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Sinan Ünver

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# On the additive dilogarithm

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Let $k$ be a field of characteristic zero, and let $k[\varepsilon]_n := k[\varepsilon]/(\varepsilon^n)$. We construct an additive dilogarithm $\mathrm{Li}_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$, where $B_2$ is the Bloch group which is crucial in studying weight two motivic cohomology. We use this construction to show that the Bloch complex of $k[\varepsilon]_n$ has cohomology groups expressed in terms of the K-groups $K_{(\cdot)}(k[\varepsilon]_n)$ as expected. Finally we compare this construction to the construction of the additive dilogarithm by Bloch and Esnault defined on the complex $T_n\mathbb{Q}(2)(k)$.

## 1. Introduction

**1.1.** For any scheme $S$ one expects a category $\mathcal{M}_S$ of motivic (perverse) sheaves over $S$, which should be an abelian tensor category that satisfies all the formalism of mixed sheaf theory [Beĭlinson 1987, 5.10]. The Tate sheaves $\mathbb{Z}_{\mathcal{M}}(n)$ should play a special role. Namely, letting

$$H^i(S, \mathbb{Z}_{\mathcal{M}}(n)) := \mathrm{Ext}^i_{\mathcal{M}_S}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n)),$$

the Chern character map

$$K_{2n-i}(S)^{(n)}_{\mathbb{Q}} \to H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \tag{1.1.1}$$

from the $n$-th graded piece of Quillen's K-theory tensored with $\mathbb{Q}$, defined as the $k^n$-eigenspace for the $k$-th Adams operator (Remark 3.1.2), to motivic cohomology of weight $n$ should be an isomorphism when $S$ is regular (loc. cit.). Since $\mathcal{M}_S$ is to have realizations corresponding to various cohomology theories, the regulator map

$$K_{2n-i}(S)^{(n)}_{\mathbb{Q}} \to H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \to H^i_*(S, \mathbb{Q}_*(n)),$$

where $*$ is the relevant realization, gives arithmetically important information.

The complexes $\underline{\mathrm{RHom}}_{\mathrm{Zar}}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))$ of sheaves on the Zariski site should have the property that $H^i(S_{\mathrm{Zar}}, \underline{\mathrm{RHom}}_{\mathrm{Zar}}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))) = H^i(S, \mathbb{Z}_{\mathcal{M}}(n))$. Hence the motivic cohomology of $S$ of weight $n$ could be computed as the hypercohomology of a complex of sheaves on $S_{\mathrm{Zar}}$.

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*Keywords:* polylogarithms, additive polylogarithms, mixed Tate motives, Hilbert's 3rd problem.

Recently Voevodsky and others have made progress in motivic cohomology [Mazza et al. 2006]. If $S = \mathrm{Spec}(k)$, where $k$ is a field of characteristic zero, Voevodsky constructs a triangulated category $DM^{\mathrm{eff},-}_{\mathrm{Nis}}(k)$ [loc. cit., Chapter 14] and a complex of sheaves $\underline{\mathbb{Z}}(n)$ on the big Zariski site over $k$, which should be isomorphic to the hypothetical $\underline{\mathrm{RHom}}_{\mathrm{Zar}}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))$ above, such that for any smooth scheme $X$ over $k$,

$$H^i(X_{\mathrm{Zar}}, \underline{\mathbb{Z}}(n)) \simeq \mathrm{Ext}^i_{DM^{\mathrm{eff},-}_{\mathrm{Nis}}}(M(X), \underline{\mathbb{Z}}(n))$$

(see [loc. cit., 14.16]), where $M(X)$ is the motive of $X$ [loc. cit., Definition 14.1]. Since $\underline{\mathbb{Z}}(n)$ and Bloch's complex of algebraic cycles of codimension $n$ are isomorphic [loc. cit., Chapter 19], the Bloch–Grothendieck–Riemann–Roch theorem [Bloch 1986] implies that the hypercohomology of $\underline{\mathbb{Q}}(n)$ on $X_{\mathrm{Zar}}$ is expressed in terms of the K-groups of $X$ as above:

$$K_{2n-i}(X)^{(n)}_{\mathbb{Q}} \simeq H^i(X_{\mathrm{Zar}}, \underline{\mathbb{Q}}(n)). \tag{1.1.2}$$

