*Optical and Quantum Electronics* 9 *(1977) 223–232*

# On the accuracy of the WKB approximation in optical dielectric waveguides

A. H. HARTOG, M. J. ADAMS
*Department of Electronics, University of Southampton, Southampton, SO9 5NH*

*Received 1 September 1976; revised 4 January 1977*

Eigenvalues and field distributions for optical waveguides obtained from the WKB approximation have been compared with those found from geometrical optics and from more rigorous weakly-guiding LP mode theories in the cases of the step-index and parabolic-index profiles. In all cases it is found that the zero-order WKB approximation yields very accurate eigenvalues provided care is taken in the choice of phase factors in the eigenvalue equation. Expressions are deduced for the required phase factors for guides of arbitrary index profile in both two- and three-dimensions, and physical interpretations are given in terms of ray optics. The first-order WKB field distributions are found to give good agreement with the mode fields everywhere except in the vicinity of the caustics.

## 1. Introduction

The WKB method is easily adapted to the analysis of multimode graded-index fibres and has been used recently to deal with a number of problems of practical importance [1–9]. However, there is no clear criterion for its accuracy in a particular problem other than the well-known requirement that the refractive index variation must be small in distances of the order of a wavelength. The object of the present work is therefore to review certain aspects of the theory and to examine the validity of the method for two important cases where it can be compared with exact analytical results, viz. the step-index and parabolic-index profiles. In particular, the connections between WKB theory and geometric optics are emphasized, and the problems of determination of phase factors is discussed.

We will be concerned here with the solutions of the scalar wave equation. Since we deal only with optical fibres of the 'weakly-guiding' type [10] (small relative index of refraction differences), we know that linearly-polarized (LP), quasi-mode solutions of the scalar wave equation are good approximations for the accurate vector modes [10–14]. In addition Kurtz [13] has recently shown that the hybrid vector modes may be constructed from solutions of the scalar wave equations in the weakly-guiding case for any refractive index distribution. Therefore we will compare solutions found by the WKB approximation with the LP modes rather than the rigorous vector modes.

The scalar wave equation for weakly-guiding optical fibres [10] may be written in cylindrical polar coordinates as:

$$\frac{\partial^2 E}{\partial r^2}+\frac{1}{r}\frac{\partial E}{\partial r}+\left[k^2n^2(r)-\beta^2-\frac{\nu^2}{r^2}\right]E = 0 \qquad (1)$$

where $E$ is an electric field component, $r$ is the radius, $\beta$ is the longitudinal propagation constant, $\nu$ is the azimuthal mode number, $n(r)$ gives the refractive index variation, and $k = 2\pi/\lambda$ ($\lambda$ = wavelength). Writing $E$ as $E_0 \exp\,[ikS(r)]$ and expanding $S$ in powers of $1/k$ we obtain

$$\begin{aligned} S &= S_0(r)+\frac{1}{k}S_1(r)+\ldots \\ E &= E_0 \exp\,[ikS_0(r)+iS_1(r)+\ldots] \end{aligned} \qquad (2)$$

where terms of higher order may be neglected provided the variation of $n(r)$ is small over a wavelength; this is the fundamental WKB assumption noted above.

Using Equation 2 in Equation 1 and equating terms of similar order in $k$, the zero and first order WKB approximations are obtained:

$$S_0(r) = \frac{1}{k}\int^r \left[k^2n^2(r) - \beta^2 - \frac{\nu^2}{r^2}\right]^{1/2} \mathrm{d}r \tag{3}$$

$$S_1(r) = \frac{i}{4}\ln\left[r^2n^2(r) - \frac{\beta^2 r^2}{k^2} - \frac{\nu^2}{k^2}\right] \tag{4}$$

where the constants of integration have been omitted. The lower limit on the integral in Equation 3 is a radius at which the integrand vanishes and represents a 'caustic' or turning point, separating regions of oscillatory and evanescent field variations. At the caustics the function $S_1$ in Equation 4 possesses a pole, and hence the first-order WKB approximation fails at these radii, and must be corrected by a further approximation [15–17] which will not be discussed here.

