QUARTERLY OF APPLIED MATHEMATICS
VOLUME LIV, NUMBER 4
DECEMBER 1996, PAGES 677-685

ON THE ACCURATE COMPUTATION OF THE PROLATE SPHEROIDAL RADIAL FUNCTIONS OF THE SECOND KIND

By

T. DO-NHAT AND R. H. MACPHIE

Department of Electrical and Computer Engineering, University of Waterloo, Canada

Abstract. The series expansion of the prolate radial functions of the second kind, expressed in terms of the spherical Neumann functions, converges very slowly when the spheroid's surface coordinate ξ approaches 1 (thin spheroids). In this paper an analytical series expansion in powers of $(\xi^2 - 1)$ is obtained to facilitate the convergence. Then, by using the Wronskian test, it is shown that this newly developed expansion has been computed with a double precision accuracy.

Introduction. The prolate spheroidal radial functions satisfy the following differential equation [1], [2]:

$$\frac{d}{d\xi}\left((\xi^2-1)\frac{d}{d\xi}R_{mn}(h,\xi)\right) - \left(\lambda_{mn} - h^2\xi^2 + \frac{m^2}{\xi^2-1}\right)R_{mn}(h,\xi) = 0, \qquad m = 0,1,2,\ldots,\ n = m, m+1, m+2, \ldots \tag{1}$$

where ξ is the spheroid's radial coordinate ($\xi \geq 1$), and λ_{mn} is the spheroid's eigenvalue for the given h parameter, i.e., $h = kF$ where $k = 2\pi/\lambda$ is the operating wavenumber and F is the semi-interfocal distance of the spheroid. According to Flammer [2] the second solution of the above differential equation, which can be expressed in terms of the associated Legendre functions of the first and second kinds, is given by

$$R^{(2)}_{mn}(h,\xi) = \frac{1}{\kappa^{(2)}_{mn}(h)}\left\{\sum_{r=2m,2m+1}^{\infty}{}' d_r^{mn}(h)Q^m_{m+r}(\xi) + \sum_{r=2m+2,2m+1}^{\infty}{}' d_{\rho|r}^{mn}(h)P^m_{r-m-1}(\xi)\right\} \tag{2}$$

where $\sum'$ denotes the summation over even or odd values of r if $n-m$ is even or odd. Also, the expansion coefficients d_r^{mn} and $d_{\rho|r}^{mn}$ follow the recursion relations given in [2] along with the spheroid's joining factor $\kappa^{(2)}_{mn}(h)$ [2, p. 33]. Another representation of the prolate spheroidal wave function of the second kind is given in terms of the following spherical Neumann expansion [2]:

$$R^{(2)}_{mn}(h,\xi) = \left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\sum_{r=0,1}^{\infty}{}' a_r^{mn}(h)n_{m+r}(h\xi), \tag{3}$$

Received May 2, 1994.

where a_r^{mn} are the normalized expansion coefficients, and $n_{m+r}(h\xi)$ are the spherical Neumann functions.

Equations (2) and (3) are the main interest. It is historically well known that the series in (3) converges very slowly when $h\xi$ is small. According to Morse and Feshbach [3, p. 1506], "the series does not converge well for $h\xi$ small, in fact it is an asymptotic series not being absolutely convergent for any finite value of $h\xi$." Recently, Sinha and MacPhie [4] summed this series up to 40 terms and replaced the residual series by an integral. However, the integrand of this integral is a curve-fitting function which may not be reliable for large m or n.

In this paper we focus on the series given by (2) expressed in terms of the associated Legendre functions of the first and second kinds. Here, due to the lack of the development of the $Q^m_{m+r}(\xi)$ function, Flammer [2] expanded the prolate spheroidal function of the second kind in powers of (ξ^2-1) by using that of the first kind and the Wronskian of $R^{(1)}_{mn}(h,\xi)$ and $R^{(2)}_{mn}(h,\xi)$. However, Flammer's prolate spheroidal wave expression of the second kind is cumbersome and complicated, and is limited to some lower values of m.

