On the Achievable Delay Margin Using LTI Control for Unstable Plants

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Abstract—Handling delays in control systems is difficult and is of long-standing interest. It is well known that, given a finite-dimensional linear time-invariant (FDLTI) plant and controller forming a strictly proper stable feedback connection, closed-loop stability will be maintained under a small delay in the feedback loop, although most closed loop systems become unstable for large delays. One previously unsolved fundamental problem in this context is whether, for a given FDLTI plant, an arbitrarily large delay margin can be achieved using LTI control. Here, we adopt a frequency domain approach and demonstrate that, for a strictly proper real rational plant, there is a uniform upper bound on the delay that can be tolerated when using an LTI controller, if and only if the plant has at least one closed right half plane pole not at the origin. We also give several explicit upper bounds on the achievable delay margin, and, in some special cases, demonstrate that these bounds are tight.

Index Terms—Delay margin, frequency domain, linear systems, time delay.

I. INTRODUCTION

TIME delays are common in many control processes. These delays arise from a variety of sources, including signal transmission delay, computational delay (e.g., in a system which uses image processing), and physical transport delay. The presence of time delays in a control system may cause degraded performance, poor robustness or instability in a feedback control system. There is, therefore, a large literature on topics relating to control of such processes, e.g., see [1], [8], [10], [11], and [14] for collections of recent results.

In many situations, the delay is uncertain, although the maximum delay that can occur may be known. In this context, some authors have discussed stability analysis for time delay systems, e.g., see [21]. Others have examined synthesis problems (e.g., [15]) including robust synthesis (e.g., see [22]). Another area of research, as distinct from robust synthesis, focuses on fundamental limitations in control. In this context, it is natural to pose questions such as that in the recent book on open problems in control [5]: "For a fixed FDLTI plant, is there an upper bound on the uncertain delay which can be tolerated using an LTI controller?" This paper focuses on this latter type of question.

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For the case of a stable plant, clearly the zero controller provides tolerance of all delays. When considering open loop unstable plants with static state feedback only, the results of [15, Theorem 2] show that indeed, it is not possible to achieve an arbitrarily large delay margin. Also, for the case of an unstable FDLTI plant, recent results (see [16]–[18]) have established that when using linear time varying (LTV) feedback control, an arbitrarily large delay margin is possible. This can be compared with work on the gain margin problem: when using LTI controllers, it is shown in [23] and [13] that there is an upper bound on the gain margin for an unstable nonminimum phase plant, but that this bound can be dispensed with by moving to linear periodic controllers, e.g., see [9], [12], [20], and [26]. Here, we wish to obtain comparable results on the use of LTI controllers for the delay margin problem. Indeed, for the LTI control case, it has been conjectured that there is an upper bound on the achievable delay margin for unstable plants (see, for example, [6], in which a lower bound on the upper bound is given). Until now, to the best of our knowledge, there have been no firm results to this effect. Here we adopt the frequency domain approach and demonstrate that there is indeed an upper bound on the achievable delay margin when using a LTI controller if and only if the plant has a nonzero closed right half plane pole. Furthermore, we provide an explicit upper bound in terms of the plant poles and zeros, and demonstrate that this bound is tight in several special cases, including that of having a single unstable pole with no nonminimum phase zeros.

We use standard notation throughout the paper. We let $\mathbf{R}$ denote the set of real numbers, $\mathbf{C}$ denote the set of complex numbers, $\mathbf{C}^-$ denotes the set of complex numbers with negative real parts, and $\mathbf{C}^+$ denote the set of complex numbers with a positive real part. We use the Holder 2-norm to measure the size of $x \in \mathbf{C}^n$

$$\|x\| := \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}.$$

With $A \in \mathbf{C}^{m \times n}$, we let $\|A\|$ denote the corresponding induced norm, namely the largest singular value of A.

$H_\infty^{n \times m}$ denotes the set of $n \times m$ complex-valued functions which are analytic and bounded in $\mathbf{C}^+$, and $RH_\infty^{n \times m}$ denotes the subset of real rational elements. It is a fact that $H_\infty^{n \times m}$ is a Banach space: the norm of $G \in H_\infty^{n \times m}$ is given by

$$\|G\|_\infty := \sup_{s \in \mathbf{C}^+} \|G(s)\|.$$

Using the Maximum Modulus Theorem, it can be shown that

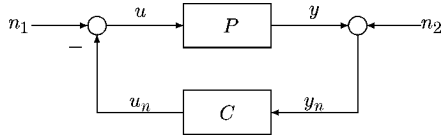$$\|G\|_\infty = \sup_{\omega \in \mathbf{R}} \|G(j\omega)\|.$$

Fig. 1. Standard feedback control structure.

In this paper, we will be dealing with single-input single-output systems, so we will be almost exclusively interested in $H_\infty^{1\times 1}$; henceforth, when the dimensions of the space are 1×1 we simply write H_∞.

The quotient field of H_∞ is defined by

$$\mathcal{F}(H_\infty) := \left\{ \frac{U}{V} : U, V \in H_\infty, V \neq 0 \right\}.$$

We say that $N, D \in H_\infty$ are *coprime* if there exist $X, Y \in H_\infty$ so that

$$(NX + DY)(s) = 1, \quad s \in \mathbf{C}^+.$$

With $U, V \in H_\infty$ satisfying $V \neq 0$, we say that U/V is a *stable coprime factorization* of $F \in \mathcal{F}(H_\infty)$ if:
  i) $F(s) = U(s)/V(s), s \in \mathbf{C}^+$;
  ii) U and V are coprime.

II. THE PROBLEM

In this paper, we work in the transfer function domain. Our nominal plant is single-input, single-output, real-rational and strictly proper, and denoted by $P_0(s)$. We are considering the problem of robust stabilization of a plant with an unknown time delay in the feedback loop; for convenience, we combine this delay with the nominal plant P_0 to yield our modified plant model. Hence, with $\bar{\tau} > 0$, the set of admissible plants is given by

$$\mathcal{P}_{\bar{\tau}} := \left\{ e^{-s\tau} P_0(s) : \tau \in [0, \bar{\tau}] \right\}.$$

While the nominal plant is finite dimensional, the delayed version is infinite dimensional, so we allow for infinite-dimensional LTI controllers. Since we are working in the frequency domain, we adopt the methodology of Vidyasagar [24, Chapter 8] to describe this class: the set of admissible controllers is the quotient field of H_∞, which we have labelled $\mathcal{F}(H_\infty)$. We consider the standard feedback structure—see Fig. 1; we say that $C \in \mathcal{F}(H_\infty)$ *stabilizes* $P \in \mathcal{F}(H_\infty)$ if the transfer function from $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$ lies in $H_\infty^{2\times 2}$, i.e.,

$$(1 + PC)^{-1}, \quad P(1 + PC)^{-1}, \quad C(1 + PC)^{-1} \in H_\infty$$

and that C *stabilizes* $\mathcal{P}_{\bar{\tau}}$ if C stabilizes $e^{-s\tau} P_0(s)$ for every $\tau \in [0, \bar{\tau}]$.

If $C \in \mathcal{F}(H_\infty)$ stabilizes P_0 then the *delay margin* is

$$DM(P_0, C) := \sup\{\bar{\tau} > 0 : C \text{ stabilizes } \mathcal{P}_{\bar{\tau}}\}.$$

The maximum delay margin achievable by a stabilizing controller is given by

$$DM(P_0) := \sup \{DM(P_0, C) : C \in \mathcal{F}(H_\infty) \text{ stabilizes } P_0\}. \tag{1}$$

So, there are two natural questions: How do we compute or obtain bounds on $DM(P_0, C)$ and $DM(P_0)$, respectively?

It is well known (see for example [19, §4.1] or [11, §2.1]) that the problem of computing $DM(P_0, C)$ is clearly the easier of the two problems.

Remark 1: Note that computing the exact delay margin $DM(P_0, C)$ provided by a given real rational controller $C(s)$ that stabilizes the nominal plant $P_0(s)$ is relatively straight-forward. First, we let $\{\omega_1, \ldots, \omega_p\}$ denote the critical frequencies where

$$|P_0(j\omega)C(j\omega)| = 1$$

and then let $\phi_i \in (0, 2\pi)$ denote the corresponding phase margins, i.e., $P_0(j\omega_i)C(j\omega_i) = e^{j(\pi + \phi_i)} = -e^{j\phi_i}$. Stability will be maintained when a delay of $e^{-s\tau}$ is placed in the loop as long as the number of encirclements of -1 remains the same; this will clearly be the case for all $\tau \in [0, \bar{\tau}]$ iff

$$\omega_i \bar{\tau} < \phi_i, \quad i = 1, \ldots, p$$

so

$$DM(P_0, C) = \min \left\{ \frac{\phi_i}{\omega_i} : i = 1, \ldots, p \right\}.$$

This observation will play a critical role in some of the forthcoming examples.

Computing $DM(P_0)$ is much harder. It is well known (see, for example, [19, §4.2] or [11, §3.3]) that a lower bound on $DM(P_0)$ can be computed by solving an H_∞ robust control problem as follows:

Proposition 2: Suppose that $C \in \mathcal{F}(H_\infty)$ stabilizes P_0 and define $T_0(s) = (C(s)P_0(s))/(1 + C(s)P_0(s))$. Then

$$DM(P_0, C) \geq \|sT_0(s)\|_\infty^{-1} \tag{2}$$

and

$$DM(P_0) \geq \left(\inf_{C \text{ stabilizes } P_0} \|sT_0(s)\|_\infty \right)^{-1}. \tag{3}$$

Proof: The proof uses a standard robust control argument based on "uncertainty embedding," e.g., see [19, §4.2]. ∎

It has been shown recently [16]–[18] that for every real rational strictly proper plant $P_0(s)$, we can obtain a linear periodic controller to make the delay margin as large as desired. Here the focus is on LTI compensators: the main contributions of this paper is to show that $DM(P_0)$ is finite iff P_0 has a nonzero pole in the closed right half plane. We prove this by considering a number of special cases, which are quantitative in nature, that lead to the main result.

To proceed we need to characterize the set of all LTI stabilizing compensators. With $N_0, D_0 \in RH_\infty$, let N_0/D_0 denote

a stable coprime factorization of P_0, and let $X_0, Y_0 \in RH_\infty$ represent the solution to the Bezout identity

$$N_0 X_0 + D_0 Y_0 = 1.$$

It is well known (see Theorem 8.3.12 of [24]) that the set of all admissible controllers in $\mathcal{F}(H_\infty)$ which stabilize P_0 is given by the following Youla parametrization:

$$\left\{ \frac{X_0 + D_0 Q}{Y_0 - N_0 Q} : Q \in H_\infty \right\}.$$

The subset of all real rational stabilizing controllers can be obtained by restricting Q to RH_∞.