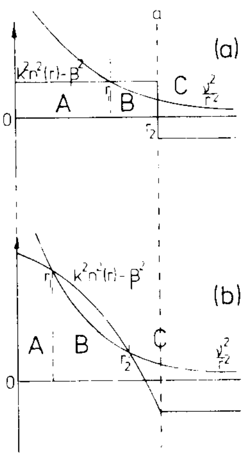


*Figure 1* Square magnitudes of local plane wave components for (a) a step-index fibre and (b) a general graded-index fibre, showing regions A and C of evanescent field, and region B of oscillatory field:
A is region $0 \leqslant r \leqslant r_1$
B is region $r_1 \leqslant r \leqslant r_2$
C is region $r_2 \leqslant r$

A physical interpretation of the WKB theory in terms of local plane waves has been given by Gloge and Marcatili [1]. The squared magnitudes of the local plane wave components for a bound mode are shown schematically in Fig. 1 for the cases (a) of a step-index fibre, and (b) of a graded-index fibre. From the figure it is clear that there exist two regions of evanescent fields (A and C) and one region of oscillatory variation (B), corresponding to bound electromagnetic power between the two caustics $r_1$ and $r_2$. From Equations 2–4, the forms of first-order field variation in the two types of region are:

$$\text{oscillatory:}\quad E(r) \propto \left[r^2n^2(r) - \frac{\beta^2r^2}{k^2} - \frac{\nu^2}{k^2}\right]^{-1/4} \cos\left\{\int^r \left[k^2n^2(r) - \beta^2 - \frac{\nu^2}{r^2}\right]^{1/2} \mathrm{d}r - \frac{\pi}{4}\right\} \tag{5}$$

$$\text{evanescent:}\quad E(r) \propto \frac{1}{2}\left[\frac{\nu^2}{k^2} + \frac{\beta^2r^2}{k^2} - r^2n^2(r)\right]^{-1/4} \exp\left\{-\int^r \left[\frac{\nu^2}{r^2} + \beta^2 - k^2n^2(r)\right]^{1/2} \mathrm{d}r\right\} \tag{6}$$

where the phase $-\pi/4$ in Equation 5 has been chosen so that the standard 'connection formulae' of WKB theory [15, 16] are obeyed, i.e. the waves on either side of the turning point are consistent in amplitude and phase. It also follows from consideration of an oscillatory field region between two evanescent field regions [15] that

$$\int_{r_1}^{r_2}\left[k^2n^2(r)-\beta^2-\frac{\nu^2}{r^2}\right]^{1/2}\mathrm{d}r = \left(m+\frac{1}{2}\right)\pi \tag{7}$$

where $m$ is the radial mode number (= 0, 1, 2 . . .). Hence Equation 7, the WKB resonance condition, is also an eigenvalue equation for the fibre modes characterised by $(m, \nu)$.

## 2. Step-index fibres

### 2.1. Zero-order WKB approximation

In a step-index fibre the outer caustic $r_2$ (see Fig. 1a) occurs at an abrupt variation of refractive index, viz. the core-cladding interface at radius $a$. Hence in the zero-order WKB approximation the fields in the core and cladding regions may be expressed (taking the lower limit of the integral in Equation 3 as $r_1$) as:

$$\text{Region B:}\quad E(r) = A\cos\left[\left(\frac{u^2r^2}{a^2}-\nu^2\right)^{1/2}-\nu\cos^{-1}\left(\frac{\nu a}{ur}\right)-\frac{\pi}{4}\right] \tag{8}$$

$$\text{Region C:}\quad E(r) = B\left|\frac{[\nu^2+(r^2/a^2)w^2]^{1/2}+\nu}{(\nu^2+w^2)^{1/2}+\nu}\;\frac{a}{r}\right|^{\nu}\exp\left[(\nu^2+w^2)^{1/2}-\left(\nu^2+\frac{w^2r^2}{a^2}\right)^{1/2}\right] \tag{9}$$

where the conventional notation $u^2 = a^2(k^2n_1^2-\beta^2)$, $w^2 = a^2(\beta^2-k^2n_2^2)$ has been used, with $n_1, n_2$ the refractive indices of core and cladding respectively. To obtain an eigenvalue equation it is now only necessary to match the fields and their radial derivatives in the usual way [10] at $r = a$. Hence we obtain

$$(u^2-\nu^2)^{1/2}-\nu\cos^{-1}\left(\frac{\nu}{u}\right) = \cos^{-1}\left[\frac{(u^2-\nu^2)^{1/2}}{v}\right]+\frac{\pi}{4}+m\pi \tag{10}$$

where $v = ak(n_1^2-n_2^2)^{1/2} = (u^2+w^2)^{1/2}$.

