ON THE ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES I

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1. 1. Definitions. Let $\sum a_n$ be a given infinite series, and let s_n^α and t_n^α denote the n-th Cesàro means of order α $(\alpha > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively, where s_n is the n-th partial sum. The series $\sum a_n$ is said to be absolutely summable (C, α), or summable $|C, \alpha|$, if the sequence $\{s_n^\alpha\}$ is of bounded variation, that is, if the infinite series $\sum |s_n^\alpha - s_{n-1}^\alpha|$ is convergent ([4], [6]).

1. 2. In what follows we shall require the following identities.

(1. 2. 1) $$t_n^\alpha = n(s_n^\alpha - s_{n-1}^\alpha) \qquad ([6], [7]);$$

(1. 2. 2) $$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu,$$

where

(1. 2. 3) $$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1} \qquad (|x| < 1);$$

and, by definition ([5]),

(1. 2. 4) $$A_{-1}^\alpha = 0, \; A_0^{-1} = 1, \; A_n^{-1} = 0 \quad (n \geqq 1); \; A_n^{-2} = 0 \quad (n \geqq 2);$$

(1. 2. 5) $$A_n^\alpha = \begin{cases} \binom{n+\alpha}{n} & (\alpha > -1), \\ (-1)^n \binom{-\alpha-1}{n} & (\alpha \leqq -1); \end{cases}$$

(1. 2. 6) $$A_n^\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\} \sim n^\alpha/\Gamma(\alpha+1) \; (\alpha \neq -1, -2, \ldots\ldots\ldots).$$

For any sequence $\{\varepsilon_n\}$, we write

(1. 2. 7) $$\Delta^0 \varepsilon_n = \varepsilon_n, \; \Delta\varepsilon_n = \Delta^1 \varepsilon_n = \varepsilon_n - \varepsilon_{n+1},$$

and

(1.2.8) $$\Delta^{\rho}\varepsilon_n = \sum_{\nu=}^{\infty} A_\nu^{-\rho-1}\varepsilon_{\nu+n},$$

provided this series is convergent.

If h and k are positive integers, we have

(1.2.9) $$\Delta^h\Delta^k\varepsilon_n = \Delta^{h+k}\varepsilon_n,$$

and

(1.2.10) $$\Delta^k(\delta_n\varepsilon_n) = \sum_{r=0}^{k}\binom{k}{r}\Delta^r\delta_n\Delta^{k-r}\varepsilon_{n+r}.$$

We also write

(1.2.11) $$\bar{\varepsilon}_n = \Delta\varepsilon_n,$$

(1.2.12) $$E^{\beta,\sigma}(n,\mu) = \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} A_\nu^{-\sigma-1}\varepsilon_{\nu+\mu};\ E^{\beta,\sigma}(-1,\mu) = 0;$$

(1.2.13) $$\bar{E}^{\beta,\sigma}(n,\mu) = \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} A_\nu^{-\sigma-1}\bar{\varepsilon}_{\nu+\mu};$$

and

(1.2.14) $$T_n^k = \sum_{\nu=1}^{n} \nu^{-1}t_\nu^k,\ T_0^k = 0.$$

2.1. Introduction. The present paper is the first of a series of papers which we intend to devote to the study of the absolute Cesàro summability factors of infinite series. We take our start from a recent paper of Bosanquet and Chow ([2]) in which the following theorem is shown to be true by virtue of its equivalence to a previous result of Chow ([3], Theorem 2).

THEOREM A. *If* $\kappa \geqq -1$, $\rho \geqq 0$, $p \geqq 0$, *necessary and sufficient conditions for* $\sum a_n\varepsilon_n$ *to be summable* $|C,\rho|$ *whenever* $s_n^\kappa = O(n^p)$ *are*:

(I)a $\sum n^{\kappa-\rho+p}|\varepsilon_n| < \infty$, (I)b $\sum n^{-1+p}|\varepsilon_n| < \infty$,

(II) $\sum n^{\kappa+p}|\Delta^{\kappa+1}\varepsilon_n| < \infty$.