Lemma 3: (Lemma 3.1 of [25]) With $U, V, N_p, D_p \in H_\infty$, if U/V and N_p/D_p are stable coprime factorizations of C and P, respectively, then C stabilizes P iff

$$(N_p U + D_p V)^{-1} \in H_\infty.$$

Remark 4: With $N_0, D_0 \in RH_\infty$ and N_0/D_0 a stable coprime factorization of the real rational, strictly proper, plant P_0, it follows from [3] (Theorem 2.1) that $e^{-s\tau} N_0(s)/D_0(s)$ is a stable coprime factorization of the plant $e^{-s\tau} P_0(s)$. Hence, if U/V is a stable coprime factorization of the controller $C \in \mathcal{F}(H_\infty)$, then by Lemma 3, C stabilizes $e^{-s\tau} P_0(s)$ iff

$$(e^{-s\tau} N_0 U + D_0 V)^{-1} \in H_\infty.$$

Now we turn to a technical result. Since a delay is irrational, which is difficult to analyse in closed-loop, instead we consider a class of complex-valued all-pass functions, chosen because they also have unity gain on the imaginary axis; from this, we can infer the behaviour of the closed loop system with a delay. While the following result can be presented and proven in much greater generality, it is at the expense of clarity, so we have chosen the simplest version suitable for our needs.

Proposition 5: With P_0 real rational and strictly proper and $C \in \mathcal{F}(H_\infty)$, suppose that C stabilize P_0. Consider the all-pass function $B_\alpha(s)$, parametrized by the real variable $\alpha \geq 0$, satisfying three conditions.

1) The transfer function $B_\alpha(s)$ has one of the following three forms.
   i) $B_\alpha(s) = (1 - c_\alpha s)/(1 + c_\alpha s)$.
   ii) $B_\alpha(s) = ((1 - c_\alpha s)/(1 + c_\alpha s))((s - p)/(s + p))((s + p + \alpha)/(s - p - \alpha))$ and $P_0(s)$ has a pole at $p > 0$.
   iii) $B_\alpha(s) = ((1 - c_\alpha s)/(1 + c_\alpha s))((s + z)/(s - z))((s - z - \alpha)/(s + z + \alpha))$ and P_0 has a zero at $z > 0$.
2) c_α is a complex-valued continuous function of α satisfying $c_0 = 0$ and either i) c_α is identically zero or ii) $Re(c_\alpha) > 0$ for $\alpha > 0$.
3) There exists an $\hat{\alpha} > 0$ for which $B_{\hat{\alpha}}(s)$ and $P_0(s)$ have an unstable pole-zero cancellation in $\mathbf{C}^+$ [excluding the obvious one at p if 1 ii) holds and the obvious one at z if 1 iii) holds].

With the modified plant defined by $P_\alpha(s) := P_0(s) B_\alpha(s)$, the following is true.

a) There exists an $\bar{\alpha} \in (0, \hat{\alpha})$ so that C stabilizes P_α for all $\alpha \in [0, \bar{\alpha}]$.
b) If C is real rational, then there exists a critical value $\alpha^* \in (0, \hat{\alpha})$ and $\omega^* \in \mathbf{R}$ so that

$$P_{\alpha^*}(j\omega^*) C(j\omega^*) = B_{\alpha^*}(j\omega^*) P_0(j\omega^*) C(j\omega^*) = -1. \quad (4)$$

c) If C is irrational, then there exists a critical value $\alpha^* \in (0, \hat{\alpha}), \omega^* \in \mathbf{R}$ and a sequence $\{s_i\} \subset \mathbf{C}^+$ converging to $j\omega^*$ satisfying

$$\lim_{i \to \infty} B_{\alpha^*}(s_i) P_0(s_i) C(s_i) = \lim_{i \to \infty} B_{\alpha^*}(j\omega^*) P_0(s_i) C(s_i) = -1. \quad (5)$$

Proof: Since the proof is not central to the paper we relegate it to Appendix VIII-A. ∎

Remark 6: Suppose that $\alpha^* > 0$, $\omega^* \in \mathbf{R}$ and $\{s_i\} \subset \mathbf{C}^+$ converging to $j\omega^*$ are chosen so that (4) or (5) hold, as appropriate. It follows immediately that $C(s)$ does not stabilize $P_0(s) B_{\alpha^*}(j\omega^*)$. Since $B_{\alpha^*}(j\omega)$ has negative phase for all $\omega > 0$ (and positive phase for all $\omega < 0$) and $\alpha^* > 0$, there exists a $T^* > 0$ so that

$$e^{-j\omega^* T^*} = B_{\alpha^*}(j\omega^*)$$

which means that

$$\left[1 + e^{-sT^*} P_0 C \right]^{-1} \notin H_\infty$$

so

$$DM(P_0) \leq T^*.$$

The difficulty lies in choosing the c_α term (in the definition of B_α) in such a way as to make T^* as small as possible.

In the following sections, we shall use this proposition to establish various upper bounds on the achievable delay margin. First, we look at the real pole case, followed by the complex pole case. Thereafter, we consider the special case of imaginary axis poles. We finish with an analysis of the effect of having nonminimum phase zeros as well as unstable plant poles.

III. PLANTS WITH A REAL UNSTABLE POLE

Theorem 7: If P_0 has a real unstable pole at p, then $DM(P_0) \leq (2/p)$.

Proof: Suppose that $C \in \mathcal{F}(H_\infty)$ stabilizes P_0. Define the all-pass transfer function

$$B_\alpha(s) := \frac{1 - \alpha s}{1 + \alpha s}, \quad \alpha \geq 0$$

and observe that B_α has the form required in Proposition 5. Since there is a pole zero cancellation between $B_\alpha(s)$ and $P_0(s)$ at $s = p$ when $\alpha = 1/p$, it follows that there exists $\alpha^* \in (0, 1/p), \omega^* \in \mathbf{R}$, and $\{s_i\} \subset \mathbf{C}^+$ converging to $j\omega^*$ so that

$$\lim_{i \to \infty} B_{\alpha^*}(j\omega^*) P_0(s_i) C(s_i) = -1.$$

which means that $C(s)$ does not stabilize $B_{\alpha^*}(j\omega^*) P_0(s)$. Note that clearly $\omega^* \neq 0$, for otherwise, $C(s)$ would not stabilize $P_0(s)$.

We would now like to find, if possible, a $T > 0$ for which the delay element e^{-sT} evaluated at $s = j\omega^*$ exactly equals $B_{\alpha^*}(j\omega^*)$, for then

$$\lim_{i \to \infty} \left[e^{-s_i T} P_0(s_i) C(s_i) \right] = -1$$

which would mean that $C(s)$ does not stabilize $e^{-sT} P_0(s)$. To proceed, observe that both $B_{\alpha^*}(s)$ and e^{-sT} have magnitude one on the imaginary axis, so we need to match their phase

$$-\omega^* T = -2\tan^{-1}(\alpha^* \omega^*)$$

which clearly has a positive solution, which we label T^*, namely

$$
\begin{aligned}
T^* &= \frac{2}{\omega^*} \tan^{-1}(\alpha^* \omega^*) \\
&= \frac{2}{|\omega^*|} \tan^{-1}\left(\alpha^* |\omega^*|\right) < \frac{2}{|\omega^*|} \alpha^* |\omega^*| < \frac{2}{p}.
\end{aligned}
$$

Hence, $C(s)$ does not stabilize $e^{-sT^*} P_0(s)$, so

$$DM(P_0, C) < T^* < \frac{2}{p}.$$

Since this is true for every stabilizing controller in $\mathcal{F}(H_\infty)$, then from (1), we obtain

$$DM(P_0) \leq \frac{2}{p}.$$

∎

Remark 8: If the plant has only one pole $p > 0$ in the closed right half plane, and no zeros in the closed right-half plane, then the bound is tight. To see this, partition the plant as

$$P_0(s) = \frac{1}{(s-p)} W(s)$$

with $W(s)$ stable, minimum-phase, and having relative degree $r \geq 0$. We would like to apply the controller

$$\left((1 - \varepsilon^2)^{1/2} s + (1 + \varepsilon^4)^{1/2} p \right) W(s)^{-1}$$

for then the loop gain is easy to analyse; since the controller is improper and, hence, inadmissible, we roll it off at high frequency. To this end, consider the FDLTI controller

$$
\begin{aligned}
C_\varepsilon(s) &= \frac{(1 - \varepsilon^2)^{1/2} s + (1 + \varepsilon^4)^{1/2} p}{(1 + \varepsilon s)^{r+1}} W(s)^{-1} (1 + \varepsilon^4 p^2)^{\frac{r+1}{2}} \\
&\in RH_\infty, \quad \varepsilon \in (0, 1).
\end{aligned} \tag{6}
$$

Then for $\varepsilon \in (0, 1)$, there are no unstable pole-zero cancellations between $C_\varepsilon(s)$ and $P(s)$ and the loop gain is

$$P_0(s) C_\varepsilon(s) = \frac{(1-\varepsilon^2)^{1/2} s + (1+\varepsilon^4)^{1/2} p}{(s-p)} \left(\frac{\sqrt{1+\varepsilon^4 p^2}}{(1+\varepsilon s)} \right)^{r+1}.$$

It is easy to verify that for $\varepsilon \in (0, 1)$ this loop transfer function has a magnitude that is a strictly decreasing function of ω on the range $[0, \infty)$, going from $(1 + \varepsilon^4)^{1/2}(1 + \varepsilon^4 p^2)^{(r+1)/2} > 1$ to zero, so there is a unique $\omega_c > 0$ for which

$$|P_0(j\omega_c) C_\varepsilon(j\omega_c)| = 1.$$

Indeed, it is easy to verify that $\omega_c = \varepsilon p$. This yields a corresponding phase margin of

$$\phi^* = 2\varepsilon + \mathcal{O}(\varepsilon^2).$$

Now observe that the Nyquist plot encircles -1 exactly one time for all sufficiently small $\varepsilon > 0$, so the closed loop system is stable. Using Remark 1, it follows that the delay margin provided by this controller is

$$DM(P_0, C_\varepsilon) = \frac{\phi^*}{w_c} = \frac{2\varepsilon + \mathcal{O}(\varepsilon^2)}{p\varepsilon} = \frac{2}{p} + \mathcal{O}(\varepsilon)$$

for small $\varepsilon > 0$. Hence, we can get a delay margin as close to $2/p$ as desired by choosing ε appropriately. Note that the controller proposed here may suffer from difficulties such as poor sensitivity, phase margin and/or gain margin and, therefore, is not intended as a practical controller. The result does serve, however, to delineate the achievable delay margin for this case.

IV. PLANTS WITH COMPLEX POLES

We now consider several different cases involving complex poles. The first case we consider is that of complex poles with positive real part.