On the other hand, the alternative eigenvalue equation derivation, i.e. that via the resonance condition in Equation 7 yields

$$(u^2-\nu^2)^{1/2}-\nu\cos^{-1}\left(\frac{\nu}{u}\right) = \frac{\pi}{2}+m\pi. \tag{11}$$

Equations 10 and 11 represent alternative forms of the eigenvalue equation in the zero-order WKB approximation for the step-index fibre. They yield similar results for large $m$ values, but are significantly different for low $m$, e.g. 0, 1, 2. We will examine the physical meaning of these differences in Section 2.2.

### 2.2. Geometrical optics

On the ray optics picture an eigenvalue equation may be obtained by calculating phase changes along ray paths and requiring that the total transverse phase difference after one round trip between successive contacts with one caustic be an integral multiple of $2\pi$ [16, 18–21]*. For one such round trip, the transverse phase difference due to optical path length only (neglecting reflections and refractions at caustics) is given by [16, 18]:

$$\delta\phi = 2\int_{r_1}^{r_2}\left[k^2n^2(r)-\beta^2-\frac{\nu^2}{r^2}\right]^{1/2}\mathrm{d}r. \tag{12}$$

However, it is also necessary to add the phase changes $2\phi_1, 2\phi_2$ suffered at the caustics $r_1, r_2$. Hence the geometrical optics eigenvalue equation for the step-index fibre becomes:

$$(u^2-\nu^2)^{1/2}-\nu\cos^{-1}\left(\frac{\nu}{u}\right) = m\pi-\phi_1-\phi_2. \tag{13}$$

*This approach has been used extensively in its two-dimensional form, especially for applications in integrated optics e.g. [19–21]. However, the three-dimensional version used here does not seem to have yet received much attention.

We may now compare the three alternative eigenvalue Equations 10, 11 and 13 to see their relationship. Since it is known [18–21] that the phase change suffered at the inner caustic of a step-index waveguide is $\phi_1 = -\pi/4$, it is clear that $\phi_2$ in Equation 13 may be identified with the corresponding term in Equation 10:

$$\phi_2 = -\cos^{-1}\left[\frac{(u^2-\nu^2)^{1/2}}{v}\right]. \tag{14}$$

Equation 14 gives therefore the expression for half the phase shift on reflection at a cylindrical dielectric interface. Love and Snyder have recently derived exactly this expression and the corresponding eigenvalue Equation 10 via a purely geometrical optics approach [22].

In addition, it is clear from a comparison of Equations 11 and 13 that the usual WKB resonance condition Equation 7, assumes $\phi_2 = -\pi/4$, i.e. that refraction occurs at both caustics and no reflections occur. This assumption is of course valid for graded-index distributions but is a poor approximation for any distribution which includes a discontinuity in refractive indices.

An alternative approximation which has been used [4, 18] is to replace $\phi_2$ by its limit far from cut-off, viz. $-\pi/2$. It is clear from the above remarks that this latter approximation is rather poor near cut-off where the error term in the eigenvalue equation will be $\cos^{-1}(\nu/v)$. In fact if this approximation were used to predict cut-off values the results would be in error by about one azimuthal number.

## 2.3. Comparisons with weakly-guiding mode theory

The rigorous vector mode theory for cylindrical dielectric waveguides of homogeneous core was developed by Snitzer [23]. However, in the weakly-guiding case it has been shown that the LP eigenvalues and field distributions [10] provide very good approximations to the hybrid modes [24]. Since we have applied the WKB theory to the scalar wave equation, it is clearly appropriate to compare the predictions of this theory to the results of the LP mode approximation. In this theory, the radial variation of the fields is given by:

$$E(r) \propto \begin{cases} J_\nu\left(\dfrac{ur}{a}\right), & r \leqslant a; \\[2ex] K_\nu\left(\dfrac{wr}{a}\right), & r > a \end{cases} \tag{15}$$

and the corresponding eigenvalue equation is:

$$u\frac{J_{\nu\pm1}(u)}{J_\nu(u)} = \pm w\frac{K_{\nu\pm1}(w)}{K_\nu(w)}. \tag{16}$$