The demonstration of this theorem by Bosanquet and Chow is naturally circuitous, involving the proof of a number of results which are apparently out of context. Our main object in the proposed series of papers, is to replace such a roundabout technique of proof by one which is direct and straightforward. In this paper we give a direct proof of a generalized version of Theorem A in the case: $\kappa = \rho =$ a positive integer, replacing the sequence $\{n^p\}$ by a wider class of sequences $\{\lambda_n\}$. It may be observed that our generalization of the necessity part of Theorem A holds in the general case $\kappa=\rho=k\geqq 0$, since in our proof we do not confine k to integral values only.

2. 2. We establish the following theorem.

THEOREM. *If k be an integer $\geqq 0$, then necessary and sufficient conditions that $\sum a_n\varepsilon_n$ should be summable $|C, k|$ whenever*

(2. 2. 1) $$s_n^k=O(\lambda_n), \text{ as } n\to\infty,$$

where $\{\lambda_n\}$ is a positive, monotonic non-decreasing sequence, are:

(i) $\sum \lambda_n|\varepsilon_n|<\infty$

and (ii) $\sum n^k\lambda_n|\Delta^{k+1}\varepsilon_n|<\infty$.

3. 1. We require the following lemmas for the proof of the sufficiency part of this theorem.

LEMMA 1. *If $p>0$, $\{\lambda_n\}$ is a positive, monotonic nondecreasing sequence, and $\sum n^p\lambda_n|\Delta^{p+1}\varepsilon_n|<\infty$, then $\sum n^q\lambda_n|\Delta^{q+1}\varepsilon_n|<\infty$ for every q such that $0\leqq q\leqq p$.*

This is proved by an easy adaptation of the proof of Lemma 2 of [1].

LEMMA 2 ([3], Lemma 1). *If $\sigma>-1$ and $\sigma-\delta>0$, then*

$$\sum_{n=\mu}^{\infty}\frac{A_{n-\mu}^{\delta}}{nA_n^{\sigma}}=\sum_{n=0}^{\infty}\frac{A_n^{\delta}}{(n+\mu)A_{n+\mu}^{\sigma}}=\frac{1}{\mu A_{\mu}^{\sigma-\delta-1}}.$$

LEMMA 3 ([3], Lemma 9). *If $\beta>0$, $\sigma>0$, then*

$$\frac{E^{3,\sigma}(n,\mu)}{\overline{E}^{3,\sigma}(n,\mu)}=\sum_{r=0}^{k}\binom{k}{r}\sum_{\nu=0}^{n}A_{n-\nu}^{\beta-r-1}A_{\nu}^{-\sigma+k-1}\frac{\Delta^{k-r}\varepsilon_{\nu+\mu+r}}{\Delta^{k-r}\overline{\varepsilon}_{\nu+\mu+r}}.$$

3. 2. Proof of the theorem: Sufficiency. Let $\tau_n=\tau_n^0=na_n\varepsilon_n$, and let τ_n^k denote the n-th Cesàro mean of order k of the sequence $\{\tau_n\}$. By hypothesis, and by (1. 2. 1),

$$T_n^k=O(\lambda_n),$$

as $n\to\infty$. And, again by (1. 2. 1), we have to show that

$$\sum_n n^{-1}|\tau_n^k|<\infty.$$

The case $k=0$ is obvious. We therefore take $k\geqq 1$. We have

$$A_n^k\tau_n^k=\sum_{\nu=1}^{n}A_{n-\nu}^{k-1}\nu a_\nu\varepsilon_\nu$$

$$= \sum_{\nu=1}^{n} A_{n-\nu}^{k-1} \varepsilon_\nu \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{-k-1} A_\mu^k t_\mu^k$$

$$= \sum_{\mu=1}^{n} A_\mu^k t_\mu^k \sum_{\nu=0}^{n-\mu} A_{n-\mu-\nu}^{k-1} A_\nu^{-k-1} \varepsilon_{\nu+\mu}$$