A. Open RHP Complex Poles

Theorem 9: If P_0 has unstable poles at $re^{\pm j\phi}$ with $\phi \in (0, \pi/2)$, then

$$DM(P_0) \leq \frac{\pi}{r} \sin \phi + \max \left\{ \frac{2}{r} \cos \phi, \frac{2}{r} \phi \sin \phi \right\}.$$

Proof: Suppose that $C \in \mathcal{F}(H_\infty)$ stabilizes P_0. Define the complex valued all-pass transfer function

$$B_\alpha(s) := \frac{1 - \alpha s/(re^{-j\phi})}{1 + \alpha s/(re^{j\phi})}$$

and observe that B_α has the form required in Proposition 5. Since there is a pole zero cancellation between $B_\alpha(s)$ and $P_0(s)$ at $s = re^{-j\phi}$ when $\alpha = 1$, it follows that there exists $\alpha^* \in (0, 1)$, $\omega^* \in \mathbf{R}$, and $\{s_i\} \subset \mathbf{C}^+$ converging to $j\omega^*$ so that

$$\lim_{i \to \infty} B_{\alpha^*}(j\omega^*) P_0(s_i) C(s_i) = -1$$

which means that $C(s)$ does not stabilize $B_{\alpha^*}(j\omega^*) P_0(s)$. Note that clearly $\omega^* \neq 0$, for otherwise, $C(s)$ would not stabilize $P_0(s)$. We now consider three different ranges of ω^*. In each case, we would like to find, if possible, a $T > 0$ for which the delay element e^{-sT} evaluated at $s = j\omega^*$ exactly equals $B_{\alpha^*}(j\omega^*)$, for then

$$\lim_{i \to \infty} e^{-s_i T} P_0(s_i) C(s_i) = -1$$

which would mean that $C(s)$ does not stabilize $e^{-sT}P_0(s)$. Before proceeding, notice that $|B_{\alpha^*}(j\omega^*)| = 1 = |e^{-j\omega T}|$ for all $T \in \mathbf{R}$, so all we need is to find $T > 0$ to match the phase.

Case 1: $\omega^* > 0$: For $T > 0$ to satisfy $e^{-j\omega^* T} = B_{\alpha^*}(j\omega^*)$, we need

$$-\omega^* T = -2\tan^{-1}\left(\frac{\frac{\alpha^*}{r}\omega^*\cos\phi}{1 + \frac{\alpha^*}{r}\omega^*\sin\phi}\right)$$

which clearly has a positive solution, which we label T^*, namely

$$T^* = \frac{2}{\omega^*}\tan^{-1}\left(\frac{\frac{\alpha^*}{r}\omega^*\cos\phi}{1 + \frac{\alpha^*}{r}\omega^*\sin\phi}\right).$$

Using the properties of $\tan^{-1}$ and the constraint on ϕ, it follows that

$$T^* \le \frac{2}{\omega^*}\frac{\frac{\alpha^*}{r}\omega^*\cos\phi}{1 + \frac{\alpha^*}{r}\omega^*\sin\phi} \le \frac{2\alpha^*}{r}\cos\phi < \frac{2}{r}\cos\phi.$$

This means that in the case where $\omega^* > 0$, $C(s)$ cannot stabilize $e^{-sT^*}P_0(s)$ and, therefore

$$DM(P_0, C) < \frac{2}{r}\cos\phi. \tag{7}$$

To delineate two different cases when ω^* is negative, we define

$$\omega_1 := \frac{r}{\alpha^*\sin\phi}. \tag{8}$$

Case 2: $-\omega_1 < \omega^* < 0$: For $T > 0$ to satisfy $e^{-j\omega^* T} = B_{\alpha^*}(j\omega^*)$, we need

$$|\omega^*|T = 2\mathrm{Arg}\left[\left(1 - \frac{\alpha^*}{r}|\omega^*|\sin\phi\right) + j\left(\frac{\alpha^*}{r}|\omega^*|\cos\phi\right)\right].$$

Since the RHS lives in the interval $(0, \pi)$, it is clear that there does indeed exist a $T^* > 0$ which satisfies this equation, namely

$$T^* = \frac{2}{|\omega^*|}\mathrm{Arg}\left[\left(1 - \frac{\alpha^*}{r}|\omega^*|\sin\phi\right) + j\left(\frac{\alpha^*}{r}|\omega^*|\cos\phi\right)\right]. \tag{9}$$

From (8) and the assumed range for ω^*, the real part inside the above expression for the Argument is positive, so it follows that

$$T^* = \frac{2}{|\omega^*|}\tan^{-1}\left(\frac{\frac{\alpha^*}{r}|\omega^*|\cos\phi}{1 - \frac{\alpha^*}{r}|\omega^*|\sin\phi}\right).$$

Now introduce the new quantities

$$\rho^* := \frac{r}{\alpha^*|\omega^*|\sin\phi}, \quad c := \frac{\cos\phi}{\sin\phi} > 0.$$

From (8), it follows that $\rho^* \in (1, \infty)$. Substituting this into the formula for T^* and simplifying, we end up with

$$T^* = 2\frac{\alpha^*\sin\phi}{r}\rho^*\tan^{-1}\left(\frac{c}{\rho^* - 1}\right).$$

Using the fact that $\rho^* > 1$, it follows that

$$\rho^*\tan^{-1}\left(\frac{c}{\rho^* - 1}\right) = (\rho^* - 1)\tan^{-1}\left(\frac{c}{\rho^* - 1}\right)$$
$$+ \tan^{-1}\left(\frac{c}{\rho^* - 1}\right)$$
$$< c + \frac{\pi}{2}.$$

Since $\alpha^* \in (0, 1)$, it follows that

$$T^* < 2\frac{\alpha^*\sin\phi}{r}\left(c + \frac{\pi}{2}\right) = 2\frac{\alpha^*\sin\phi}{r}\left(\frac{\cos\phi}{\sin\phi} + \frac{\pi}{2}\right)$$
$$= \frac{2}{r}\cos\phi + \frac{\pi}{r}\sin\phi.$$

So, in this case

$$DM(P_0, C) < \frac{2}{r}\cos\phi + \frac{\pi}{r}\sin\phi. \tag{10}$$

Case 3: $\omega^* \in (-\infty, -\omega_1]$: For this range of ω^* the critical value of $T > 0$ is once again given by (9), which simplifies to

$$T^* = \frac{2}{|\omega^*|}\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{\frac{\alpha^*}{r}|\omega^*|\sin\phi - 1}{\frac{\alpha^*}{r}|\omega^*|\cos\phi}\right)\right)$$
$$< \frac{\pi}{|\omega^*|} + \frac{2}{|\omega^*|}\tan^{-1}(\tan\phi) \le \frac{\pi}{r}\sin\phi + \frac{2}{r}\phi\sin\phi$$

where we have used both (8) and $\alpha^* \in (0, 1)$. So, in this case

$$DM(P_0, C) < \frac{\pi}{r}\sin\phi + \frac{2}{r}\phi\sin\phi. \tag{11}$$

Combining the above three cases, (7), (10), and (11), and noting that at least one of these three cases must hold, we obtain

$$DM(P_0, C) \le \max\left\{\frac{2}{r}\cos\phi, \frac{2}{r}\cos\phi + \frac{\pi}{r}\sin\phi\right.$$
$$\left.\frac{\pi}{r}\sin\phi + \frac{2}{r}\phi\sin\phi\right\}.$$

Since this holds for every $C \in \mathcal{F}(H_\infty)$, it follows that

$$DM(P_0) \le \frac{\pi}{r}\sin\phi + \max\left\{\frac{2}{r}\cos\phi, \frac{2}{r}\phi\sin\phi\right\}.$$

∎

Note that the above argument in Theorem 9 does not cover the case of purely imaginary plant poles, since when $\phi = \pi/2$, B_α selected above is not asymptotically stable. We now turn to consider the case of purely imaginary poles.

B. Imaginary Axis Poles

Theorem 10: If P_0 has poles at $s = \pm j\omega_0 \ne 0$ with $\omega_0 > 0$, then $DM(P_0) \le 2\pi/\omega_0$.

Proof: First suppose that $C \in \mathcal{F}(H_\infty)$ is real rational and stabilizes P_0. Since the loop gain is continuous except at its

poles, and it is zero at $s = \infty$, it follows that there exists $w^* > w_0$ for which

$$|P_0(j\omega^*)C(j\omega^*)| = 1.$$

Let $\phi^* \in [0, 2\pi)$ denote the phase margin associated with this frequency

$$P_0(j\omega^*)C(j\omega^*) = e^{j(\pi + \phi^*)} = -e^{j\phi^*}.$$

With $T^* = \phi^*/\omega^*$, we have

$$P_0(j\omega^*)C(j\omega^*)e^{-j\omega^* T^*} = -1$$

which means that

$$DM(P_0, C) < T^* = \frac{\phi^*}{\omega^*} \leq \frac{2\pi}{\omega_0}.$$

Now suppose that $C \in \mathcal{F}(H_\infty)$ is infinite dimensional and stabilizes P_0; the proof here is more involved, since C may have discontinuities on the imaginary axis. We adopt any stable coprime factorization and choose the Youla parameter $Q \in H_\infty$ which corresponds to the controller, so $P_0 = N_0/D_0$

$$N_0 X_0 + D_0 Y_0 = 1 \tag{12}$$

and a stable coprime factorization of the controller is $C = (X_0 + D_0 Q)/(Y_0 - N_0 Q)$. So with $s \in \bar{C}^+$, we have[1]

$$\begin{aligned} |P_0 C(s)| = 1 &\Leftrightarrow \left| \frac{N_0(X_0 + D_0 Q)}{D_0(Y_0 - N_0 Q)}(s) \right| = 1 \\ &\Leftrightarrow \left| \frac{N_0(X_0 + D_0 Q)}{-1 + N_0(X_0 + N_0 D_0 Q)}(s) \right| = 1 \\ &\Leftrightarrow \operatorname{Re}\left[N_0(X_0 + D_0 Q) \right](s) = \frac{1}{2}. \end{aligned}$$

Since $N_0(X_0 + D_0 Q)$ is analytic in $\mathbf{C}^+$ and $D_0(j\omega_0) = 0$, from the Bezout identity, it follows that

$$\lim_{s \to j\omega_0, s \in \mathbf{C}^+} [N_0(X_0 + D_0 Q)](s) = 1.$$

It follows that there exists a $\bar{\varepsilon} > 0$ so that

$$\operatorname{Re}[N_0(X_0 + D_0 Q)](\varepsilon + j\omega_0) > \frac{1}{2}, \quad \varepsilon \in (0, \bar{\varepsilon}). \tag{13}$$

Now let $\{\varepsilon_i\} \subset (0, \bar{\varepsilon})$ denote any positive sequence converging to zero. Then for each $i \in \mathbf{N}$

$$\operatorname{Re}[N_0(X_0 + D_0 Q)](\varepsilon_i + j\omega)$$

is a continuous function of $\omega \in \mathbf{R}$ and tends to zero as $\omega \to \infty$, so together with (13) there clearly exists an $\omega_i > \omega_0$ so that

$$\operatorname{Re}[N_0(X_0 + D_0 Q)](\varepsilon_i + j\omega_i) = \frac{1}{2}. \tag{14}$$

[1]Notice that $|P_0(s)C(s)| = 1$ with $s \in \bar{\mathbf{C}}^+$ automatically rules out s being a pole of P_0 or C.