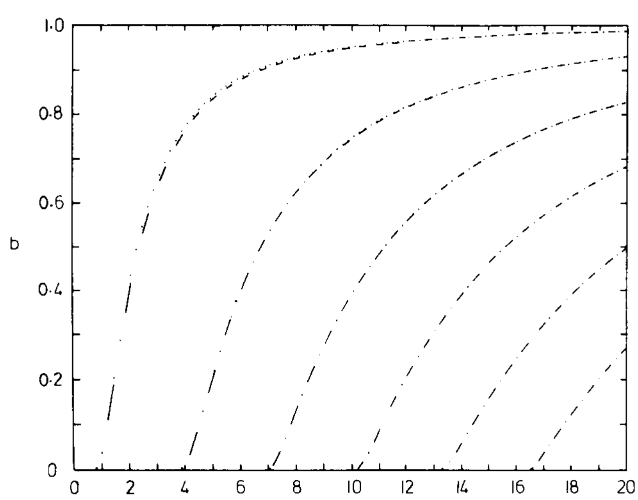


*Figure 2* Comparison between eigenvalues obtained by (a) the WKB approximation (dotted lines) Equation 10, and (b) the weakly guiding mode theory [10] (broken lines). The curves show $b$ versus $v$, where $b = 1 - u^2/v^2$, for meridional rays ($\nu = 0$).

We note first that the asymptotic forms of Equations 10 and 16 are identical. In order to determine the accuracy of the approximation represented by Equation 10 we have plotted results obtained from Equations 10 and 16 in Figs. 2 and 3 for $\nu = 0$ and 1, respectively. The results are given in the form of $b$–$v$ curves [10], where $b = 1 - u^2/v^2$. The figures show that Equation 10 provides a very good approximation to the weakly-guiding mode eigenvalues even for lower-order modes. The r.m.s. difference between each pair of curves (calculated from Equations 10 and 16) is less than $10^{-2}$ for all the cases of Figs. 2 and 3. However, this level of agreement would not be achieved if Equation 11 were to be used, or if the limit $\phi_2 \to -\pi/2$ were employed.

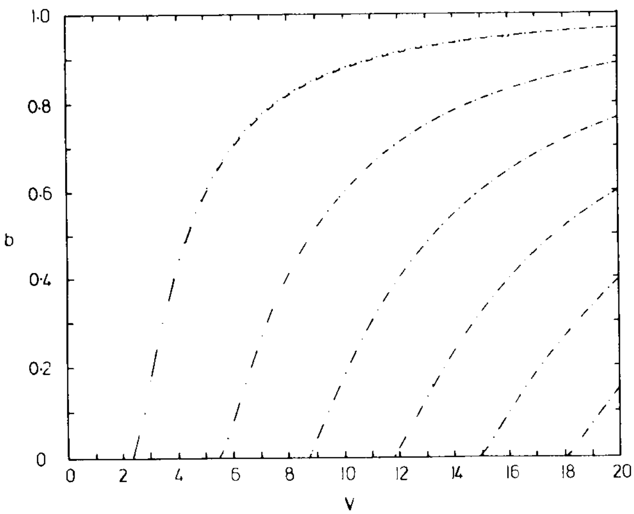


*Figure 3* As Fig. 2, except that here $\nu = 1$.

The accuracy of the WKB approximation has also been tested by comparing the calculated field distributions with the more exact ones from Equation 15. It is found that the field zeros given by the zero and first-order WKB solutions are usually within 0.6% of those determined from the Bessel function zeros. In the cladding region, the WKB and modified Bessel field are indistinguishable. In the core, the zero-order WKB fields are in some error since they give fixed amplitude oscillations whereas the first-order WKB solutions agree well with the Bessel functions from the core-cladding interface at '$a$' up to a distance of about $0.1a$ from the inner caustic. As would be expected the pole at the caustic introduces errors within distances of the order $0.1a$ from the caustic.