$$= \sum_{\mu=1}^{n} A_\mu^k t_\mu^k E^{k,k}(n-\mu, \mu)$$

$$= \sum_{\mu=1}^{n} \mu A_\mu^k E^{k,k}(n-\mu, \mu) \frac{t_\mu^k}{\mu}$$

Hence, by Abel's transformation,

$$A_n^k \tau_n^k = \sum_{\mu=1}^{n-1} \Delta_\mu \{\mu A_\mu^k E^{k,k}(n-\mu, \mu)\} T_\mu^k + n A_n^k E^{k,k}(0, n) T_n^k$$

$$= \sum_{\mu=1}^{n} \Delta_\mu \{\mu A_\mu^k E^{k,k}(n-\mu, \mu)\} T_\mu^k + (n+1) A_{n+1}^k E^{k,k}(-1, n+1) T_n^k$$

$$= \sum_{\mu=1}^{n} \Delta_\mu \{\mu A_\mu^k E^{k,k}(n-\mu, \mu)\} T_\mu^k$$

$$= \sum_{\mu=1}^{n} \Delta_\mu (\mu A_\mu^k) E^{k,k}(n-\mu, \mu) T_\mu^k$$

$$+ \sum_{\mu=1}^{n} (\mu+1) A_{\mu+1}^k \Delta_\mu E^{k,k}(n-\mu, \mu) T_\mu^k.$$

Now,

$$\Delta_\mu E^{k,k}(n-\mu, \mu)$$

$$= \sum_{\nu=0}^{n-\mu} A_{n-\mu-\nu}^{k-1} A_\nu^{-k-1} \varepsilon_{\nu+\mu} - \sum_{\nu=0}^{n-\mu-1} A_{n-\mu-\nu-1}^{k-1} A_\nu^{-k-1} \varepsilon_{\nu+\mu+1}$$

$$= \sum_{\nu=0}^{n-\mu-1} A_\nu^{-k-1} \Delta_\mu (A_{n-\mu-\nu}^{k-1} \varepsilon_{\nu+\mu}) + A_{n-\mu}^{-k-1} \varepsilon_n$$

$$= \sum_{\nu=0}^{n-\mu-1} A_\nu^{-k-1} A_{n-\mu-\nu}^{k-1} \Delta_\mu \varepsilon_{\nu+\mu}$$

$$+ \sum_{\nu=0}^{n-\mu-1} A_\nu^{-k-1} A_{n-\mu-\nu}^{k-2} \varepsilon_{\nu+\mu+1} + A_{n-\mu}^{-k-1} \varepsilon_n$$

$$= \sum_{\nu=0}^{n-\mu} A_\nu^{-k-1} A_{n-\mu-\nu}^{k-1} \Delta \varepsilon_{\nu+\mu} + \sum_{\nu=0}^{n-\mu} A_\nu^{-k-1} A_{n-\mu-\nu}^{k-2} \varepsilon_{\nu+\mu+1}$$

$$- A_{n-\mu}^{-k-1}\Delta\varepsilon_n - A_{n-\mu}^{-k-1}\varepsilon_{n+1} + A_{n-\mu}^{-k-1}\varepsilon_n$$
$$= \overline{E}^{k,k}(n-\mu,\mu) + E^{k-1,k}(n-\mu,\mu+1).$$

