbf{A}_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{24}$$

In this matrix, a 0 in the ith row indicates a null tension in cable i and 1 a tension being equal to Δt_i. For example, the vector $[1\ 1\ 0]^T$ represents the situation where cables 1 and 2 have a maximum tension while cable 3 has minimum tension. Thus, all the columns of matrix $\mathbf{C}$ represent vertices of the hyper-rectangle projected in the wrench-space. If $n = d$, all these wrenches will be vertices of the zonotope. If $n > d$, then only some of them will be vertices of the zonotope. The others will be included inside the convex hull. The convex hull of the column vectors in $\mathbf{C}$ added to $\mathbf{W}\underline{\mathbf{t}}$ can be calculated to obtain A. A numerical procedure such as *quickhull* [2], which is implemented in *Matlab*, can be utilized. This algorithm is widely used to find the convex hull of a finite set of points in an arbitrary number of dimensions.

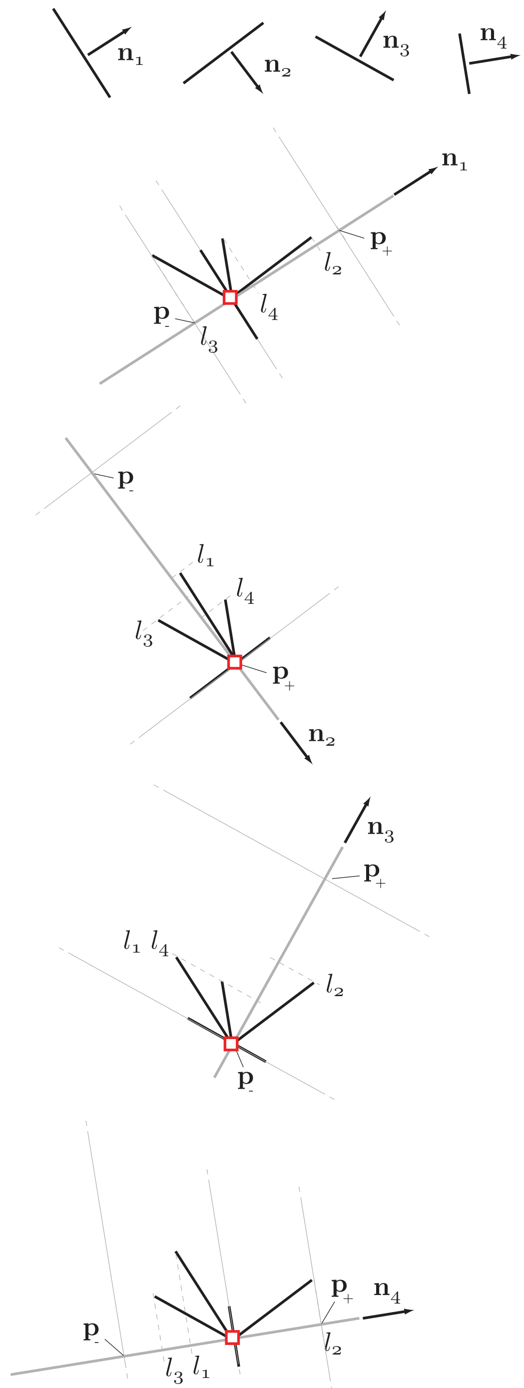

Figure 8 Shifting of the four initial hyperplanes for a $d = 2, n = 4$ mechanism.

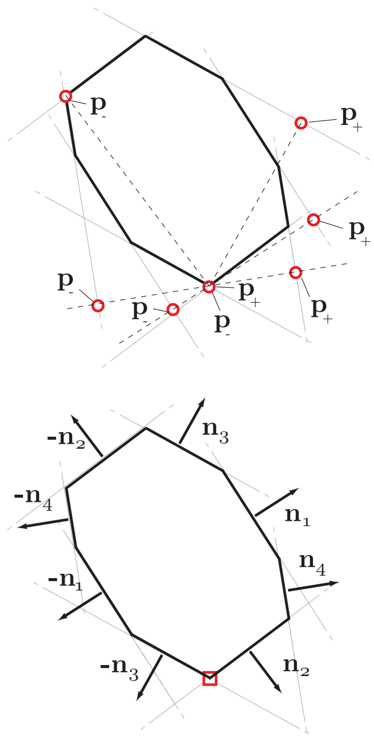

Figure 9 The intersection of all the half spaces defined by the hyperplanes form the zonotope.