## 3. Parabolic-index fibres

### 3.1. Zero-order WKB approximation

We deal here with a refractive index distribution of the form:

$$n^2(x) = \begin{cases} n_1^2(1 - 2\Delta x^2), & x \leqslant 1 \\ n_1^2(1 - 2\Delta) \equiv n_2^2, & x \geqslant 1 \end{cases} \tag{17}$$

where $x = r/a$, $n_1$ is the index at core centre, and $\Delta$ is the relative index difference between core centre and cladding. For this index profile the caustics for a guided mode occur at radii given by

$$x^2 = \frac{u^2 \pm (u^4 - 4v^2\nu^2)^{1/2}}{2v^2} \tag{18}$$

where the conventional $u, v$ notation has again been used.

*Region B*: Between the two caustics, given by Equation 18, the field has an oscillatory nature and is given in the zero-order WKB approximation by

$$E(x) = A\cos\left\{\frac{1}{2}(u^2x^2-\nu^2-v^2x^4)^{1/2}+\frac{u^2}{4v}\cos^{-1}\left[\frac{u^2-2v^2x^2}{(u^4-4v^2\nu^2)^{1/2}}\right]\right.$$
$$\left.-\frac{\nu}{2}\cos^{-1}\left[\frac{2\nu^2-u^2x^2}{x^2(u^4-4v^2\nu^2)^{1/2}}\right]-\frac{\pi}{4}\right\}. \quad (19)$$

*Region C*: Outside the second caustic but still within the core region, the evanescent field is given by

$$E(x) = B\exp\left[-\frac{1}{2}(v^2x^4-u^2x^2+\nu^2)^{1/2}\right]\left|\frac{2\nu(v^2x^4-u^2x^2+\nu^2)^{1/2}+2\nu^2-u^2x^2}{x^2(u^4-4v^2\nu^2)^{1/2}}\right|^{\nu/2}$$
$$\times\left|\frac{2x^2v^2-u^2+2v(v^2x^4-u^2x^2+\nu^2)^{1/2}}{(u^4-4v^2\nu^2)^{1/2}}\right|^{u^2/4v}. \quad (20)$$

In addition the remaining evanescent field in the cladding region is given by a third expression.

From Equations 19 and 20 it may be shown that the derivative of $E(r)$ vanishes at the inner caustic and at the outer caustic whether approached from inside or outside. Hence we may not derive an eigenvalue equation for the zero-order approximation by matching fields and derivative at the outer caustic, as we did in the step-index case above. Hence the only approach to obtaining an eigenvalue equation within the framework of the zero-order WKB picture is to use the resonance condition, Equation 7. It follows, using Equation 7 that the required eigenvalue equation is

$$u^2 = 2v(2m+\nu+1). \quad (21)$$

Note that this equation was derived without any specific assumption about the cladding region ($x \geqslant 1$) and therefore applies equally to the infinite parabolic-index medium.

## 3.2. Geometrical optics

As in Section 2.2 above, the eigenvalue equation in the ray optics picture is obtained from phase changes along the ray path and at the caustics. For one round trip in a graded-index fibre, this equation reads

$$\delta\phi + 2\phi_1 + 2\phi_2 = 2m\pi \quad (22)$$

where $\delta\phi$ is given by Equation 12 and $2\phi_1, 2\phi_2$ are phase changes at the caustics $r_1$ and $r_2$; in this case $\phi_1 = -\pi/4 = \phi_2$. Hence in any graded-index fibre, Equation 22 is equivalent to the WKB resonance condition, Equation 7. It follows that, in the case of the parabolic-index fibre, Equation 21 is found as the eigenvalue equation from geometrical optics. This result has also been obtained recently by Love and Snyder [22].

## 3.3. Comparison with the scalar mode theory of the infinite parabolic medium

To compare the results of Sections 3.1 and 3.2 with a more rigorous weakly-guiding mode theory, we consider the parabolic-index profile of Equation 17 as an approximation to the infinite parabolic medium, as given by

$$n^2(x) = n_1^2(1-2\Delta x^2) \quad (23)$$

for all $x$. The scalar mode theory of this medium is deficient in that it neglects the effect of the gradient of dielectric permittivity in the wave equation [25, 26]. However, it has been shown by perturbation techniques [27–29] that the neglect of this term is (at worst) no more inaccurate than the assumption of weak-guidance. The scalar modes of the parabolic medium are well-known and are given by

$$E(x) = C\tilde{v}^{\nu}x^{\nu}\exp\left(\frac{-vx^2}{2}\right)L_m^{\nu}(vx^2) \quad (24)$$