For the above purpose we first derive the representations of the associated Legendre function of the second kind $Q^m_{m+r}(\xi)$ for any integer $m+r$ ($r=-2m,-2m+1,\dots$, $m=0,1,2,\dots$). By using the linear hypergeometric transformation, $Q^m_{m+r}(\xi)$ is given in closed form for $-2m\le r\le -1$. However, for $r\ge 0$, $Q^m_{m+r}(\xi)$ is explicitly expressed in terms of the associated Legendre functions of the first kind. By using these representations it will be proved that the prolate spheroidal radial function of the second kind can be expressed in terms of its first kind. Nevertheless, when ξ is near to 1 an analytical series expansion in powers of (ξ^2-1) is obtained and all the expansion coefficients are expressed in closed forms in terms of the d_r^{mn} and $d_{\rho|r}^{mn}$ coefficients.

2. Closed form expression of $Q^m_\nu(\xi)$ ($\nu\ge -m$). First, for $\nu=-m,-m+1,\dots,-1$, we start with the general definition of Barnes [5, Ch. XV, p. 326] for positive integers m as follows:

$$Q^m_\nu(\xi)=\frac{\sin(\mu+m)\pi}{\sin\nu\pi}\frac{\Gamma(\nu+m+1)\Gamma(1/2)}{2^{\nu+1}\Gamma(\nu+3/2)}\frac{(\xi^2-1)^{m/2}}{\xi^{\nu+m+1}} \cdot F\left(\frac{\nu}{2}+\frac{m}{2}+1,\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2};\nu+\frac{3}{2};\xi^{-2}\right),\quad |\xi|\ge 1, \tag{4}$$

where $F(a,b;c;z)$ is the hypergeometric function with $z=\xi^{-2}$, $a=\nu/2+m/2+1, b=\nu/2+m/2+1/2$, and $c=a+b-m$. By using the linear hypergeometric transformation for $m=1,2,3,\dots$, $F(a,b;c;z)$ is given [6, p. 560, Eq. 15.3.12] as follows:

$$\begin{aligned}F(a,b;a+b-m;z)&=\frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)}(1-z)^{-m}\sum_{k=0}^{m-1}\frac{(a-m)_k(b-m)_k}{k!\,(1-m)_k}(1-z)^k\\&-(-1)^m\frac{\Gamma(a+b-m)}{\Gamma(a-m)\Gamma(b-m)}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!\,(k+m)!}(1-z)^k[\ln(1-z)\\&-\Psi(k+1)-\Psi(k+m+1)+\Psi(k+a)+\Psi(k+b)]\end{aligned}\tag{5}$$

for $m = 1, 2, \ldots$, $|\arg(1-z)| < \pi$, $|1-z| < 1$, where $\Psi(x) = \frac{d}{dx}[\ln \Gamma(x)]$ is the Digamma function. In our case $\Gamma(a-m) = \Gamma(\nu/2 - m/2 + 1), \Gamma(b-m) = \Gamma(\nu/2 - m/2 + 1/2)$ and, since $\nu = -m, -m+1, \ldots, -1$, the Gamma function $\Gamma(a-m)$, or $\Gamma(b-m)$ with argument $0, -1, -2, \ldots$, tends to infinity. Hence, (5) is reduced to

$$F(a,b;a+b-m;z) = \frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)}(1-z)^{-m}\sum_{k=0}^{m-1}\frac{(a-m)_k(b-m)_k}{k!\,(1-m)_k}(1-z)^k, \quad m = 1, 2, \ldots . \tag{6}$$

If we now substitute (6) into (4), then $Q_\nu^m(\xi)$ $(\nu = m + r)$ is given in *closed form* as follows:

$$Q_{m+r}^m(\xi) = (-1)^m 2^{m-1}(m-1)!\frac{(\xi^2-1)^{-m/2}}{\xi^{r+1}}\sum_{k=0}^{m-1}\frac{(\frac{r}{2}+1)_k(\frac{r+1}{2})_k}{k!\,(1-m)_k}\frac{(\xi^2-1)^k}{\xi^{2k}}, \quad r = -2m, -2m+1, \ldots, -1. \tag{7}$$

If $m = 1$ and $r = -2$, by using (7) we obtain $Q_{-1}^1(\xi) = -\xi(\xi^2-1)^{-1/2}$, which agrees with that which is derived by using the recursion formula of the associated Legendre functions [6, p. 334, Eq. 8.5.3].