Indeed, since N_0 is strictly proper and $X_0 + D_0 Q \in H_\infty$, it follows that there exists an $\bar{\omega} > 0$ so that

$$\omega_i < \bar{\omega}, \quad i \in \mathbf{N}.$$

The sequence $\{(j\omega_i, [N_0(X_0 + D_0 Q)](\varepsilon_i + j\omega_i)\}$ lies in a compact set[2] so there exists a convergent subsequence, say $\{(j\omega_{i_k}, [N_0(X_0 + D_0 Q)](\varepsilon_{i_k} + j\omega_{i_k})\}$, with $\{\omega_{i_k}\}$ converging to $\omega^* \geq \omega_0$. If we combine this with (14), we can conclude that

$$PC(\varepsilon_{i_k} + j\omega_{i_k}) = \frac{N_0(X_0 + D_0 Q)}{1 - N_0(X_0 + D_0 Q)}(\varepsilon_{i_k} + j\omega_{i_k})$$

converges as well; indeed, it converges to a number on the unit circle, say $e^{j(\pi + \phi^*)}$ with $\phi^* \in [0, 2\pi)$. As in the finite-dimensional case, choose $T^* = \phi^*/\omega^*$; clearly

$$e^{-(\varepsilon_{i_k} + j\omega_{i_k})T^*} \to e^{-j\omega^* T^*}$$

which means that

$$e^{-(\varepsilon_{i_k} + j\omega_{i_k})T^*} PC(\varepsilon_{i_k} + j\omega_{i_k}) \to e^{-j\omega^* T^*} e^{j(\pi + \phi^*)} = -1$$

or equivalently that

$$1 + e^{-(\varepsilon_{i_k} + j\omega_{i_k})T^*} PC(\varepsilon_{i_k} + j\omega_{i_k}) \to 0$$

which means that $[1 + e^{-sT^*}P(s)C(s)]^{-1} \notin H_\infty$, so $C(s)$ does not stabilize $e^{-sT^*}P(s)$. Hence

$$DM(P_0, C) < T^* = \frac{\phi^*}{\omega^*} < \frac{2\pi}{\omega_0}.$$

Since this is true for every stabilizing controller in $\mathcal{F}(H_\infty)$, it follows that

$$DM(P_0) \leq \frac{2\pi}{\omega_0}.$$

■

It turns out that for the case where there are no other CRHP plant poles or zeros, the lower bound given in Theorem 10 is in fact tight, as established in the following result.

Corollary 11: Suppose that the nominal plant, $P_0(s)$, can be factored as

$$P_0(s) = \frac{1}{(s^2 + \omega_0^2)} W(s) \tag{15}$$

with $\omega_0 > 0$, and with $W(s)$ stable, minimum phase, and with relative degree $r \geq -1$. Then

$$DM(P_0) = \frac{2\pi}{\omega_0}. \tag{16}$$

Proof: From Theorem 10, we have an upper bound on the achievable delay margin. It remains to prove, therefore, that this bound is tight. Motivated by the proof of Theorem 10, we design a loop transfer function that has two gain cut-off frequencies $\omega_c^- < \omega_0 < \omega_c^+$, both of which can be made arbitrarily close to ω_0, whilst simultaneously achieving a phase margin for each

[2]Notice that $\omega_i \in [\omega_0, \bar{\omega}]$ and $|[N_0(X_0 + D_0 Q)](\varepsilon_i + j\omega)| \leq \|N_0(X_0 + D_0 Q)\|_\infty$.

cut-off frequency that may be made arbitrarily close to $360°$. Details of the proof are given in Appendix VIII-B. ∎

Remark 12: The bounds produced in Theorems 9, 7, and 10 are compatible in the following sense: with a pole of the form $re^{j\phi}$ with $r > 0$ and $\phi \in (0, \pi/2)$, the bound provided for the complex case in Theorem 9 tends to that of Theorem 7 as $\phi \to 0$ and tends to that of Theorem 10 as $\phi \to (\pi/2)$.

This leaves one other case of imaginary axis poles, namely, poles at the origin.

C. Poles at the Origin

If all of the unstable poles are at zero, then we can make the delay margin as large as desired, as demonstrated in the following Theorem.

Theorem 13: If the only unstable poles of $P_0(s)$ are at zero, then $DM(P_0) = \infty$.

Proof: The proof, given in the Appendix, uses the Youla parametrization of all stabilizing controllers together with Proposition 2. In particular, we construct $Q_\varepsilon \in RH_\infty$ such that the corresponding complementary sensitivity function $T_\varepsilon(s)$ satisfies $\|sT_\varepsilon(s)\|_\infty = \mathcal{O}(\sqrt{\varepsilon})$, and, therefore, we are able to achieve an arbitrarily large delay margin. Details are given in Appendix VIII-C. ∎

D. General Case

We are now in a position to combine the previous results to give a complete answer to whether or not $DM(P_0)$ is finite or not.

Theorem 14: $DM(P_0)$ is finite if and only if P_0 has a nonzero unstable pole.

Proof: First, observe that the zero controller yields an infinite delay margin for all stable plants, and Theorem 13 demonstrate that $DM(P_0) = \infty$ if the only unstable pole(s) is at zero. If there is a nonzero unstable pole, then Theorems 7, 9, and 10 demonstrate that the $DM(P_0)$ is finite by considering the three possible cases: the real case, the complex case in the open RHP, and the imaginary axis case. ∎

V. PLANTS WITH NMP ZEROS

Up to this point, we have considered only unstable plant poles in our discussions. If the only unstable poles are at the origin in the complex plane, then the previous results show that the achievable delay margin is arbitrarily large, regardless of the plant zeros. In addition, if there are nonzero unstable plant poles, then the achievable delay margin can be shown to be finite, without considering plant zeros. However, intuitively we know that systems with unstable poles near nonminimum phase zeros are more difficult to control. Here, we demonstrate that additional constraints arise if we have both a real unstable pole and a real nonminimum phase zero.

Theorem 15: Suppose that $P_0(s)$ has a real pole at $p > 0$ and a real zero at $z > 0$.
  i) If $p < z$, then $DM(P_0) \leq (2/p) - (2/z)$.
  ii) If $p > z$, then $DM(P_0) \leq \min\{(2/z) - (2/p), (2/p), 2/(3z)\}$.

Proof: The result follows by considering several different all-pass factorizations, and utilizing Proposition 5. Details are given in Appendix VIII-D. ∎

Remark 16: From Theorem 15, we see that as we approach a real unstable pole-zero cancellation, the achievable delay margin necessarily tends to zero.

Remark 17: If the plant has one real pole at $p > 0$, one real zero at $z > p$, and no other poles or zeros in $\mathbf{C}^+$, then the bound given in Theorem 11 i) is tight. To see this, write

$$P_0(s) = \left(\frac{s-z}{s-p}\right) W(s)$$

with $W(s)$ stable, minimum phase, and having relative degree $r \geq 1$. We would like to apply the controller

$$\left((1 - \varepsilon^2)^{1/2}s + (1 + \varepsilon^4)^{1/2}p\right) \times \left(\frac{-1}{s+z}\right) W(s)^{-1}$$

for then the loop gain is easy to analyse; since the controller is improper and, hence, inadmissible, we roll it off at high frequency. To this end, consider the FDLTI controller

$$C_\varepsilon(s) = \left((1 - \varepsilon^2)^{1/2}s + (1 + \varepsilon^4)^{1/2}p\right) \left(\frac{-1}{s+z}\right)$$
$$\times \frac{(1 + \varepsilon^4 p^2)^{r/2}}{(1 + \varepsilon s)^r} W(s)^{-1} \quad (17)$$

which lies in RH_∞ for $\varepsilon \in (0, 1)$. Then for $\varepsilon \in (0, 1)$, there are no unstable pole-zero cancellations and the loop gain is

$$P_0(s)C_\varepsilon(s) = \frac{(1 - \varepsilon^2)^{1/2}s + (1 + \varepsilon^4)^{1/2}p}{(s - p)} \times \frac{(z - s)}{(s + z)}$$
$$\times \frac{(1 + \varepsilon^4 p^2)^{r/2}}{(1 + \varepsilon s)^r}. \quad (18)$$

It is easy to verify that for $\varepsilon \in (0, 1)$ this loop transfer function has a magnitude that is a strictly decreasing function of ω on the range $[0, \infty)$, going from $(1 + \varepsilon^4)^{1/2}(1 + \varepsilon^4 p^2)^{r/2} > 1$ to zero, so there is a unique $\omega_c > 0$, for which

$$|P(j\omega_c)C_\varepsilon(j\omega_c)| = 1.$$

Indeed, it is easy to verify that $\omega_c = \varepsilon p$. This yields a corresponding phase margin of

$$\phi^* = 2\varepsilon - 2\varepsilon\frac{p}{z} + \mathcal{O}(\varepsilon^2).$$

Now observe that the Nyquist plot encircles -1 exactly one time for all sufficiently small $\varepsilon > 0$, so the closed loop system is stable. Using Remark 1, it follows that the delay margin provided by this controller is

$$DM(P_0, C_\varepsilon) = \frac{\phi^*}{w_c} = \frac{2\varepsilon - \varepsilon\frac{p}{z} + \mathcal{O}(\varepsilon^2)}{p\varepsilon} = \frac{2}{p} - \frac{2}{z} + \mathcal{O}(\varepsilon)$$

for small $\varepsilon > 0$. Hence, we can get a delay margin as close to $2((1/p) - (1/z))$ as desired by choosing ε appropriately.

TABLE I
COMPARISON OF FREQUENCY DOMAIN PERFORMANCE FOR THE EXAMPLE SYSTEM

Plant/ Controller	Gain Margin (dB)	ϕ Margin (degrees)	Gain Crossover (rad/sec)	Example Delay Margin	Maximum Delay Margin
Static State FB	-1.12	26.6	0.06	7.746	9.254
Min ϕ Output FB	-0.007	22.1	0.0216	17.88	18.51
Non-Min ϕ Output FB	-0.007	15.7	0.0216	12.70	13.33

VI. EXAMPLE

Consider the example system in [15, Section 5], with a state space realization of

$$\dot{x} = \underbrace{\begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}}_{=:A} x + \underbrace{\begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}}_{=:B} u(t-\tau). \quad (19)$$

The open loop poles are given by $\lambda(A) = \{0.1081, -0.1490 \pm j0.2015\}$. Reference [15] gives several procedures for using static full state feedback to perform eigenvalue assignment for this system. One of the methods proposed uses a static state feedback $u = +Kx$ with a gain of

$$K = [0.471 \quad 0.504 \quad 0.602]. \quad (20)$$

A. Minimum Phase Output Feedback

To employ the results of this paper, we start with a system in output feedback form, and take (for example) the output definition as the second state

$$y = [0 \quad -1 \quad 0]x =: Cx. \quad (21)$$

Combining (21) with (19) gives a nominal plant transfer function of

$$P_0(s) = \frac{0.2(s + 0.2311)(s + 0.02142)}{(s - 0.1081)(s^2 + 0.2981s + 0.06281)}. \quad (22)$$

Using the results of Theorem 7, we conclude that $DM(P_0) \leq (2/0.1081) = 18.51$; Remark 8 says that we can obtain a delay margin as close to this as desired using a controller of the form

$$C(s) = 5(1 + \varepsilon^4 p^2)^{1/2} \frac{((1 - \varepsilon^2)^{1/2}s + (1 + \varepsilon^4)^{1/2}0.1081)}{(\varepsilon s + 1)}$$
$$\times \frac{(s^2 + 0.2981s + 0.06281)}{(s + 0.2311)(s + 0.02142)} \quad (23)$$

with $\varepsilon > 0$ small.