Hyperplane Shifting Method

Even if $quickhull$ is usually fast, a method that is not iterative is desirable, particularly in the context of robot control or optimization. This hyperplane shifting method is inspired from [16]. The different steps are explained below. Step 2 is illustrated in Figure 8 with the same generators as in Figure 2. The result of this example, the support hyperplanes and their normal vectors, is shown in Figure 9. If two unit wrenches are linearly dependent and $n = d$, the mechanism is in a singularity and the robot can no longer control all the DOFs. In this case, the method will not work. If $n > d$ the method will work as long as there is a minimum of d linearly independent unit wrenches.

1. **Defining an Initial Hyperplane** – Take a combination of $d-1$ linearly independent cable unit wrenches $\mathbf{w}_i$ that can define $\mathbf{n}$, a unit vector perpendicular to a hyperplane that includes all these unit wrenches. The remaining $n-d+1$ unit wrenches are noted $\mathbf{w}_j$. To obtain $\mathbf{n}$, we normalize the generalized cross product taken among the chosen $\mathbf{w}_i$ so

$$\mathbf{n} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{25}$$

where

$$\mathbf{v} = \mathbf{w}_1 \times \cdots \times \mathbf{w}_{d-1} \tag{26}$$

whose kth component is

$$v_k = (-1)^{k+1} det \begin{bmatrix} {}^k\mathbf{w}_1 & \cdots & {}^k\mathbf{w}_{d-1} \end{bmatrix} \tag{27}$$

and the notation ${}^k\mathbf{w}_i$ represents $\mathbf{w}_i$ with its kth component removed. For example, for $d=3$, $\mathbf{v} = \mathbf{w}_1 \times \mathbf{w}_2$.

2. **Shifting the initial hyperplane** – Because of the nature of the zonotope, two of its faces will have a normal parallel to $\mathbf{n}$. To define completely their distinct supporting hyperplanes, we also need one point included in each of them, namely $\mathbf{p}_+$ and $\mathbf{p}_-$. We start from an initial hyperplane that includes the origin and whose normal is $\mathbf{n}$. The unit wrenches $\mathbf{w}_i$ chosen at step 1 define the orientation of the two faces parallel to this hyperplane. The remaining $\mathbf{w}_j$ will define the position of the faces, or how the initial hyperplane is shifted along $\mathbf{n}$ to coincide with the two supporting hyperplanes. The points $\mathbf{p}_+$ and $\mathbf{p}_-$ can be defined as a distance along $\mathbf{n}$. We want to find these two distances, h_+ and h_-, that are the projection of the vertices inside the two faces on vector $\mathbf{n}$. The first step is to take l_j, the individual projections of the $\mathbf{w}_j$ on $\mathbf{n}$ using the dot product:

$$l_j = \mathbf{w}_j^T \mathbf{n}. \tag{28}$$

Again, all vertices correspond to combinations where all the tensions are maximum or minimum so the $\mathbf{w}_j$ are all weighted by 0 or Δt_j. For this reason, the two distances will be the maximum and minimum combinations of the l_j weighted by 0 or Δt_j:

$$h_+ = max \left(\sum_{j=1}^{n-(d-1)} \alpha_j \Delta t_j l_j,\ \alpha_j = \{0,1\} \right), \tag{29}$$

$$h_- = min \left(\sum_{j=1}^{n-(d-1)} \alpha_j \Delta t_j l_j,\ \alpha_j = \{0,1\} \right). \tag{30}$$

The support hyperplane with $\mathbf{n}$ as outward pointing normal will include the point

$$\mathbf{p}_+ = h_+\mathbf{n} + \mathbf{W}\underline{\mathbf{t}}, \tag{31}$$

and the hyperplane with $-\mathbf{n}$ as outward pointing normal will include the point

$$\mathbf{p}_- = h_-\mathbf{n} + \mathbf{W}\underline{\mathbf{t}}. \tag{32}$$

Note that these points are not necessarily on the faces of the zonotope, they can be on the extension of the faces. Every time this step is completed, two hyperplanes are completely defined.

3. **Obtaining $\mathbf{d}$ for the $\mathcal{H}$-representation** – Steps 1 and 2 are repeated to define all the possible hyperplanes and then step 3 is performed: Vector $\mathbf{d}$ is determined using eq. (19) to obtain the complete $\mathcal{H}$-representation as in eq. (18).