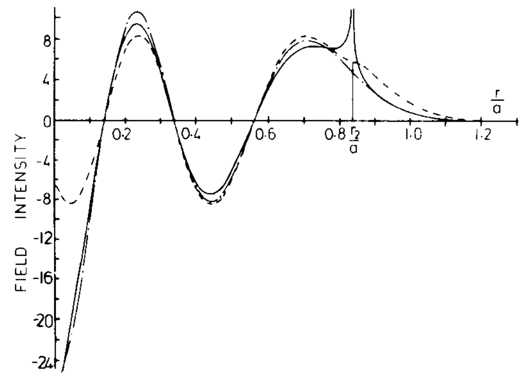


*Figure 4* Field distributions in the infinite parabolic-index medium calculated by (a) zero-order WKB approximation (broken line), (b) first-order WKB approximation (solid line), and (c) scalar mode theory [30] (dotted line). Parameter values are $v = 20$, $\nu = 0$, $m = 3$.

where $L_m^\nu$ is the generalized Laguerre polynomial. The eigenvalue equation for this medium [30] is identical to Equation 21. The fact that the WKB method gives exactly the eigenvalues predicted by the mode theory was pointed out by Messiah [15] in his treatment of the harmonic oscillator.

A comparison of the fields given by both zero and first-order WKB approximations for the parabolic medium with the Laguerre–Gaussian modes of Equation 24 is given in Figs. 4 and 5 for $\nu = 0$ and 1 respectively. The parameters used were $v = 20$ and $m = 3$, and the fields are normalized to carry the same power in each case. It is seen that the zeros of the WKB eigenfunctions are in good agreement with those of the mode fields, as was also the case in the step-index fibre (Section 2.3). These comparisons may be seen as an extension of the two-dimensional comparison carried out by Arnaud [31].

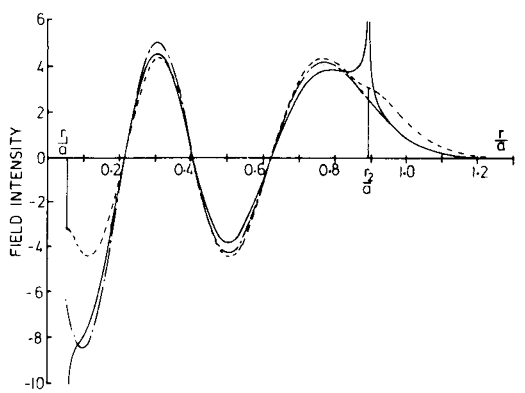


*Figure 5* As Fig. 4, except that here $\nu = 1$.

In the oscillatory region, sufficiently far from the turning points, the first-order WKB method is seen to give an excellent approximation to the Laguerre–Gaussian modes. The slight error observed at the peaks is believed to be due to the problem of normalization of the WKB fields, rather than to an inaccuracy of the method. As in the step-index fibre, the zero-order case is deficient in that it predicts constant amplitude field oscillations.

The first-order WKB approximation cannot be very accurate in the vicinity of the turning points, particularly between the outer caustic and the cladding; in this region other asymptotic expressions may be used [16, 17]. Alternatively in some practical applications, the zero-order WKB method may be sufficiently accurate [7–9, 32].

## 4. Generalizations for arbitrary index profiles

We may summarize our results by use of one master equation utlizing the results of WKB theory and geometric optics in a consistent manner. From Equations 12 and 22 this equation is:

$$\int_{r_1}^{r_2} \left(k^2 n^2(r) - \beta^2 - \frac{\nu^2}{r^2}\right)^{1/2} \mathrm{d}r = m\pi - \phi_1 - \phi_2 \tag{25}$$

where the phase changes occurring at $r_1$ and $r_2$ are $2\phi_1, 2\phi_2$, respectively. Equation 25 may be applied to a variety of waveguides, provided that the choice of $r_1, r_2$ and $\phi_1, \phi_2$ is made appropriately. The most common examples of two- and three-dimensional guides with abrupt and graded-index profiles are illustrated by the WKB diagrams in Figs. 6 and 1. Note that we use here a definition of planar guide width $= a =$ cylindrical guide radius; the definitions of $u$, $v$, etc., are then directly comparable for the two- and three-dimensional cases [33]. It follows that the number of meridional modes on a fibre is one half the total number supported by the equivalent slab, since only modes of even symmetry are permitted on the fibre.