B. Nonminimum Phase Output Feedback

Now suppose that the output is the third state

$$y = [0 \quad 0 \quad 1]x =: Cx. \quad (24)$$

Combining (24) with (19) gives a nominal plant transfer function of

$$P_0(s) = 0.1 \frac{(s - 0.3859)(s + 0.1659)}{(s - 0.1081)(s^2 + 0.2981s + 0.06281)} \quad (25)$$

with one unstable pole $p = 0.1081$ and one nonminimum phase zero $z = 0.3859$ which satisfy $0 < p < z$. In this case, using the results of Theorem 15, we conclude that $DM(P_0) \leq (2/p) - (2/z) = 13.33$; Remark 17 says that we can obtain a delay margin as close to this as desired using a controller of the form

$$C_\varepsilon(s) = -10(1 + \varepsilon^4 p^2)^{1/2} \frac{(s^2 + 0.2981s + 0.06281)}{(s + 0.3859)(s + 0.1659)}$$
$$\times \frac{((1 - \varepsilon^2)^{1/2}s + (1 + \varepsilon^4)^{1/2}0.1081)}{(\varepsilon s + 1)} \quad (26)$$

with $\varepsilon > 0$ small.

C. Comparison

In this section, we consider the three controllers in closed loop, namely the following.
1) **State FB**: Static full state feedback using the gain given in (20).
2) **Min ϕ**: Dynamic feedback from a minimum phase output (21), using the controller given in (23) with $\varepsilon = 0.2$.
3) **Nonmin ϕ**: Dynamic feedback from a nonminimum phase output (24), using the controller given in (26) with $\varepsilon = 0.2$.

We computed the gain margin, the phase margin, gain crossover frequency and delay margin achieved; in addition, using the results of [15] and Theorems 7 and 15 we can also compute the maximum achievable delay margin which we include for comparison. This information is displayed in Table I. Note that the controllers that achieve near optimal delay margin, in this case, suffer from very poor gain margin and related poor sensitivity properties. The static state feedback gain designed from the perspective of optimizing delay margin also suffers from these problems, though not to the same extent as the dynamic controllers exhibited in this example. Therefore, it appears that there may be a tradeoff between maximizing the delay margin and other sensitivity and robustness properties.

VII. SUMMARY AND CONCLUSIONS

Handling time delays in feedback control systems is a difficult problem of long-standing interest. A previously unsolved technical problem in this area is that of obtaining general bounds on

the maximum delay margin achievable for an FDLTI plant when using an LTI controller. Here, we adopt the frequency domain approach, and demonstrate that for a strictly proper real rational plant, there is a uniform upper bound on the delay that can be tolerated when using an LTI controller if and only if the plant has an unstable nonzero pole. Furthermore, we provide a quantitative bound on the so-called delay margin in terms of plant poles and zeros, and have proven that it is tight in several special cases. It has already been proven ([16]–[18]) that there are no constraints when we move to linear time-varying controllers.

An open research problem is that of computing the exact (tight) bound on the maximum delay margin achievable using an LTI controller, or at least computing it in more special cases than was carried out herein. For example, if a plant P_0 has two positive real unstable poles at p_1 and p_2, the results of Theorem 7 show that $DM(P_0) \leq min\{2/p_1, 2/p_2\}$. We conjecture that this bound is not tight in general, but at this stage, we have not been able to either prove or disprove this conjecture. It is also of interest to examine the tradeoffs between the delay margin, gain margin, and other measures of robustness. The example given illustrates this potential trade-off, as well as the superior delay margin achievable via dynamic output feedback as compared to static state feedback.

APPENDIX

A. Proof of Proposition 5

First, we write $P_\alpha(s)$ as the ratio of two polynomials which have no common zeros in $\bar{\mathbf{C}}^-$ for small $\alpha \geq 0$.

i) If 1 i) holds, write $P_0(s) = g(\bar{n}_0(s)/\bar{d}_0(s))$ with $\bar{n}_0(s)$ and $\bar{d}_0(s)$ monic and coprime, and define

$$ (n_\alpha(s), d_\alpha(s)) := (\bar{n}_0(s), \bar{d}_0(s)) . $$

ii) If 1 ii) holds, write $P_0(s) = g\bar{n}_0(s)/((s-p)\bar{d}_0(s))$ with $\bar{n}_0(s)$ and $(s-p)\bar{d}_0(s)$ monic and coprime and set

$$ n_\alpha(s) := (s + p + \alpha)\bar{n}_0(s) $$
$$ d_\alpha(s) := (s + p)(s - p - \alpha)\bar{d}_0(s). $$

iii) If 1 iii) holds, write $P_0(s) = g((s-z)\bar{n}_0(s)/\bar{d}_0(s))$ with $(s-z)\bar{n}_0(s)$ and $\bar{d}_0(s)$ monic and coprime and set

$$ n_\alpha(s) := (s + z)(s - z - \alpha)\bar{n}_0(s) $$
$$ d_\alpha(s) := (s + z + \alpha)\bar{d}_0(s). $$

It is routine to verify that

$$ P_\alpha(s) = g\frac{n_\alpha(s)}{d_\alpha(s)}\frac{1 - c_\alpha s}{1 + c_\alpha s} $$

and that $n_\alpha(s)$ and $d_\alpha(s)$ have no common zeros in $\bar{\mathbf{C}}^+$ for small $\alpha \geq 0$. Using Hypothesis 2), it is easy to confirm that $n_\alpha(s)(1 - c_\alpha s)$ and $d_\alpha(s)(1 + c_\alpha s)$ also have no common zeros in $\bar{\mathbf{C}}^+$ for small $\alpha \geq 0$; by Hypothesis 3), there exists an $\alpha > 0$ for which this property is lost, so using the continuity of c_α we conclude that these exists a smallest such α, which we label $\hat{\alpha}$. Hence, $n_\alpha(s)(1 - c_\alpha s)$ and $d_\alpha(s)(1 + c_\alpha s)$ have no common

zeros in $\bar{\mathbf{C}}^+$ for $\alpha \in [0, \hat{\alpha})$, so with p the degree of $d_0(s)$, it follows that

$$ N_\alpha(s) = \frac{gn_\alpha(s)(1 - c_\alpha s)}{(s + 1)^p(1 + c_\alpha s)}, \quad D_\alpha(s) = \frac{d_\alpha(s)}{(s + 1)^p} $$

is a stable coprime factorization of $P_\alpha(s)$ for $\alpha \in [0, \hat{\alpha})$; since the set of all stabilizing controllers is independent of the coprime factorization, we may as well assume that this procedure was used to construct D_0 and N_0.

Proof of (a): Since C stabilizes P_0, it follows that there exists a $Q \in H_\infty$ so that

$$ C = \frac{X_0 + D_0Q}{Y_0 - N_0Q}. $$

A stable coprime factorization of the controller is N_c/D_c with

$$ N_c = X_0 + D_0Q \in H_\infty, \quad D_c = Y_0 - N_0Q \in H_\infty. $$

It follows from Lemma 3 that $C(s)$ stabilizes $P_\alpha(s)$ at $\alpha \in [0, \hat{\alpha})$ iff

$$ (D_cD_\alpha + N_cN_\alpha)^{-1} \in H_\infty. $$

Since $C(s)$ stabilizes P_0, it follows that $(D_cD_0 + N_cN_0)^{-1} \in H_\infty$; indeed, it is easy to check that

$$ D_cD_0 + N_cN_0 = 1. $$

Now define

$$ \tilde{N}_\alpha(s) := N_\alpha(s) - N_0(s) $$
$$ = g\frac{n_\alpha(s) - n_0(s)}{(s + 1)^p} - 2gc_\alpha\frac{sn_\alpha(s)}{(s + 1)^p} \times \frac{1}{1 + c_\alpha s} $$

so

$$ \left\|\tilde{N}_\alpha(s)\right\|_\infty \leq \left\|g\frac{n_\alpha(s) - n_0(s)}{(s + 1)^p}\right\|_\infty + c_\alpha\left\|\frac{2gsn_\alpha(s)}{(s + 1)^p(1 + c_\alpha s)}\right\|_\infty. $$

Similarly

$$ \tilde{D}_\alpha(s) := D_\alpha(s) - D_0(s) = \frac{d_\alpha(s) - d_0(s)}{(s + 1)^p} $$

which means that

$$ \left\|\tilde{D}_\alpha(s)\right\|_\infty \leq \left\|\frac{d_\alpha(s) - d_0(s)}{(s + 1)^p}\right\|_\infty. $$

Now observe that

$$ D_cD_\alpha + N_cN_\alpha = D_c[D_0 + (D_\alpha - D_0)] $$
$$ + N_c[N_0 + (N_\alpha - N_0)] $$
$$ = (D_cD_0 + N_cN_0) + (D_c\tilde{D}_\alpha + N_c\tilde{N}_\alpha) $$
$$ = 1 + (D_c\tilde{D}_\alpha + N_c\tilde{N}_\alpha). \qquad (27) $$

Our goal is to prove that this is stably invertible for small $\alpha > 0$. Using the facts that the coefficients of the monic polynomials

$n_\alpha(s)$ and $d_\alpha(s)$ are continuous functions of $\alpha > 0$ and that $\lim_{\alpha \to 0} c_\alpha = 0$, it follows that

$$\lim_{\alpha \to 0} \|\tilde{N}_\alpha\|_\infty = 0 \quad \text{and} \quad \lim_{\alpha \to 0} \|\tilde{D}_\alpha\|_\infty = 0.$$

Hence, there exists an $\bar{\alpha} \in (0, \hat{\alpha})$ so that for all $\alpha \in (0, \bar{\alpha})$

$$\|D_c \tilde{D}_\alpha + N_c \tilde{N}_\alpha\|_\infty \le \frac{1}{2}. \tag{28}$$

Using the Small Gain Theorem together with (28), it follows immediately that

$$\left\| \left[1 + (D_c \tilde{D}_\alpha + N_c \tilde{N}_\alpha) \right]^{-1} \right\|_\infty$$
$$\le \left(1 - \left\| (D_c \tilde{D}_\alpha + N_c \tilde{N}_\alpha) \right\|_\infty \right)^{-1}$$
$$\le 2. \tag{29}$$

Combining this with (27), we have

$$\left\| (D_c D_\alpha + N_c N_\alpha)^{-1} \right\|_\infty \le 2 < \infty, \quad \alpha \in (0, \bar{\alpha})$$

so closed loop stability is maintained for this range of α.