Hence

$$A_n^k \tau_n^k = \sum_{\mu=1}^{n} \Delta_\mu(\mu A_\mu^k) E^{k,k}(n-\mu,\mu) T_\mu^k$$
$$+ \sum_{\mu=1}^{n} (\mu+1) A_{\mu+1}^k \overline{E}^{k,k}(n-\mu,\mu) T_\mu^k$$
$$+ \sum_{\mu=1}^{n} (\mu+1) A_{\mu+1}^k E^{k-1,k}(n-\mu,\mu+1) T_\mu^k$$
$$= \Delta_n(nA_n^k) E^{k,k}(0,n) T_n^k + (n+1) A_{n+1}^k \overline{E}^{k,k}(0,n) T_n^k$$
$$+ (n+1) A_{n+1}^k E^{k-1,k}(0,n+1) T_n^k + nA_n^k E^{k-1,k}(1,n) T_{n-1}'$$
$$+ \sum_{\mu=1}^{n-1} \Delta_\mu(\mu A_\mu^k) E^{k,k}(n-\mu,\mu) T_\mu^k$$
$$+ \sum_{\mu=1}^{n-1} (\mu+1) A_{\mu+1}^k \overline{E}^{k,k}(n-\mu,\mu) T_\mu^k$$
$$+ \sum_{\mu=1}^{n-2} (\mu+1) A_{\mu+1}^k E^{k-1,k}(n-\mu,\mu+1) T_\mu^k.$$

Therefore

$$\sum_{n=2}^{\infty} n^{-1} |\tau_n^k|$$
$$\leqq \sum (nA_n^k)^{-1} |\Delta_n(nA_n^k)|\, |\varepsilon_n|\, |T_n^k|$$
$$+ \sum (nA_n^k)^{-1}(n+1) A_{n+1}^k |\Delta\varepsilon_n|\, |T_n^k|$$
$$+ \sum (nA_n^k)^{-1}(n+1) A_{n+1}^k |\varepsilon_{n+1}|\, |T_n^k|$$
$$+ k \sum |\Delta\varepsilon_n|\, |T_{n-1}^k| + \sum |\varepsilon_n|\, |T_{n-1}^k|$$
$$+ \sum (nA_n^k)^{-1} \sum_{\mu=1}^{n-1} |\Delta_\mu(\mu A_\mu^k)|\, |E^{k,k}(n-\mu,\mu)|\, |T_\mu^k|$$
$$+ \sum (nA_n^k)^{-1} \sum_{\mu=1}^{n-1} (\mu+1) A_{\mu+1}^k |\overline{E}^{k,k}(n-\mu,\mu)|\, |T_\mu^k|$$

$$+ \sum (nA_n^k)^{-1} \sum_{\mu=0}^{n-2} (\mu + 1)A_{\mu+1}^k |E^{k-1,k}(n - \mu, \mu + 1)| \, |T_\mu^k|.$$

Thus, it suffices for our purpose to show that

(3. 2. 1) $$\sum n^{-1}\lambda_n |\varepsilon_n| < \infty;$$

(3. 2. 2) $$\sum \lambda_n |\Delta\varepsilon_n| < \infty;$$

(3. 2. 3) $$\sum \lambda_n |\varepsilon_{n+1}| < \infty;$$

(3. 2. 4) $$\sum \lambda_{n-1} |\Delta\varepsilon_n| < \infty;$$

(3. 2. 5) $$\sum \lambda_{n-1} |\varepsilon_n| < \infty;$$

(3. 2. 6) $$\sum_{\mu=1}^{\infty} A_\mu^k \lambda_\mu \sum_{n=1}^{\infty} \frac{|E^{k,k}(n, \mu)|}{(n + \mu)A_{n+\mu}^k} < \infty;$$

(3. 2. 7) $$\sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu \sum_{n=1}^{\infty} \frac{|E^{k,k}(n, \mu)|}{(n + \mu)A_{n+\mu}^k} < \infty;$$

(3. 2. 8) $$\sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu \sum_{n=2}^{\infty} \frac{|E^{k-1,k}(n, \mu + 1)|}{(n + \mu)A_{n+\mu}^k} < \infty.$$