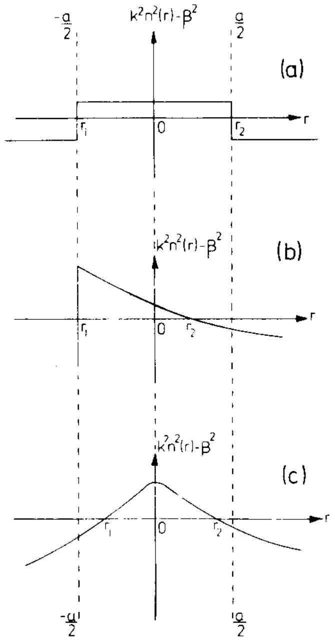


*Figure 6* WKB diagrams for two dimensional planar waveguide:
(a) step-index guide: $\phi_1 = \phi_2 = -\cos^{-1}(u/v)$
(b) graded-index guide (maximum index at dielectric interface):

$$\phi_1 = -\cos^{-1}(u/v)$$

$$\phi_2 = -\frac{\pi}{4}$$

(c) 'buried' graded-index guide: $\phi_1 = \phi_2 = -\frac{\pi}{4}$

$2\phi_1, 2\phi_2$ are the phase shifts occurring at $r_1, r_2$ respectively (see Equation 25).

In each of the cases of Figs. 6 and 1 the position of the limits $r_1$ and $r_2$ are shown; for a reflection at a dielectric discontinuity the phase change is

$$-2\cos^{-1}\left[\frac{(u^2 - \nu^2)^{1/2}}{v}\right]$$

($\nu = 0$ for planar cases of Fig. 6 and for meridional rays in the fibre, Fig. 1), and for a caustic, i.e. a zero of the integrand in Equation 25, the phase change is $-\pi/2$. In the planar graded-index case we distinguish between guides whose index maximum occurs at a dielectric interface (Fig. 6b) and those where the maximum index occurs within one medium (the 'buried' guide, Fig. 6c); this distinction was first made by Gedeon [34] and Conwell [35].

It should be emphasized that the results obtained here are valid only for 'weakly guiding' situations where, in the planar case TE and TM modes are degenerate, and in the cylindrical case the LP mode assumption [10] may be used. Equation 25 may easily be applied to many other cases of interest, including other dielectric distributions, e.g. the ring-profile [36], or mode families other than guided, e.g. leaky modes [37].

## 5. Conclusions

The WKB method provides a simple and convenient means for the analysis of optical dielectric waveguides. When combined with the techniques of geometrical optics it yields a general eigenvalue Equation 25 which may be applied to guides of arbitrary index profile in two- or three-dimensions. The only restrictions appear to be

(a) slow variation of index within distances of order of a wavelength,

(b) weak-guidance, i.e. $n_1 - n_2 \gg n_1, n_2$,

(c) accurate choice of the phase shifts on reflection or refraction.

Under these limitations, and taking especial account of (c), the eigenvalues thus obtained would appear to be extremely good even for small $v$-values and low-order modes. This result has already been noted for the exponential index distribution in planar guides by Conwell [38]. In addition, for more general sytems it was previously known that the WKB method gave reasonably accurate results [39] even for low quantum numbers.

For the field distributions it was found in the cylindrical cases examined here that the first-order WKB method gives a good approximation to the waveguide modes except near the turning points. For the step-index profile the zero-order approximation is also very good, but only in the cladding region. For applications where accurate field distributions are required, it appears necessary to use further asymptotic methods [16, 17, 40] in the vicinity of the turning points.

## Acknowledgements

The authors wish to thank Professor W. A. Gambling, Dr R. A. Sammut and D. N. Payne for helpful discussions and their comments on the manuscript. We are also grateful to J. D. Love and A. W. Snyder for supplying a preprint of [22]. One of us (MJA) is indebted to the UK Science Research Council for the award of a Research Fellowship.