For $\nu = 0, 1, 2, \ldots$, from the definition of Hobson for the associated Legendre functions of argument greater than 1 [5, p. 325],

$$Q_\nu^m(\xi) = (\xi^2-1)^{m/2}\frac{d^m Q_\nu(\xi)}{d\xi^m}, \tag{8}$$

where the Legendre function of the second kind $Q_\nu(\xi)$ [2, p. 36] is given by

$$Q_\nu(\xi) = \frac{1}{2}P_\nu(\xi)\ln\left(\frac{\xi+1}{\xi-1}\right) - \sum_{k=0}^{[\frac{1}{2}(\nu-1)]}\frac{2\nu-4k-1}{(\nu-k)(2k+1)}P_{\nu-2k-1}(\xi) \tag{9}$$

wherein $[\frac{1}{2}(\nu-1)]$ denotes Gauss' notation of the largest integer in $\frac{1}{2}(\nu-1)$. By substituting (9) into (8) and then using Leibniz's theorem [6, p. 12] for the mth derivative of the product of two functions $P_\nu(\xi)\ln(\frac{\xi+1}{\xi-1})$, the associated Legendre function of the second kind $Q_\nu^m(\xi)$ is expressed in terms of the associated Legendre function of the first kind as follows:

$$Q_\nu^m(\xi) = \frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\right)P_\nu^m(\xi) + \frac{1}{2}\sum_{k=1}^{m}(-1)^{k-1}\frac{m!}{k(m-k)!}\left[\left(\frac{\xi-1}{\xi+1}\right)^{k/2} - \left(\frac{\xi+1}{\xi-1}\right)^{k/2}\right] \cdot P_\nu^{m-k}(\xi) - \sum_{k=0}^{[\frac{1}{2}(\nu-1)]}\frac{2\nu-4k-1}{(\nu-k)(2k+1)}P_{\nu-2k-1}^m(\xi), \quad \nu = 0, 1, 2, \ldots . \tag{10}$$

Of course, in (10) $P_\mu^l(\xi)$ follows the Hobson definition [5, p. 325] for $|\xi| > 1$, i.e., $P_\mu^l(\xi) = (\xi^2-1)^{1/2}d^l P_\mu(\xi)/d\xi^l$.

3. Power series representation of $R^{(2)}_{mn}(h,\xi)$. If the expressions for $Q^m_\nu(\xi)$, given by (7) and (10), are used in (2), the prolate spheroidal function of the second kind is computed with ease for arbitrary argument ξ. However, as ξ approaches 1, it is very useful to express $R^{(2)}_{mn}(h,\xi)$ as a power series in (ξ^2-1). However, the associated Legendre functions are first expressed in terms of the hypergeometric function of argument $1-\xi^2$ [2, p. 36] as follows:

$$P^l_{l+2s}(\xi) = \frac{(2l+2s)!}{2^l l!\,(2s)!}(\xi^2-1)^{1/2}F(-s,1/2+s+l;l+1;1-\xi^2), \tag{11}$$

$$P^l_{l+2s+1}(\xi) = \frac{(2l+2s+1)!}{2^l l!\,(2s+1)!}\xi(\xi^2-1)^{1/2}F(-s,3/2+s+l;l+1;1-\xi^2). \tag{12}$$

If we use $Q^m_\nu(\xi)$ given by (10) and (7) in (2) it is straightforward to prove that the prolate spheroidal radial function of the second kind $R^{(2)}_{mn}(h,\xi)$ has the following form:

$$R^{(2)}_{mn}(h,\xi) = \begin{cases} \frac{1}{2}\frac{\kappa^{(1)}_{mn}(h)}{\kappa^{(2)}_{mn}(h)}R^{(1)}_{mn}(h,\xi)\ln\left(\frac{\xi+1}{\xi-1}\right)+\frac{1}{\kappa^{(2)}_{mn}(h)}\xi(\xi^2-1)^{-\frac{m}{2}}\sum_{\mu=0}^{\infty}\delta^{mn}_\mu(\xi^2-1)^\mu, & (n-m)\text{ even},\\ \frac{1}{2}\frac{\kappa^{(1)}_{mn}(h)}{\kappa^{(2)}_{mn}(h)}R^{(1)}_{mn}(h,\xi)\ln\left(\frac{\xi+1}{\xi-1}\right)+\frac{1}{\kappa^{(2)}_{mn}(h)}(\xi^2-1)^{-\frac{m}{2}}\sum_{\mu=0}^{\infty}\delta^{mn}_\mu(\xi^2-1)^\mu, & (n-m)\text{ odd},\end{cases} \tag{13}$$

where $R^{(1)}_{mn}(h,\xi)$ is the prolate spheroidal radial function of the first kind and is obtained from the prolate spheroidal angle function of the first kind $S_{mn}(h,\xi)$ by the relation [2, pp. 32–34] $R^{(1)}_{mn}(h,\xi) = S_{mn}(h,\xi)/\kappa^{(1)}_{mn}(h)$ with $\kappa^{(1)}_{mn}(h)$ being the joining factor. Flammer deduced the above form without recognizing his factor $Q_{mn}(h) = \kappa^{(1)}_{mn}(h)/\kappa^{(2)}_{mn}(h)$ [2, p. 35]. In view of (2) and (13), the power series expansion given in (13), i.e., $h_{mn}(h,\xi) = \sum_{\mu=0}^{\infty}\delta^{mn}_\mu(\xi^2-1)^\mu$, has the following form from (2) and (13):

$$\begin{aligned} h_{mn}(h,\xi) &= \sum_{\mu=0}^{\infty}\delta^{mn}_\mu(\xi^2-1)^\mu \\ &= (\xi^{-1}u_{mn}+1-u_{mn})(\xi^2-1)^{m/2}\left[\sum_{r=-2m,-2m+1}^{\infty}{}' d^{mn}_r(h)\right.\\ &\quad\left.\cdot\left\{Q^m_{m+r}(\xi)-\frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\right)P^m_{m+r}(\xi)\right\}+\sum_{r=2m+2,2m+1}^{\infty}{}' d^{mn}_{\rho|r}(h)P^m_{r-m-1}(\xi)\right] \end{aligned} \tag{14}$$

where u_{mn} is defined by

$$u_{mn} = \begin{cases} 1, & (n-m)\text{ even},\\ 0, & (n-m)\text{ odd}.\end{cases} \tag{15}$$

From (7), (10), (11), and (12), with the hypergeometric functions expressed in powers of ξ^2-1, the right-hand side of (14) can also be verified as a power series of (ξ^2-1). To find the expansion coefficients δ^{mn}_μ, both sides of (14) are differentiated μ times with respect to (ξ^2-1) with $\xi^2\to 1$. Hence,

$$\delta^{mn}_\mu = \frac{1}{\mu!}\lim_{x\to 0}\frac{d^\mu}{dx^\mu}h_{mn}(h,\xi), \quad x=\xi^2-1. \tag{16}$$

It is shown in Appendix A that the differentiation process (μ times) is rather tedious but straightforward. After the differentiation, by retaining only the coefficient associated with the zeroth power of (ξ^2-1), the expansion coefficients δ_μ^{mn} can be expressed in terms of the coefficients $d_r^{mn}(h)$ and $d_{\rho|r}^{mn}(h)$ as follows:

$$\delta_\mu^{mn}=\sum_{r=-2m,-2m+1}^{-2,-1}{}' D_r^{1\mu mn}d_r^{mn}(h)+\sum_{r=0,1}^{\infty}{}'(D_r^{2\mu mn}+D_r^{3\mu mn})d_r^{mn}(h)+\sum_{r=2m+2,2m+1}^{\infty}{}' D_r^{4\mu mn}d_{\rho|r}^{mn}(h). \tag{17}$$