Just like the convex hull method, the hyperplane shifting method is combinatorial and can be treated using permutation matrices.

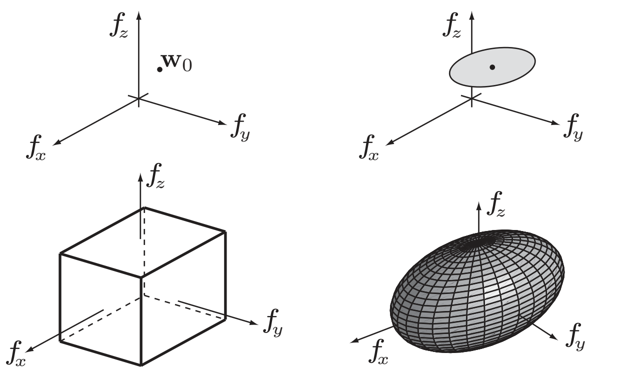

Figure 10 Different possible shapes of Task Wrench Set.

RELATION WITH THE TASK WRENCH SET

We have now defined A using its $\mathcal{H}$-representation. From this description and the definition of T, the final step is to verify if eq. (1) is valid in order to determine if all the possible task wrenches can be generated. The Task Wrench Set can take many forms, depending on the use of the robot. If the robot operates in quasi-static conditions, T can be approximated by a unique wrench, namely the wrench required to balance gravity. If one wants to exert efforts in all directions up to a maximum magnitude [23], T will be a sphere. If the goal is to generate plus or minus a given range of efforts, then T will be a hyper-rectangle [12]. Another possible shape of T is a disk around a given point if the mechanism has to move a platform along a surface [9]. These examples are illustrated in Figure 10. Some other shapes are obviously possible. In practice, most T will fall into three types of shapes, namely the point, the convex polytope and the ellipsoid. The next three subsections will explain how to test if a Task Wrench Set with these shapes can be generated.

Point

If only one wrench $\mathbf{w}_0$ is considered, e.g. the wrench required to balance gravity, it is the simplest case where T is a unique point, namely:

$$T = \{\mathbf{w}_0\}. \tag{33}$$

Using the $\mathcal{H}$-representation, this problem is straight forward. It consists in verifying the vector inequality of eq. (18):

$$\mathbf{N}\mathbf{w}_0 \leq \mathbf{d}. \tag{34}$$

If the inequality is verified, $\mathbf{w}_0$ can be generated because it is on the side of all support hyperplanes toward the interior of A.

Convex Polytope

The convex polytope that would most probably be encountered is the hyper-rectangle shown in Figure 10. Because A is a convex set, if two points are inside A, then all the points on the segment between those two points will also be inside A. If T is also a convex polytope, then if its vertices are inside A, so will be all the points in T. Eq. (34) can thus be modified to obtain a matrix inequality equation:

$$\mathbf{N}\mathbf{V} \leq \mathbf{D}, \tag{35}$$

where each column of the matrix $\mathbf{V}$ is a vertex of polytope T and $\mathbf{D}$ is a matrix whose columns are all identical to $\mathbf{d}$. To determine if a convex polytope T is inside A, the best way is to have T in its $\mathcal{V}$-representation and A in its $\mathcal{H}$-representation in order to be able to use eq. (35) directly.

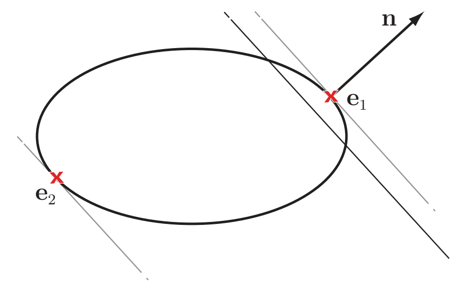

Figure 11 Ellipse crossing a support plane of A.

Ellipsoid

A ball-shaped Task Wrench Set will appear when a mechanism has to generate efforts of a given magnitude in all directions. In some cases, different axes can have different physical units (forces and torques) so the ball will deform to become an ellipsoid. In general, the axes of the ellipsoid will be aligned with the axes of the wrench-space reference frame. This is the case considered here.