## References

1. D. GLOGE and E. A. J. MARCATILI, *Bell Syst. Tech. J.* **52** (1973) 1563–78.
2. C. C. TIMMERMANN, *Arch. Elekt. & Uebertr.* **28** (1974) 186–8.
3. *Idem, ibid* **28** (1974) 344–6.
4. D. GLOGE, *IEEE Trans.* **MTT-23** (1975) 106–20.
5. K. PETERMANN, *Arch. Elekt. & Uebertr.* **29** (1975) 345–8.
6. *Idem, ibid* **29** (1975) 485–7.
7. M. J. ADAMS, D. N. PAYNE and F. M. E. SLADEN, *Opt. Commun.* 17 (1976) 204–9.
8. *Idem, Elect. Lett.* **12** (1976) 281–3.
9. *Idem, ibid* **12** (1976) 348.
10. D. GLOGE, *Appl. Opt.* **10** (1971) 2252–8.
11. C. N. KURTZ and W. STREIFER, *IEEE Trans.* **MTT-17** (1969) 10–14.
12. *Idem, ibid* **MTT-17** (1969) 250–3.
13. C. N. KURTZ, *J. Opt. Soc. Amer.* **65** (1975) 1235–40.
14. G. L. YIP and S. NEMETO, *IEEE Trans.* **MTT-23** (1975) 260–3.
15. A. MESSIAH, 'Mecanique Quantique' **1** (Dunod, Paris, 1964).
16. J. P. GORDON, *Bell Syst. Tech. J.* **46** (1966) 321–32.
17. C. C. TIMMERMANN, *Arch. Elekt. & Uebertr.* **29** (1975) 186–8.
18. J. B. KELLER and S. I. RUBINOW, *Ann. Phys.* **9** (1960) 24–75.
19. P. K. TIEN, *Appl. Opt.* **10** (1971) 2395–413.
20. G. B. HOCKER and W. K. BURNS, *IEEE J. Quant. Elect.* **QE-11** (1975) 270–6.
21. G. B. HOCKER, *ibid* **QE-12** (1976) 232–6.
22. J. D. LOVE and A. W. SNYDER, *Appl. Opt.* **15** (1976) 2121–5.
23. E. SNITZER, *J. Opt. Soc. Amer.* **51** (1961) 491–8.
24. G. BIERNSON and D. J. KINSLEY, *IEEE Trans.* **MTT-13** (1965) 345–56.
25. Y. SUEMATSU and K. FURAYA, *Electron. Commun. Japan* **54-B** (1971) 39–46.
26. M. MATSUHARA, *J. Opt. Soc. Amer.* **63** (1973) 135–8.
27. D. MARCUSE, *IEEE J. Quant. Elect.* **QE-9** (1973) 958–60.
28. R. YAMADA and Y. INABE, *J. Opt. Soc. Amer.* **64** (1974) 964–8.

29. K. THYAGARAJAN and A. K. GHATAK, *Opt. Commun.* **11** (1974) 417–21.
30. W. STREIFER and C. N. KURTZ, *J. Opt. Soc. Amer.* 57 (1967) 779–86.
31. J. A. ARNAUD, in 'Progress in Optics', edited by E. WOLF **XI** (North-Holland, Amsterdam, 1973) p. 258.
32. W. J. STEWART, Topical Meeting on Optical Fibre Transmission, Williamsburg, Virginia (7–9 January, 1975).
33. H. KOGELNIK and V. RAMASWAMY, *Appl. Opt.* **13** (1974) 1857–62.
34. A. GEDEON, *Opt. Commun.* **12** (1974) 329–32.
35. E. M. CONWELL, *IEEE J. Quant. Elect.* **QE-11** (1975) 217–8.
36. D. GLOGE and E. A. J. MARCATILI, *Bell Syst. Tech. J.* **52** (1973) 1161–8.
37. R. SAMMUT and A. W. SNYDER, *Appl. Opt.* **15** (1976) 477–82.
38. E. M. CONWELL, *J. Appl. Phys.* **46** (1975) 1407.
39. L. I. SCHIFF, 'Quantum Mechanics', (McGraw-Hill, New York, 1949), p. 192.
40. D. MARCUSE, *IEEE J. Quant. Elect.* **QE-9** (1973) 1000–6.