4. Closed form expressions of $D_r^{i\mu mn}$ $(i = 1,2,3,4)$. The coefficients $D_r^{i\mu mn}$ $(i=1,2,3,4)$, which are independent of h, are derived in Appendix A. Their closed form expressions are given by

$$D_r^{1\mu mn}=(-1)^{m+\mu}2^{m-\mu-1}(m-1)!\sum_{k=0}^{m-1}\frac{(-1)^k(r+1)_{2k}}{2^k k!\,(\mu-k)!\,(1-m)_k}(r+2k+1+u_{mn})'_{2(\mu-k)-1},\quad \mu-k\ge 0, \tag{18}$$

$$\begin{aligned}D_r^{2\mu mn}=\frac{m!}{2^{2\mu-m}}\Bigg[&-\sum_{k=1}^{m}\frac{(k-1)!\,(2m+r-k)!}{2^k(k+r)!\,(m-k)!}\Bigg\{(1-u_{k,0})\sum_{s=0}^{k}{}'\sum_{t=0}^{[s/2]}\frac{(s/2)!}{s!\,(k-s)!\,(s/2-t)!\,t!}\\ &\cdot\frac{{}^-(k+r-u_{mn})'_{-2(\mu-m+k-t)+1}(r-k+2m+1+u_{mn})'_{2(\mu-m+k-t)-1}}{2^{-2t}(\mu-m+k-t)!\,(m-k+1)_{\mu-m+k-t}}\\ &-u_{k,0}\sum_{s=1}^{k}{}'\sum_{t=0}^{[\frac{s+1}{2}-u_{mn}]}\frac{(\frac{s+1}{2}-u_{mn})!}{s!\,(k-s)!\,(\frac{s+1}{2}-u_{mn}-t)!\,t!}\\ &\cdot\frac{1}{2^{-2t}(\mu-m+k-t)!\,(m-k+1)_{\mu-m+k-t}}\\ &\cdot{}^-(k+r-1+u_{mn})'_{-2(\mu-m+k-t)+1}(r-k+2m+2-u_{mn})'_{2(\mu-m+k-t)-1}\Bigg\}\Bigg],\\ &\mu-m+k-t\ge 0,\end{aligned} \tag{19}$$

$$\begin{aligned}D_r^{3\mu mn}=&\frac{(-1)^{\mu-m+1}}{2^{2\mu-m}m!\,(\mu-m)!\,(m+1)_{\mu-m}}\sum_{k=0}^{[\frac{1}{2}(m-1+r)]}\frac{2(m+r)-4k-1}{(m+r-k)(2k+1)}\frac{(2m+r-2k-1)!}{(r-2k-1)!}\\ &\cdot{}^-(r-2k-1-u_{mn})'_{-2(\mu-m)+1}(r-2k+2m-u_{mn})'_{2(\mu-m)-1},\quad \mu\ge m,\end{aligned} \tag{20}$$

$$D_r^{4\mu mn}=\frac{(r-1)!}{2^{2\mu-m}m!\,(r-2m-1)!}\frac{(r+u_{mn})'_{2(\mu-m)-1}\,{}^-(r-2m-1-u_{mn})'_{-2(\mu-m)+1}}{(\mu-m)!\,(m+1)_{\mu-m}},\quad \mu\ge m, \tag{21}$$

where the notations $(\alpha)_k, (\alpha)'_{2k-1}, {}^-(\alpha)'_{-2k+1}$ are defined by

$$\begin{aligned}(\alpha)_0&=1, & (\alpha)_k&=\alpha(\alpha+1)\cdots(\alpha+k-1), & k&=1,2,\dots,\\ (\alpha)'_{-1}&=1, & (\alpha)'_{2k-1}&=\alpha(\alpha+2)(\alpha+4)\cdots(\alpha+2k-2), & k&=1,2,\dots,\\ {}^-(\alpha)'_1&=1, & {}^-(\alpha)'_{-2k+1}&=\alpha(\alpha-2)(\alpha-4)\cdots(\alpha-2k+2), & k&=1,2,\dots.\end{aligned} \tag{22}$$

Therefore, having the coefficients $D_r^{i\mu mn}$ $(i = 1, 2, 3, 4)$, the expansion coefficients δ_μ^{mn} are determined from (17) and the power series of $R_{mn}^{(2)}(h, \xi)$ given by (13) is completely specified.