Proof of Part (b): The proof of the real rational controller case is straight-forward: it is based on the continuity of the zeros of the characteristic polynomial as a function of the free parameter α. Briefly, we know that it has all of its zeros in $\mathbf{C}^-$ for $\alpha \in [0, \bar{\alpha})$ and has a zero in $\mathbf{C}^+$ for $\alpha = \hat{\alpha}$. By continuity, there must exist an $\alpha^* \in [\bar{\alpha}, \hat{\alpha}]$ so the characteristic polynomial has a zero on the imaginary axis, say at $s = j\omega^*$, which means that

$$B_{\alpha^*}(j\omega^*) P_0(j\omega^*) C(j\omega^*) = -1.$$

The proof of the infinite-dimensional case, which we present here, requires a more involved proof, and utilizes Runge's Theorem.

To proceed we first examine the coprime factors N_α and D_α. Adopting the definition of $\tilde{N}_\alpha$ and $\tilde{D}_\alpha$ given above, we have

$$D_c D_\alpha + N_c N_\alpha = D_c D_0 + N_c N_0 + D_c \tilde{D}_\alpha + N_c \tilde{N}_\alpha$$
$$= 1 + D_c \tilde{D}_\alpha + N_c \tilde{N}_\alpha \tag{30}$$

but $\tilde{N}_\alpha$ and $\tilde{D}_\alpha$ are strictly proper transfer functions, and we can easily prove that they tend to zero uniformly (over $\alpha \in [0, \hat{\alpha}]$) as $|s| \to \infty$; in particular, there exists $\rho > 0$ so that

$$\left\| \begin{bmatrix} \tilde{D}_\alpha(s) & \tilde{N}_\alpha(s) \end{bmatrix} \right\| \times \left\| \begin{pmatrix} D_c \\ N_c \end{pmatrix} \right\|_\infty \le \frac{1}{2}$$
$$|s| \ge \rho, \alpha \in [0, \hat{\alpha}]. \tag{31}$$

Using (30), this means that

$$\inf_{|s| \ge \rho, s \in \mathbf{C}^+} \inf_{\alpha \in [0, \hat{\alpha}]} |D_c(s) D_\alpha(s) + N_c(s) N_\alpha(s)| \ge \frac{1}{2}. \tag{32}$$

Now define the compact rectangles in $\mathbf{C}^+$ by

$$S_i = \left\{ s = a + jb : a \in \left[\frac{1}{i}, \rho \right], b \in [-\rho, \rho] \right\}$$

and

$$S_\infty = \{ s = a + jb : a \in [0, \rho], b \in [-\rho, \rho] \}.$$

Notice that Q (and, hence, N_c and D_c) are analytic on $S_i, i \in \mathbf{N}$, but may be discontinuous on the left edge of S_∞, which lies on the imaginary axis.

Claim: For every $i \ge \bar{i}$, there exists an $\alpha_i \in [\bar{\alpha}, \hat{\alpha}]$ and $s_i \in \mathbf{C}^+$ satsfying

$$\text{Re}(s_i) = \frac{1}{i}, \quad \text{Im}(s_i) \in [-\rho, \rho]$$

and

$$[D_c D_{\alpha_i} + N_c N_{\alpha_i}](s_i) = 0.$$

Proof of Claim: Let $i > \bar{i}$. For every $\varepsilon > 0$, by Runge's Theorem (e.g., see [4, p. 198]) there exists an element $Q_\varepsilon \in RH_\infty{}^3$ so that

$$\sup_{s \in S_i} |(Q - Q_\varepsilon)(s)| < \varepsilon.$$

This gives rise to the corresponding real rational controller C_ε, which clearly stabilizes P_0. Now write $C_\varepsilon(s) = (a(s)/b(s))$ with a and b coprime and with b monic. The characteristic polynomial of the closed-loop system arising from C_ε applied to P_α is given by

$$\Delta_\alpha(s) := b(s) d_\alpha(s) (1 + c_\alpha s) + g a(s) n_\alpha(s) (1 - c_\alpha s).$$

For $\alpha \ne 0$, we can make this monic by an appropriate scaling

$$\bar{\Delta}_\alpha(s) = \begin{cases} \Delta_\alpha(s), & \text{if } c_\alpha = 0 \text{ for all } \alpha \ge 0 \\ \frac{1}{c_\alpha} \Delta_\alpha(s), & \text{otherwise.} \end{cases}$$

Notice that the monic property follows from the fact that $C_\varepsilon(s)$ is proper, $n_\alpha(s)/d_\alpha(s)$ is strictly proper, and $b(s)$ and $d_\alpha(s)$ are monic. Since $g n_\alpha(s)(1 - c_{\hat{\alpha}} s)$ and $d_{\hat{\alpha}}(s)(1 + c_{\hat{\alpha}} s)$ has a common zero at $s = \hat{s} \in \mathbf{C}^+$, it follows that

$$\bar{\Delta}_{\hat{\alpha}}(\hat{s}) = 0$$

as well. From part i), we know that the zeros of $\bar{\Delta}_\alpha$ are in $\mathbf{C}^-$ for small $\alpha > 0$. Since the zeros of a monic polynomial are continuous functions of its coefficients, it follows from the continuity of c_α and the parameters of the monic polynomials $n_\alpha(s)$ and $d_\alpha(s)$ that there must exist an $\alpha_\varepsilon \in (0, \hat{\alpha})$ for which $\bar{\Delta}_{\hat{\alpha}}(s)$ has a zero in ∂S_i,[4] say at $s = s_\varepsilon$. Hence, it must be that

$$[(X_0 + D_0 Q_\varepsilon) N_{\alpha_\varepsilon} + (Y_0 - N_0 Q_\varepsilon) D_{\alpha_\varepsilon}](s_\varepsilon) = 0. \tag{33}$$

Now let $\{\varepsilon_j\}$ denote any positive sequence converging to zero; Q_{ε_j}, s_{ε_j} and α_{ε_j} are defined accordingly, using the above procedure. Then $(s_{\varepsilon_j}, \alpha_{\varepsilon_j}) \subset \partial S_i \times [0, \hat{\alpha}]$, and since $S_i \times [0, \hat{\alpha}]$ is

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[3]Runge's Theorem guarantees the existence of a (possibly improper) rational approximation with all poles in $\mathbf{C}^-$; it is straightforward to roll it off at high frequency to generate an arbitrarily good approximation which lies in RH_∞.

[4]We use the standard notation of ∂S_i to denote the boundary of S_i.

compact, there is a convergent subsequence; for notational simplicity, we may as well assume that $(s_{\varepsilon_j}, \alpha_{\varepsilon_j})$ enjoys this property with a limit of (s_i, α_i). Since $Q_{\varepsilon_j} \to Q$, $N_{\alpha_{\varepsilon_j}} \to N_{\alpha_i}$, and $D_{\alpha_{\varepsilon_j}} \to D_{\alpha_i}$ uniformly on S_i as $j \to \infty$, by continuity of all terms on S_i it follows from (33) that

$$[(X_0 + D_0 Q)N_{\alpha_i} + (Y_0 - N_0 Q)D_{\alpha_i}](s_i) = 0. \tag{34}$$

If s_i is not on the left-hand face of ∂S_i, then we will have $|s_i| = \rho$, so from (32)

$$|[(X_0 + D_0 Q)N_{\alpha_i} + (Y_0 - N_0 Q)D_{\alpha_i}](s_i)| \geq 1/2$$

which violates (34). Last of all, since C stabilize P_α for all $\alpha \in [0, \bar{\alpha})$, it must be that $\alpha_i > \bar{\alpha}$. QED (End of proof of Claim)

At this point, we have $\alpha_i \in [\bar{\alpha}, \hat{\alpha}]$ and $s_i \in \mathbf{C}^+$ satisfying

$$\mathrm{Re}(s_i) = \frac{1}{i}, \mathrm{Im}(s_i) \in [-\rho, \rho], \text{ and } [D_c D_{\alpha_i} + N_c N_{\alpha_i}](s_i) = 0.$$

Now the sequence $\{(\alpha_i, s_i)\}$ is restricted to the compact set $[\bar{\alpha}, \hat{\alpha}] \times S_\infty$, so it has a convergent subsequence; for notational simplicity we may as well assume that (α_i, s_i) enjoys this property, with a limit of (α^*, s^*). Since $\mathrm{Re}(s_i) = (1/i)$, clearly there exists a $\omega^* \in \mathbf{R}$ so that $s^* = j\omega^*$. However

$$[D_c D_{\alpha^*} + N_c N_{\alpha^*}](s_i) = [D_c D_{\alpha_i} + N_c N_{\alpha_i}](s_i)$$
$$+ [D_c (D_{\alpha^*} - D_{\alpha_i})](s_i) + [N_c (N_{\alpha^*} - N_{\alpha_i})](s_i)$$

and it follows from continuity that

$$\lim_{i \to \infty} \|D_{\alpha^*} - D_{\alpha_i}\|_\infty = \lim_{i \to \infty} \|N_{\alpha^*} - N_{\alpha_i}\|_\infty = 0.$$

Since $D_c, N_c \in H_\infty$, it follows that

$$\lim_{i \to \infty} [D_c D_{\alpha^*} + N_c N_{\alpha^*}](s_i) = 0. \tag{35}$$

Hence, $[D_c D_{\alpha^*} + N_c N_{\alpha^*}]^{-1} \notin H_\infty$; by Lemma 3, it follows that we do not have closed loop stability.