(3. 2. 1)—(3.2.5) are all true by hypothesis. We proceed to prove the resı

PROOF OF (3. 2. 6). By Lemma 3, since $n \geqq 1$,

$$E^{k,k}(n, \mu) = \sum_{r=0}^{k-1} \binom{k}{r} A_n^{k-r-1} \Delta^{k-r} \varepsilon_{\mu+r},$$

and, for $0 \leqq r \leqq k - 1$,

$$\sum_{\mu=1}^{\infty} A_\mu^k \lambda_\mu \sum_{n=1}^{\infty} A_n^{k-r-1} |\Delta^{k-r} \varepsilon_{\mu+r}| \frac{1}{(n+\mu)A_{n+\mu}^k}$$

$$\leqq \sum_{\mu=1}^{\infty} A_\mu^k \lambda_\mu |\Delta^{k-r} \varepsilon_{\mu+r}| \sum_{n=0}^{\infty} \frac{A_n^{k-r-1}}{(n+\mu)A_{n+\mu}^k}$$

$$= \sum_{\mu=1}^{\infty} A_\mu^k \lambda_\mu |\Delta^{k-r} \varepsilon_{\mu+r}| \frac{1}{\mu A_\mu^r}$$

(by Lemma 2)

$$\leqq K \sum_{\mu=1}^{\infty} \mu^{k-r-1} \lambda_\mu |\Delta^{k-r} \varepsilon_{\mu+r}| < \infty, *$$

by hypothesis and Lemma 1.

* K denotes a positive constant, not always the same.

PROOF OF (3.2.7). By Lemma 3, since $n \geqq 1$,

$$\bar{E}^{k,k}(n, \mu) = \sum_{r=0}^{k-1} \binom{k}{r} A_n^{k-r-1} \Delta^{k-r} \bar{\varepsilon}_{\mu+r},$$

and, for $0 \leqq r \leqq k-1$,

$$\sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu \sum_{n=1}^{\infty} A_n^{k-r-1} |\Delta^{k-r} \bar{\varepsilon}_{\mu+r}| \frac{1}{(n+\mu)A_{n+\mu}^k}$$

$$\leqq \sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu |\Delta^{k-r+1} \varepsilon_{\mu+r}| \sum_{n=0}^{\infty} \frac{A_n^{k-r-1}}{(n+\mu)A_{n+\mu}^k}$$

$$= \sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu |\Delta^{k-r+1} \varepsilon_{\mu+r}| \frac{1}{\mu A_\mu^r}$$

(by Lemma 2)

$$\leqq K \sum_{\mu=1}^{\infty} \mu^{k-r} \lambda_\mu |\Delta^{k-r+1} \varepsilon_{\mu+r}| < \infty,$$

by hypothesis and Lemma 1.

PROOF OF (3.2.8). By Lemma 3, since $n \geqq 2$,

$$E^{k-1,k}(n, \mu+1) = \sum_{r=0}^{k-2} \binom{k}{r} A_n^{k-r-2} \Delta^{k-r} \varepsilon_{\mu+r+1},$$

and, for $0 \leqq r \leqq k-2$,

$$\sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu \sum_{n=2}^{\infty} A_n^{k-r-2} |\Delta^{k-r} \varepsilon_{\mu+r+1}| \frac{1}{(n+\mu)A_{n+\mu}^k}$$

$$\leqq \sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu |\Delta^{k-r} \varepsilon_{\mu+r+1}| \sum_{n=0}^{\infty} \frac{A_n^{k-r-2}}{(n+\mu)A_{n+\mu}^k}$$

$$= \sum_{\mu=1}^{\infty} A_\mu^{k+1} \lambda_\mu |\Delta^{k-r} \varepsilon_{\mu+r+1}| \frac{1}{\mu A_\mu^{r+1}}$$

(by Lemma 2)

$$\leqq K \sum_{\mu=1}^{\infty} \mu^{k-r-1} \lambda_\mu |\Delta^{k-r} \varepsilon_{\mu+r+1}| < \infty,$$

by hypothesis and Lemma 1.

This completes the proof of the sufficiency part of our theorem.