5. Recursion relation among the δ_μ^{mn} coefficients. In addition, there exists a recursion relation among the δ_μ^{mn} coefficients. To find it we substitute (13) into (1) to obtain the following inhomogeneous radial differential equation:

$$\left[(\xi^2-1)\frac{d^2}{d\xi^2} + 2\xi\frac{d}{d\xi} - \frac{m^2}{\xi^2-1} - \lambda_{mn} + h^2 + h^2(\xi^2-1)\right]\{(\xi-1)u_{mn}+1\} \cdot (\xi^2-1)^{-m/2}h_{mn}(h,\xi) = 2\kappa_{mn}^{(1)}(h)\frac{d}{d\xi}R_{mn}^{(1)}(h,\xi). \quad (23)$$

It is noted that (23) was also obtained by Flammer [2, p. 35] using the $Q_{mn}(h)$ coefficients. The prolate spheroidal radial function of the first kind $R_{mn}^{(1)}(h, \xi)$ can be expanded as follows:

$$R_{mn}^{(1)}(h,\xi) = \frac{1}{\kappa_{mn}^{(1)}(h)}\{(\xi-1)u_{mn}+1\}(\xi^2-1)^{m/2}\sum_{k=0}^{\infty}(-1)^k C_{2k}^{mn}(h)(\xi^2-1)^k, \quad (24)$$

where the expansion coefficients $C_{2k}^{mn}(h)$ are given in closed forms in [2, pp. 23–24]. The final step is to substitute the expansion of $h_{mn}(h, \xi)$ into (23) with the use of (24) to obtain the following recursion relation:

$$\begin{aligned} &4\mu(\mu-m)\delta_\mu^{mn} + [(2\mu-m-2+u_{mn})(2\mu-m-1+u_{mn}) - \lambda_{mn} + h^2]\delta_{\mu-1}^{mn} + h^2\delta_{\mu-2}^{mn} \\ &= \begin{cases} (-1)^{r-m}2(2\mu-m)C_{2\mu-2m}^{mn}, & (n-m) \text{ even}, \\ (-1)^{r-m-1}2(2\mu-m-1)C_{2\mu-2m-2}^{mn} + (-1)^{r-m}2(2\mu-m)C_{2\mu-2m}^{mn}, & (n-m) \text{ odd}. \end{cases} \end{aligned} \quad (25)$$

The relation (25) was also obtained by Flammer [2, p. 36] using the Q_{mn} coefficients. It is noted that the above relation breaks down when $\mu = m$. In this case δ_m^{mn} must be calculated from (17) with D_r^{immn} $(i = 1, 2, 3, 4)$ given by (18), (19), (20), and (21).

6. Numerical results. For thin spheroids Table 1 shows the convergence characteristics of the prolate spheroidal radial function of the second kind in terms of the number of truncated terms N of the infinite series of (13) with $h = 2$ for $m = 1$ and $n = 11$. The table also indicates that the computation time of the prolate spheroidal radial functions becomes faster as ξ approaches 1 since only a few terms are needed. Table 2 records the values of $R_{mn}^{(2)}(\xi)$ for $\xi = 1.005, 1.00005, 1.0000005$, associated with $h = 2$ and $m = 1$ when $(n - m)$ is even. To verify the accuracy of the prolate spheroidal radial functions, the computed Wronskian $R_{mn}^{(1)}(h, \xi)R_{mn}^{(2)'}(h, \xi) - R_{mn}^{(2)}(h, \xi)R_{mn}^{(1)'}(h, \xi)$ is compared to the theoretical Wronskian $\frac{1}{h(\xi^2-1)}$. There is agreement to double precision accuracy.

TABLE 1: Convergence characteristics of the prolate spheroidal radial function of the second kind $R^{(2)}_{mn}(h,\xi)$ in terms of N (number of terms in the series of (13)) with $h = 2$ for $m = 1$ and $n = 11$ in the neighborhood of $\xi = 1$.