We would now like to infer from this that

$$\lim_{i \to \infty} [1 + B_{\alpha^*}(s_i)P_0(s_i)C(s_i)] = 0. \tag{36}$$

Since all terms in square brackets of (35) are bounded in $\bar{\mathbf{C}}^+$, if

$$\liminf_{i \to \infty} |D_c(s_i)D_{\alpha^*}(s_i)| > 0 \tag{37}$$

then (36) follows immediately by taking the $D_c(s_i)D_{\alpha^*}(s_i)$ outside the square brackets. So suppose that this is not the case; then there exists a strictly increasing subsequence of integers $i_j \to \infty$ satisfying

$$\lim_{j \to \infty} [D_c(s_{i_j})D_{\alpha^*}(s_{i_j})] = 0$$

and it follows from (35) that

$$\lim_{j \to \infty} [N_c(si_j)N_{\alpha^*}(si_j)] = 0$$

as well. However, it follows from the structure of N_α and D_α that $\lim_{j \to \infty}(N_0(s_{i_j})/N_{\alpha^*}(s_{i_j}))$ and $\lim_{j \to \infty}(D_0(s_{i_j})/D_{\alpha^*}(s_{i_j}))$ exist and are finite, so

$$\lim_{j \to \infty} [D_c(s_{i_j}) D_0(s_{i_j}) + N_c(si_j)N_0(si_j)] = 0$$

which means that C does not stabilize P_0, which is a contradiction. We conclude that (36) and (37) hold. Since $B_{\alpha^*}(j\omega^*) = \lim_{i \to \infty} B_{\alpha^*}(s_i)$ has magnitude one, it follows immediately that

$$\lim_{i \to \infty} [1 + B_{\alpha^*}(j\omega^*)P_0(s_i)C(s_i)] = 0$$

as desired. Finally, if $C(s)$ is real rational, then it follows from (35) that

$$[D_c(j\omega^*)D_{\alpha^*}(j\omega^*) + N_c(j\omega^*)N_{\alpha^*}(j\omega^*)] = 0.$$

From (37), we have $D_c(j\omega^*)D_{\alpha^*}(j\omega^*) \neq 0$, so

$$1 + P_{\alpha^*}(j\omega^*)C(j\omega^*) = 1 + B_{\alpha^*}(j\omega^*)P_0(j\omega^*)C(j\omega^*) = 0$$

as desired.   QED

B. Proof of Corollary 11

Note first that, as in Remarks 8 and 17, $W(s)$ being stable and minimum phase does not pose any additional demands on the achievable delay margin and can, therefore, be ignored, i.e., we take $W(s) = 1$. Also, for simplicity and brevity, we construct a compensator that is rational, though not necessarily proper; as in Remarks 8 and 17, we can roll the controller off to achieve a proper one which provides essentially the same delay margin.

We first define some important transfer functions for the proposed family of controllers. For any $\alpha \in (0, 1/\sqrt{5})$, define $\omega_a = \sqrt{1 - 2\alpha^2}\omega_0$ and the phase lead compensator

$$H_\alpha(s) = \sqrt{1 - \alpha^2} \left(\frac{2\alpha s^2}{s^2 + 2\alpha\omega_a s + \omega_a^2} \right). \tag{38}$$

Then for any $\varepsilon > 0$, we design the (improper) controller, consisting of 3 series lead compensators, and a lightly damped resonant zero

$$C(s) = \left(s^2 + 2\varepsilon\omega_0 s + \omega_0^2\right) H_\alpha^3(s). \tag{39}$$

The resulting loop transfer function for the controller of (39) with the plant (15) is

$$L_{\varepsilon,\alpha}(s) = G_\varepsilon(s)H_\alpha^3(s) \tag{40}$$

where $G_\varepsilon(s) := ((s^2 + 2\varepsilon\omega_0 s + \omega_0^2)/(s^2 + \omega_0^2))$. The proof now follows as a series of claims relating to the delay margin associated with the loop transfer function.

1) $|G_\varepsilon(j\omega)|$ and $|H_\alpha(j\omega)|$ are both:
   a) strictly increasing for $\omega \in (0, \omega_0)$;
   b) strictly decreasing for $\omega > \omega_0$.

Note that, in the case of H_α, this claim follows by noting that

$$|H_\alpha(j\omega)|^2 = \frac{1}{\left(1 + \frac{(1-2\alpha^2)^2}{(1-\alpha^2)4\alpha^2}\left(1 - \omega_0^2/\omega^2\right)^2\right)}. \quad (41)$$

2) From (41), it also follows that $|H_\alpha(j\omega)| \le 1$ with equality if and only if $\omega = \pm\omega_0$.
3) $|G_\varepsilon(j\omega)| \ge 1$ with equality if and only if $\omega = 0$.
4) Based on parts 1, 2, and 3, there are exactly two (positive) gain crossover frequencies: $\omega_c^- \in (0, \omega_0)$ and $\omega_c^+ \in (\omega_0, \infty)$.
5) For any fixed α, the gain cross over frequencies can be made arbitrarily close to ω_0 by taking ε small, that is $\lim_{\varepsilon \to 0^+} \omega_c^- = \omega_0 = \lim_{\varepsilon \to 0^+} \omega_c^+$.
6) We now turn to consider the phase margin achieved. First, note that since $G_\varepsilon(j\omega) = 1 + j2\varepsilon\omega/(\omega_0^2 - \omega^2)$, then

$$\phi_c^- := \angle\left(G_\varepsilon\left(j\omega_c^-\right)\right) = \arctan\sqrt{\left|G_\varepsilon\left(j\omega_c^-\right)\right|^2 - 1}$$
$$\phi_c^+ := \angle\left(G_\varepsilon\left(j\omega_c^+\right)\right)$$
$$= -\arctan\sqrt{\left|G_\varepsilon\left(j\omega_c^+\right)\right|^2 - 1}. \quad (42)$$

7) Since at crossover, the loop gain equals unity, and also using parts 2 and 5 above, then as $\varepsilon \to 0^+$, we have $|G_\varepsilon(j\omega_c^\pm)| \to 1$. This fact, together with part 6, gives that for any fixed α, $\lim_{\varepsilon \to 0^+} \phi_c^\pm = 0$.
8) Note that, as $\alpha \to (\sqrt{5})^-$, we have $\angle(H_\alpha(j\omega)) \to (60°)^-$. We can also show (by a Nyquist diagram type argument) that for any $\alpha \in (0, \sqrt{5})$, for ε sufficiently small, we have nominal closed loop stability. In view of the above arguments, we have nominal closed loop stability, cut-off frequencies that approach ω_0 and phase margins that approach $360°$ or 2π(rad) and, therefore, the supremal delay margin is $2\pi/\omega_0$.

C. Proof of Theorem 13

To prove this result, we make use of Proposition 2. Suppose that $P_0(s)$ has m poles at zero but no other unstable poles. Adopt a coprime factorization of the plant $P_0(s) = N_0(s)/D_0(s)$ where $D_0(s) = s^m/d(s)$ and $d(s)$ is a Hurwitz polynomial of degree m. We then sove the corresponding Bezout identity $N_0(s)X_0(s) + D_0(s)Y_0(s) = 1$ for $X_0, Y_0 \in RH_\infty$. The class of stabilizing controllers in $\mathcal{F}(H_\infty)$ can then be expressed as

$$\left\{\frac{X_0(s) + D_0(s)Q(s)}{Y_0(s) - N_0(s)Q(s)} : Q \in H_\infty\right\}$$

which results in a closed loop complementary sensitivity function of $T(s) = N_0(s)[X_0(s) + D_0(s)Q(s)]$. We now define

$$Q_\varepsilon(s) := -\frac{X_0(s)}{D_0(s)}\frac{s^m}{(s+\varepsilon)^m}$$
$$= -X_0(s)\frac{d(s)}{(s+\varepsilon)^m} \in RH_\infty \quad (43)$$

with T_ε representing the corresponding complementary sensitivity function. Then

$$sT_\varepsilon(s) = N_0(s)X_0(s)s\left(1 - \frac{s^m}{(s+\varepsilon)^m}\right). \quad (44)$$

It follows immediately that for $|\omega| \le \sqrt{\varepsilon}$

$$\|j\omega T_\varepsilon(j\omega)\| \le 2\sqrt{\varepsilon}\|N_0(s)X_0(s)\|_\infty. \quad (45)$$

We now consider frequencies $|\omega| \ge \sqrt{\varepsilon}$. Note that since the plant is strictly proper, $N_0(s)$ is strictly proper and so $sN_0(s)X_0(s)$ is a proper stable rational transfer function. Also, note that

$$\left\|1 - \left(\frac{j\omega}{\varepsilon + j\omega}\right)^m\right\| = \left\|\left[1 - \left(\frac{j\omega}{\varepsilon + j\omega}\right)\right]\sum_{\ell=0}^{m-1}\left(\frac{j\omega}{\varepsilon + j\omega}\right)^\ell\right\|$$
$$\le m\left\|1 - \left(\frac{j\omega}{\varepsilon + j\omega}\right)\right\|$$
$$= \frac{m\varepsilon}{\sqrt{\varepsilon^2 + \omega^2}}.$$

Therefore, we can show that for $|\omega| \ge \sqrt{\varepsilon}$

$$\|j\omega T_\varepsilon(j\omega)\| \le m\sqrt{\varepsilon}\|sN_0(s)X_0(s)\|_\infty. \quad (46)$$

Using (45) and (46), we see that by choosing ε sufficiently small, we can make $\|sT(s)\|_\infty$ arbitrarily close to zero, and the result then follows.

D. Proof of Theorem 15

Before proceeding with the proof of Theorem 15, we recall some properties of the inverse tan function.
Lemma 18: Suppose that $a, b > 0$. Then:
  i) $|\tan^{-1}(a) - \tan^{-1}(b)| \le |a - b|$;
  ii) $\tan^{-1}(a + b) < \tan^{-1}(a) + \tan^{-1}b$.
Suppose that $C \in \mathcal{F}(H_\infty)$ stabilizes $P_0(s)$.
i) In this case, define the all-pass transfer function

$$B_\alpha(s) := \left(\frac{p-s}{p+s}\right) \times \left(\frac{p+\alpha+s}{p+\alpha-s}\right), \quad \alpha \ge 0.$$

Observe that B_α has the form required in Proposition 5; since there is a pole-zero cancellation between $B_\alpha(s)$ and $P_0(s)$ at $\alpha = z - p$, it follows that there exists $\alpha^* \in (0, z - p), \omega^* \in \mathbf{R}$, and $\{s_i\} \subset \mathbf{C}^+$ converging to $j\omega^*$ so that

$$\lim_{i \to \infty} B_{\alpha^*}(j\omega^*)P_0(s_i)C(s_i) = -1.$$

Note that, clearly $\omega^* \ne 0$, for otherwise $C(s)$ would not stabilize $P_0(s)$.
We would now like to find, if possible, a $T > 0$ for which the delay element e^{-sT} evaluated at $s = j\omega^*$ exactly equals $B_{\alpha^*}(j\omega^*)$, for then

$$\lim_{i \to \infty}\left[e^{-s_iT}P_0(s_i)C(s_i)\right] = -1$$

which would mean that $C(s)$ does not stabilize $e^{-sT}P_0(s)$. To proceed, observe that both $B_{\alpha^*}(s)$ and e^{-sT} have magnitude one on the imaginary axis, so we need to match their phase

$$-\omega^*T = -2\tan^{-1}\left(\frac{\omega^*}{p}\right) + 2\tan^{-1}\left(\frac{\omega^*}{p+\alpha^*}\right)$$

which has a positive solution, which we label T^*

$$T^* = \frac{2}{|\omega^*|}\left[\tan^{-1}\left(\frac{|\omega^*|}{p}\right) - \tan^{-1}\left(\frac{|\omega^*|}{p+\alpha^*}\right)\right].$$

Using Lemma 18 i), we have

$$T^* \leq \frac{2}{|\omega^*|}\left|\frac{|\omega^*|}{p} - \frac{|\omega^*|}{z}\right| = 2\left[\frac{1}{p} - \frac{1}{z}\right].$$