4.1. We require the following additional lemmas for the proof of the necessity part of the theorem.

LEMMA 4 ([3], Lemma 5). *If the sequences $\{u_n\}$ and $\{U_n\}$ are connected by the relation*

$$U_n = \sum_{\mu=1}^{\infty} S_{n,\mu}\, u_\mu \qquad (n = 1, 2, 3, \ldots\ldots),$$

a necessary condition that the series $\sum U_n$ should be convergent whenever $u_n = O(1)$ is that

$$\sum_{\mu=1}^{\infty} \left| \sum_{n=1}^{\infty} S_{n,\mu} \right| < \infty.$$

LEMMA 5 ([3], Lemma 6). *If the sequences $\{u_n\}$ and $\{U_n\}$ are connected by the relation*

$$U_n = \sum_{\mu=1}^{n} S_{n,\mu}\, u_\mu \qquad (n = 1, 2, 3, \ldots\ldots),$$

a necessary condition that the series $\sum |U_n|$ should be convergent whenever $u_n = O(1)$ is that

$$\sum_{\mu=1}^{\infty} |S_{\mu,\mu}| < \infty.$$

LEMMA 6. *Let $k > 0$, $\varepsilon_n = O(1)$, and*

$$E^{*k,k+1}(n, \mu) = \sum_{\nu=0}^{n} A_{n-\nu}^{k-1} A_\nu^{-k-2}(\nu + \mu)\, \varepsilon_{\nu+\mu}.$$

Then

$$\sum = \sum_{n=0}^{\infty} \frac{E^{*k,k+1}(n, \mu)}{(n+\mu)A_{n+\mu}^k} = \Delta^{k+1} \varepsilon_\mu.$$

PROOF. The proof runs parallel to that of Lemma 7 of Chow [3]. We give it here for completeness. By Lemma 2, we have

$$\sum = \sum_{\nu=0}^{\infty} A_\nu^{-k-2}(\nu+\mu)\varepsilon_{\nu+\mu} \sum_{n=\nu}^{\infty} \frac{A_{n-\nu}^{k-1}}{(n+\mu)A_{n+\mu}^k}$$

$$= \sum_{\nu=0}^{\infty} A_\nu^{-k-2}(\nu+\mu)\varepsilon_{\nu+\mu} \frac{1}{\nu+\mu}$$

$$= \sum_{\nu=0}^{\infty} A_\nu^{-k-2} \varepsilon_{\nu+\mu} = \Delta^{k+1} \varepsilon_\mu,$$

the inversion of the order of summation being justified by absolute convergence.

4.2. Proof of the theorem: Necessity. First we take $k > 0$. Since

$$\begin{aligned} A_n^k \tau_n^k &= \sum_{\mu=1}^{n} A_{n-\nu}^{k-1} \nu a_\nu \varepsilon_\nu \\ &= \sum_{\nu=1}^{n-1} \Delta_\nu (A_{n-\nu}^{k-1} \nu \varepsilon_\nu) s_\nu + s_n n \varepsilon_n \\ &= \sum_{\nu=1}^{n} \Delta_\nu (A_{n-\nu}^{k-1} \nu \varepsilon_\nu) s_\nu \\ &= \sum_{\nu=1}^{n} \Delta_\nu (A_{n-\nu}^{k-1} \nu \varepsilon_\nu) \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{-k-1} A_\mu^k s_\mu^k \\ &= \sum_{\mu=1}^{n} A_\mu^k s_\mu^k \sum_{\nu=\mu}^{n} \Delta_\nu (A_{n-\nu}^{k-1} \nu \varepsilon_\nu) A_{\nu-\mu}^{-k-1}, \end{aligned}$$

we have

$$\frac{\tau_n^k}{n} = \sum_{\mu=1}^{n} \frac{s_\mu^k}{\lambda_\mu} \cdot \frac{\lambda_\mu A_\mu^k}{n A_n^k} \sum_{\nu=\mu}^{n} \Delta_\nu (A_{n-\nu}^{k-1} \nu \varepsilon_\nu) A_{\nu-\mu}^{-k-1}. \tag{4.2.1}$$