Because an ellipsoid is a smooth convex set, every point on its surface will have only one support hyperplane, and this hyperplane will be tangent to the surface at this point. The intersection between a hyperplane and an ellipsoid is a subset of both. The limit case when this subset includes only one point can only arise for a point on the surface of the ellipsoid. In this case, the hyperplane will be a supporting one. There exist two points $\mathbf{e}_{1,2}$ on the surface of an ellipsoid whose normal vectors will be parallel to $\mathbf{n}$, the normal vector defining a zonotope support hyperplane. If these two points are in the same half-space limited by the hyperplane, then the hyperplane does not intersect the ellipsoid. For example, let's take the ellipsoid and hyperplane in two dimensions in Figure 11 (ellipse and black line). The line intersects the ellipse so $\mathbf{e}_1$ and $\mathbf{e}_2$ are in two distinct half spaces separated by the line. From these observations, the method to determine if an ellipsoid-shaped set T is included in A is the following:

1. Find all the points on the surface of the ellipsoid whose normals are the same as the normals of the support hyperplanes of A.
2. Test the matrix inequality, eq. (35) with a matrix $\mathbf{V}$ whose columns are the points obtained in step 1. If all these points are included in A, then $T \subset A$ and all the task wrenches can be generated.

We now explain how to find the different points $\mathbf{e}$ for the first step. The equation of an ellipsoid in $\mathbb{R}^d$ is given by

$$\frac{e_1^2}{a_1^2} + \frac{e_2^2}{a_2^2} + \cdots + \frac{e_d^2}{a_d^2} = 1, \tag{36}$$

where e_j is the coordinate of the point along axis j in the wrench-space and the different a_j are the half-axis of the ellipsoid. A vector perpendicular to the surface $\mathbf{q}$ can be obtained by taking the gradient of eq. (36):

$$\begin{aligned} \mathbf{q} &= \begin{bmatrix} \frac{e_1}{a_1^2} & \frac{e_2}{a_2^2} & \cdots & \frac{e_d}{a_d^2} \end{bmatrix}^T, & (37) \\ &= diag(\frac{1}{a_1^2}, \frac{1}{a_2^2}, \cdots, \frac{1}{a_d^2})\mathbf{e} & (38) \end{aligned}$$

This vector — which is not normalized — must be in the same direction of a given vector $\mathbf{n}$ and hence

$$\mathbf{q} = k\mathbf{n}, \tag{39}$$

where k is an unknown variable. Using eq. (38), eq. (39) becomes:

$$\mathbf{e} = diag(a_1^2, a_2^2, \cdots, a_d^2)k\mathbf{n}. \tag{40}$$

Except for k, we obtain the solution for the desired point $\mathbf{e}$. Eq. (40) is substituted in eq. (36) to determine k and define $\mathbf{e}$ completely:

$$k = \pm \frac{1}{\sqrt{(a_1 n_1)^2 + (a_2 n_2)^2 + \cdots + (a_d n_d)^2}}. \tag{41}$$

The two possible values of k account for the fact that there are two points on the surface of the ellipsoid that have a normal parallel to a vector $\mathbf{n}$. Because each support hyperplane of the zonotope has a parallel hyperplane, it is sufficient to find the points $\mathbf{e}_{1,2}$ for half of the support hyperplanes as long as no pair of chosen hyperplanes are parallel. By finding all the points $\mathbf{e}$, we find the points on the ellipse that are possibly outside A. If they are all in A, then all the points in the ellipsoid are.

CONCLUSION

In this paper, we presented a general and non-iterative method to verify if a given set of wrenches can be generated at the moving platform of a cable-driven robot. The proposed approach makes use of the geometrical concept of zonotope, a particular class of convex polytope. This method considers minimum and maximum acceptable tensions in the cables. It can be applied to an under constrained ($n = d$) or a fully-constrained cable mechanism that operates in a space from two to six degrees of freedom. We discussed how the presented approach unifies previous work on the capability of a cable-driven mechanism to generate a set of wrenches. In the last section, paths were suggested to make use of the method for three types of Task Wrench Sets that should cover the majority of the cases encountered in practice.

It is believed that this geometrical approach could be used in future work as a building block to study the more complex problem of the workspace considering bounded cable tensions. Another potential application could be the development of a non-iterative method to find the optimal cable tension distribution for cases for which $n > d$. This could be useful in the context of robot control to avoid the use of the computationally intensive quadratic programming. A continuation method to make the hyperplane shifting method faster from one pose to the next — in the context of control or architecture optimization — could also be developed.

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