N	$\xi = 1.005$	$\xi = 1.00005$
2	-8.103142281×10^9	$-2.191651531 \times 10^{11}$
3	$-1.202751218 \times 10^{10}$	$-2.191690437 \times 10^{11}$
4	$-1.215921863 \times 10^{10}$	$-2.191690450 \times 10^{11}$
5	$-1.216091045 \times 10^{10}$	$-2.191690450 \times 10^{11}$
6	$-1.216091723 \times 10^{10}$	$-2.191690450 \times 10^{11}$
7	$-1.216091723 \times 10^{10}$	$-2.191690450 \times 10^{11}$

TABLE 2: Values of the prolate spheroidal radial function of the second kind $R^{(2)}_{mn}(h,\xi)$ with $h = 2$ for $m = 1$, $n = 1, 3, \ldots, 11$ in the neighborhood of $\xi = 1$.

n	$\xi = 1.005$	$\xi = 1.00005$	$\xi = 1.0000005$
1	-4.079018848	-4.030309738×10^1	-4.029237720×10^2
3	-4.027260647×10^1	-4.406578580×10^2	-4.415280389×10^3
5	-1.792041263×10^3	-2.187624904×10^4	-2.198377442×10^5
7	-1.941032704×10^5	-2.675135913×10^6	-2.698260340×10^7
9	-3.855695141×10^7	-6.055346991×10^8	-6.134558366×10^9
11	$-1.216091723 \times 10^{10}$	$-2.191690450 \times 10^{11}$	$-2.231482592 \times 10^{12}$

Appendix A. Derivation of the coefficients $D_r^{i\mu mn}$. From (7), (10), (14), (16), and (17) the coefficients $D_r^{i\mu mn}$ are found to have the following expressions:

$$D_r^{1\mu mn} = \frac{1}{\mu!}\lim_{x\to 0}\frac{d^\mu}{dx^\mu}[\{(\xi^{-1}-1)u_{mn}+1\}(\xi^2-1)^{m/2}Q^m_{m+r}(\xi)],$$
$$x = \xi^2-1,\ r = -2m, -2m+1, \ldots, -1, \tag{A-1}$$

$$D_r^{2\mu mn} = \frac{1}{2\mu!}\lim_{x\to 0}\sum_{k=1}^{m}(-1)^{k-1}\frac{m!}{k(m-k)!}\frac{d^\mu}{dx^\mu}\{(\xi^{-1}-1)u_{mn}+1\}(\xi^2-1)^{m/2}$$
$$\cdot\left\{\left[\left(\frac{\xi-1}{\xi+1}\right)^{k/2}-\left(\frac{\xi+1}{\xi-1}\right)^{k/2}\right]P^{m-k}_{m+r}(\xi)\right\}, \tag{A-2}$$

$$D_r^{3\mu mn} = -\frac{1}{\mu!}\lim_{x\to 0}\sum_{k=0}^{[\frac{1}{2}(m+r-1)]}\frac{2(m+r)-4k-1}{(m+r-k)(2k+1)} \times \frac{d^\mu}{dx^\mu}[\{(\xi^{-1}-1)u_{mn}+1\}(\xi^2-1)^{m/2}P^m_{m+r-2k-1}(\xi)], \tag{A-3}$$

$$D_r^{4\mu mn} = \frac{1}{\mu!}\lim_{x\to 0}\frac{d^\mu}{dx^\mu}[\{(\xi^{-1}-1)u_{mn}+1\}(\xi^2-1)^{m/2}P^m_{r-m-1}(\xi)]. \tag{A-4}$$

To derive the coefficients $D_r^{1\mu mn}$ for $r=-2m,-2m+1,\ldots,-1$ the closed form expression of $Q^m_{m+r}(\xi)$, as given in (7), is substituted into (A-1) yielding

$$D_r^{1\mu mn} = (-1)^m 2^{m-1}\frac{(m-1)!}{\mu!}\sum_{k=0}^{m-1}\frac{(\frac{r}{2}+1)_k(\frac{r}{2}+\frac{1}{2})_k}{k!\,(1-m)_k}\lim_{x\to 0}\frac{d^\mu}{dx^\mu}\{(\xi^{-1}-1)u_{mn}+1\}\frac{(\xi^2-1)^k}{\xi^{2k+r+1}}. \tag{A-5}$$