THE CONDITION (i) IS NECESSARY. By virtue of Lemma 5, in order that $\sum a_n \varepsilon_n$ be summable $|C, k|$, that is, $\sum \frac{|\tau_n^k|}{n} < \infty$, whenever $s_n^k = O(\lambda_n)$, as $n \to \infty$, it is necessary that

$$\sum_{\mu=1}^{\infty} \frac{\lambda_\mu}{\mu} \left| \sum_{\nu=\mu}^{\mu} \Delta_\nu (A_{\mu-\nu}^{k-1} \nu \varepsilon_\nu) A_{\nu-\mu}^{-k-1} \right| < \infty. \tag{4.2.2}$$

Since, when $\nu = \mu$,

$$\begin{aligned} &\Delta_\nu (A_{\mu-\nu}^{k-1} \nu \varepsilon_\nu) \\ &\quad = A_{\mu-\nu}^{k-1} \nu \varepsilon_\nu - A_{\mu-\nu-1}^{k-1} (\nu + 1) \varepsilon_{\nu+1} \\ &\quad = A_0^{k-1} \mu \varepsilon_\mu - A_{-1}^{k-1} (\mu + 1) \varepsilon_{\mu+1} \\ &\quad = \mu \varepsilon_\mu, \end{aligned}$$

(4.2.2) reduces to

$$\sum_{\mu=1}^{\infty} \lambda_\mu |\varepsilon_\mu| < \infty.$$

THE CONDITION (ii) IS NECESSARY. The condition (i) which is shown above to be necessary implies that $\varepsilon_n = O(1)$. Hence, by Lemma 4, in order that $\sum a_n \varepsilon_n$ be summable $|C, k|$ whenever $s_n = O(\lambda_n)$, as $n \to \infty$, it is necessary that

$$\sum_{\mu=1}^{\infty}\left|\sum_{n=\mu}^{\infty}\frac{\lambda_\mu A_\mu^k}{nA_n^k}\sum_{\nu=\mu}^{n}\Delta_\nu\,(A_{n-\nu}^{k-1}\,\nu\varepsilon_\nu)\,A_{\nu-\mu}^{-k-1}\right|<\infty,$$

that is,

$$\sum_{\mu=1}^{\infty}\lambda_\mu\,\mu^k\left|\sum_{n=0}^{\infty}\frac{1}{(n+\mu)\,A_{n+\mu}^k}\sum_{\nu=0}^{n}\Delta_\nu\{A_{n-\nu}^{k-1}\,(\nu+\mu)\varepsilon_{\nu+\mu}\}\,A_\nu^{-k-1}\right|<\infty. \tag{4.2.3}$$

Since

$$\begin{aligned}
&E^{*k,k+1}(n,\mu)\\
&=\sum_{\nu=0}^{n}A_{n-\nu}^{k-1}\,A_\nu^{-k-2}(\nu+\mu)\,\varepsilon_{\nu+\mu}\\
&=\sum_{\nu=0}^{n-1}\Delta_\nu\{A_{n-\nu}^{k-1}\,(\nu+\mu)\varepsilon_{\nu+\mu}\}\;A_\nu^{-k-1}+(n+\mu)\,\varepsilon_{n+\mu}\,A_n^{-k-1}\\
&=\sum_{\nu=0}^{n}\Delta_\nu\{A_{n-\nu}^{k-1}\,(\nu+\mu)\,\varepsilon_{\nu+\mu}\}\;A_\nu^{-k-1},
\end{aligned}$$

(4.2.3) reduces to

$$\sum_{\mu=1}^{\infty}\lambda_\mu\,\mu^k\left|\sum_{n=0}^{\infty}\frac{1}{(n+\mu)\,A_{n+\mu}^k}E^{*k,k+1}(n,\mu)\right|<\infty,$$

which, by Lemma 6, is equivalent to

$$\sum_{\mu=1}^{\infty}\lambda_\mu\mu^k\,|\Delta^{k+1}\,\varepsilon_\mu|<\infty.$$

We finally dispose of the necessity part of the theorem in the case $k=0$, with the remark that the condition (i) (to which the condition (ii) reduces) becomes necessary as soon as we take the result of the theorem as true in the special case: $a_n=(-1)^n\lambda_n$.

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