This means that C cannot stabilize $e^{-sT^*}P_0(s)$, so

$$DM(P_0, C) < T^* < 2\left[\frac{1}{p} - \frac{1}{z}\right].$$

Since this is true for every stabilizing controller in $\mathcal{F}(H_\infty)$

$$DM(P_0) \leq 2\left[\frac{1}{p} - \frac{1}{z}\right].$$

ii) The second case is a little more complicated. We choose an all-pass function of the form

$$B_\alpha(s) := \left(\frac{s+z}{s-z}\right)\left(\frac{s-z-\alpha}{s+z+\alpha}\right)\frac{(z(z+\alpha)-\alpha s)}{(z(z+\alpha)+\alpha s)}$$
$$= \left(\frac{z+s}{z-s}\right)\left(\frac{z+\alpha-s}{z+\alpha+s}\right)\frac{(z(z+\alpha)-\alpha s)}{(z(z+\alpha)+\alpha s)}$$

with $\alpha \geq 0$. Observe that B_α has the form required in Proposition 1 with $c_\alpha := \alpha/z(z+\alpha)$. Now observe that there is an unstable pole-zero cancellation between $B_\alpha(s)$ and $P_0(s)$ both when

$$z + \alpha = p \quad \leftrightarrow \quad \alpha = p - z$$

and also when

$$c_\alpha = \frac{1}{p} \quad \leftrightarrow \quad \frac{\alpha}{z(z+\alpha)} = \frac{1}{p} \quad \leftrightarrow \quad \alpha = \frac{z^2}{p-z}.$$

Hence, define

$$\bar{\alpha} := \min\left\{p - z, \frac{z^2}{p-z}\right\}.$$

It follows from Proposition 1 that there exists an $\alpha^* \in (0, \bar{\alpha})$, $\omega^* \in \mathbf{R}$ and $\{s_i\} \subset \mathbf{C}^+$ converging to $j\omega^*$ so that

$$\lim_{i\to\infty} B_{\alpha^*}(j\omega^*)P_0(s_i)C(s_i) = -1.$$

Note that clearly $\omega^* \neq 0$, for otherwise, $C(s)$ would not stabilize $P_0(s)$.

We would now like to find, if possible, a $T > 0$ for which the delay element e^{-sT} evaluated at $s = j\omega^*$ exactly equals $B_{\alpha^*}(j\omega^*)$, for then

$$\lim_{i\to\infty}\left[e^{-s_iT}P_0(s_i)C(s_i)\right] = -1$$

which would mean that $C(s)$ does not stabilize $e^{-sT}P_0(s)$. To proceed, observe that both $B_{\alpha^*}(s)$ and e^{-sT} have magnitude one on the imaginary axis, so we need to match their phase

$$\omega^*T = 2\tan^{-1}\left(\frac{\alpha^*(\omega^*)^3}{z^2(z+\alpha^*)^2 + (\omega^*)^2(z^2+z\alpha^*+(\alpha^*)^2)}\right)$$

which clearly has a positive solution for T, which we label T^*, namely

$$T^* = \frac{2}{|\omega^*|}\tan^{-1}\left(\frac{\alpha^*|\omega^*|^3}{z^2(z+\alpha^*)^2 + \omega^{*2}(z^2+z\alpha^*+\alpha^{*2})}\right)$$
$$\leq \frac{2\alpha^*(\omega^*)^2}{z^2(z+\alpha^*)^2 + (\omega^*)^2(z^2+z\alpha^*+\alpha^{*2})}$$
$$\leq \frac{2\alpha^*}{z^2+z\alpha^*+\alpha^{*2}} =: h(\alpha^*).$$

It follows that

$$T^* \leq \sup_{\alpha^*\in(0,\bar{\alpha})} h(\alpha^*).$$

Clearly, $h(0) = 0$ and $\lim_{\alpha^*\to\infty} h(\alpha^*) = 0$, and it is easy to check that h achieves its maximum of $2/(3z)$ at $\alpha^* = z$ and is strictly increasing on $(0, z)$. We now consider the two different possibilities for the value of $\bar{\alpha}$.

Case 1: $\bar{\alpha} = p-z$: In this case, we have $p-z \leq (z^2/(p-z))$, which is equivalent to $p - z \leq z$, so h is strictly increasing on the interval $(0, p-z)$, which means that

$$\sup_{\alpha^*\in(0,\bar{\alpha})} h(\alpha^*) = \sup_{\alpha^*\in(0,p-z)} h(\alpha^*) = h(p-z)$$
$$= 2\frac{(p-z)}{p^2+z^2-pz} \leq 2\frac{(p-z)}{pz} = \frac{2}{z} - \frac{2}{p}.$$

Case 2: $\bar{\alpha} = (z^2/(p-z))$: In this case, we have $(z^2/(p-z)) \leq p-z$ which is equivalent to $p \geq 2z$. Hence, $(z^2/(p-z)) \leq (z^2/(2z-z)) = z$, so h is strictly increasing on the interval $(0, z^2/(p-z))$, which means that

$$\sup_{\alpha^*\in(0,\bar{\alpha})} h(\alpha^*) = \sup_{\alpha^*\in\left(0,\frac{z^2}{p-z}\right)} h(\alpha^*)$$
$$= h\left(\frac{z^2}{p-z}\right) = 2\frac{(p-z)}{p^2+z^2-pz}$$
$$\leq 2\frac{(p-z)}{pz} = \frac{2}{z} - \frac{2}{p}.$$

If we combine Case 1 and Case 2, we see that

$$T^* \leq \frac{2}{z} - \frac{2}{p}$$

but we also know that

$$T^* \leq \sup_{\alpha^*>0} h(\alpha^*) = h(z) = \frac{2}{3z}$$

so

$$T^* \leq \min\left\{\frac{2}{z} - \frac{2}{p}, \frac{2}{3z}\right\}.$$

This means that C cannot stabilize $e^{-sT^*}P_0(s)$, so

$$DM(P_0, C) < T^*.$$

Since this is true for every stabilizing controller in $\mathcal{F}(H_\infty)$, if we combine this with Theorem 7, we have

$$DM(P_0) \leq \min\left\{\frac{2}{z} - \frac{2}{p}, \frac{2}{p}, \frac{2}{3z}\right\}.$$

REFERENCES

[1] E. Fridman and U. Shaked, Eds., "Special Issue on time-delay systems," *Int. J. Robust Nonlin. Control*, vol. 13, 2003.

[2] J. F. Barman, F. M. Callier, and C. A. Desoer, " L^2-stability and L^2-instability of linear time-invariant distributed feedback systems perturbed by a small delay in the loop," *IEEE Trans. Autom. Control*, vol. AC-18, no. 10, pp. 479–484, Oct. 1973.

[3] F. M. Callier and C. A. Desoer, "An algebra of transfer functions for distributed linear time-invariant systems," *IEEE Trans. Circuits Syst.*, vol. CAS-25, no. 9, pp. 651–662, Sep. 1978.

[4] J. B. Conway, *Functions of One Complex Variable*, 2nd ed. New York: Springer-Verlag, 1978.

[5] D. E. Davison and D. E. Miller, "Determining the least upper bound on the achievable delay margin," in *Open Problems in Mathematical Systems and Control Theory*, V. D. Blondel and A. Megretski, Eds. Princeton , NJ: Princeton Univ. Press, 2004, pp. 276–279.

[6] R. Devanathan, "A lower bound for limiting time delay for closed-loop stability of an arbitrary SISO plant," *IEEE Trans. Autom. Control*, vol. 40, no. 4, pp. 717–721, Apr. 1995.

[7] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. New York: Macmillan, 1992.

[8] L. Dugard and E. Verriest, Eds., *Stability and Control of Time-Delay Systems*. New York: Springer-Verlag, 1997.

[9] B. A. Francis and T. T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers," *IEEE Trans. Autom. Control*, vol. AC-33, no. 9, pp. 820–832, Sep. 1988.

[10] K. Gu and S. I. Niculescu, "Survey on recent results in stability and control of time-delay systems," *Trans. ASME*, vol. 125, pp. 158–165, Jun. 2003.

[11] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Basel, Switzerland: Birkhauser Verlag AG, 2003.

[12] P. P. Khargonekar, K. Poolla, and A. Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation," *IEEE Trans. Autom. Control*, vol. AC-30, no. 11, pp. 1088–1096, Nov. 1985.

[13] P. P. Khargonekar and A. Tannenbaum, "Non-Euclidean metrics and the robust stabilization of systems with parameter uncertainty," *IEEE Trans. Autom. Control*, vol. AC-30, no. 10, pp. 1005–1013, Oct. 1985.

[14] M. S. Mahmoud, *Robust Control and Filtering for Time-Delay Systems*. New York: Marcel Dekker, 2000.

[15] W. Michiels, K. Engelborghs, P. Vansevenant, and D. Roose, "Continuous pole placement for delay equations," *Automatica*, vol. 38, pp. 747–761, 2002.

[16] D. E. Miller and D. E. Davison, "Use of time-varying control to achieve an arbitrarily large delay margin," in *Proc. IEEE Conf. Decision and Control*, Las Vegas, NV, Dec. 2002, pp. 975–980.

[17] D. E. Miller and D. E. Davison, "Use of time-varying control to achieve an arbitrarily large delay margin," presented at the IFAC Workshop on Time Delay Systems, Rocquencourt, France, Sep. 8–10, 2003, INRIA.

[18] D. E. Miller and D. E. Davison, "Stabilization in the presence of an uncertain arbitrarily large delay," *IEEE Trans. Autom. Control*, vol. 50, no. 8, pp. 1074–1089, Aug. 2005.

[19] L. Mirkin and Z. J. Plamor, "Control issues in systems with loop delays," in *The Handbook of Networked and Embedded Control Systems*, D. Hristu-Varsakelis and W. S. Levine, Eds. Basel, Switzerland: Birkhauser, 2005.

[20] M. Rossi and D. E. Miller, "Gain/phase margin improvement using static sampled-data hold functions," *Syst. Control Lett.*, vol. 37, pp. 163–172, 1999.

[21] N. Olgac and R. Sipahi, "An exact method for the stability analysis of time-delayed linear time-invariant systems," *IEEE Trans. Autom. Control*, vol. 47, no. 5, pp. 793–797, May 2002.

[22] C. E. de Souza and X. Li, "Delay-dependent robust H_∞ control of uncertain linear state-delayed systems," *Automatica*, vol. 35, pp. 1313–1321, 1999.

[23] A. Tannenbaum, "Robust stabilization of plants with uncertainty in the gain factor," *Int. J. Control*, vol. 32, pp. 1–16, 1980.

[24] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: MIT, 1985.

[25] M. Vidyasagar, H. Shneider, and B. A. Francis, "Algebraic and topological aspects of feedback stabilization," *IEEE Trans. Autom. Control*, vol. AC-27, no. 4, pp. 880–894, Apr. 1982.

[26] C. Yang and P. T. Kabamba, "Multi-channel output gain margin improvement using generalized sampled-data hold functions," *IEEE Trans. Autom. Control*, vol. 39, no. 3, pp. 657–661, Mar. 1994.

[27] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

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