But the last factor in (A-5) is

$$\begin{aligned}
&\lim_{x\to 0}\frac{d^\mu}{dx^\mu}\left[\{(\xi^{-1}-1)u_{mn}+1\}\frac{(\xi^2-1)^k}{\xi^{2k+r+1}}\right]\\
&= \lim_{x\to 0}\frac{d^\mu}{dx^\mu}(\xi^2-1+1)^{-k-(r+u_{mn}+1)/2}(\xi^2-1)^k\\
&= \lim_{x\to 0}\frac{d^\mu}{dx^\mu}x^k\sum_{v=0}^{\infty}\binom{-k-\frac{r+1+u_{mn}}{2}}{v}x^v\\
&= (-1)^{\mu-k}\frac{\mu!}{2^{\mu-k}(\mu-k)!}(r+1+u_{mn}+2k)'_{2(\mu-k)-1}, \quad \mu\ge k,
\end{aligned} \tag{A-6}$$

with $(\alpha)'_{2(\mu-k)-1}$ being defined in (22). The substitution of (A-6) into (A-5) gives (18), where the equality $(\frac{r}{2}+1)_k(\frac{r}{2}+\frac{1}{2})_k = 2^{-2k}(r+1)_{2k}$ has been used.

Next, for the coefficients $D_r^{2\mu mn}$ given by (A-2), by using

$$[(\xi-1)^k-(\xi+1)^k] = 2\begin{cases}\sum_{s=1}^{k}{}'\sum_{t=0}^{[(s+1)/2]}\binom{k}{s}\binom{(s+1)/2}{t}\frac{(\xi^2-1)^t}{\xi}, & k \text{ even, or}\\ \sum_{s=1}^{k}{}'\sum_{t=0}^{[(s-1)/2]}\binom{k}{s}\binom{(s-1)/2}{t}\xi(\xi^2-1)^t, & k\text{ even,}\\ \sum_{s=0}^{k}{}'\sum_{t=0}^{[s/2]}\binom{k}{s}\binom{s/2}{t}(\xi^2-1)^t, & k\text{ odd,}\end{cases} \tag{A-7}$$

and (from (11) and (12))

$$P^{m-k}_{m+r}(\xi) = \begin{cases}\frac{(2m+r-k)!}{2^{m-k}(m-k)!\,(k+r)!}(\xi^2-1)^{\frac{m-k}{2}}F\left(-\frac{k+r}{2}, m-\frac{k-r-1}{2}; m-k+1; 1-\xi^2\right), & (k+r)\text{ even,}\\ \frac{(2m+r-k)!}{2^{m-k}(m-k)!\,(k+r)!}\xi(\xi^2-1)^{\frac{m-k}{2}}F\left(-\frac{k+r-1}{2}, m-\frac{k-r}{2}+1; m-k+1; 1-\xi^2\right), & (k+r)\text{ odd}\end{cases} \tag{A-8}$$

in (A-2), it is straightforward to show that $D_r^{2\mu mn}$ is given by (19) after taking the μth derivative with respect to x and with $x \to 0$. Of course, to facilitate the limiting process the hypergeometric functions in (A-8) have been expanded in a power series of $z = (\xi^2 - 1)$ by using the expansion formula $F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k!\,(\gamma)_k} z^k$.

Similarly, for the coefficients $D_r^{i\mu mn}$, $i = 3, 4$, defined by (A-3) and (A-4), by expressing $P_{m+r-2k-1}^m(\xi)$ and $P_{r-m-1}^m(\xi)$ in terms of the above-mentioned power series of the hypergeometric functions, as given by (11) and (12), we can show that $D_r^{3\mu mn}$ and $D_r^{4\mu mn}$ are given by (20) and (21).

References

[1] J. A. Stratton, P. M. Morse, L. J. Chu, D. C. Little, and F. J. Carbato, *Spheroidal Wave Functions*, John Wiley and Sons, New York, 1956

[2] C. Flammer, *Spheroidal Wave Functions*, Stanford Univ. Press, Stanford, Calif., 1957

[3] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953

[4] B. P. Sinha and R. H. MacPhie, *On the computation of the prolate spheroidal radial functions of the second kind*, J. Math. Phys. **16**, 2378–2381 (1975)

[5] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, New York, 1980

[6] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1970