In order to study the motivic cohomology of $S$, it would be sufficient to restrict to a subcategory of $\mathcal{M}_S$. Let $\mathcal{MTM}_S$ denote the smallest full subcategory of $\mathcal{M}_S$ that contains the Tate motives and is closed under extensions. Then $H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \simeq \mathrm{Ext}^i_{\mathcal{M}_S}(\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n)) = \mathrm{Ext}^i_{\mathcal{MTM}_S}((\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n))$. The category $\mathcal{MTM}_S$ would be simpler than $\mathcal{M}_S$. In fact for $S = \mathrm{Spec}(k)$, where $k$ is a number field, Deligne and Goncharov [2005] have constructed a candidate for $\mathcal{MTM}_S$ as a tannakian category, using $DM^{\mathrm{eff},-}_{\mathrm{Nis}}$.

It is natural to expect that $\mathcal{MTM}_S$ can be constructed by using only the relative cohomologies of hyperplane arrangements and in turn that motivic cohomology can be computed using complexes of *linear* algebraic objects [Beĭlinson et al. 1990], rather than all algebraic cycles. Special degenerate configurations of hyperplanes, called the polylogarithmic configurations [Beĭlinson et al. 1990; Goncharov 1995], act as building blocks for all configurations and thus play a special role in describing motivic cohomology.

Using the relations satisfied by the polylogarithmic configurations, Goncharov defines a complex $\Gamma_k(n)_{\mathbb{Q}}$ by

$$\mathcal{B}_n(k) \to \mathcal{B}_{n-1}(k) \otimes k^{\times}_{\mathbb{Q}} \to \mathcal{B}_{n-2}(k) \otimes \bigwedge^2 k^{\times}_{\mathbb{Q}} \to \cdots \to \mathcal{B}_2(k) \otimes \bigwedge^{n-2} k^{\times}_{\mathbb{Q}} \to \bigwedge^n k^{\times}_{\mathbb{Q}},$$

which he conjectures can be used to compute the motivic cohomology of weight $n$ [Goncharov 1995, Conjectures A and 1.17].

If $k = \mathbb{C}$, integration over the polylogarithmic configurations can be used to define a map $\mathbb{Q}[\mathbb{P}^1(\mathbb{C})] \to \mathbb{R}$, the single-valued real analytic version of the $n$-th polylogarithmic function [Goncharov 1995, 1.0], which factors through the projection $\mathbb{Q}[\mathbb{P}^1(\mathbb{C})] \to \mathcal{B}_n(\mathbb{C})$ (loc. cit.) to give $\mathcal{L}_n : \mathcal{B}_n(\mathbb{C}) \to \mathbb{R}$, the $n$-th polylogarithm that is expected to induce the regulator $K_{2n-1}(\mathbb{C})^{(n)}_{\mathbb{Q}} \simeq H^1(\mathrm{Spec}(\mathbb{C}), \mathbb{Q}_{\mathcal{M}}(n)) \to \mathcal{B}_n(\mathbb{C}) \to \mathbb{R}$ [Goncharov 1995, page 224].

For a general field $k$, one cannot expect a polylogarithm on $\mathcal{B}_n(k)$. However, through his interpretation of hyperbolic scissor congruence groups in terms of mixed Tate motives, Goncharov expected that there should be an infinitesimal polylogarithmic function that acts like a regulator map on $K_{2n-1}(k[\varepsilon]_2, (\varepsilon))^{(n)}$, for any field $k$ of characteristic 0 [Goncharov 1999, pages 616–617; 2004], where $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$. In our notation, assuming the existence of mixed Tate motives and the complex $\Gamma_n$ over the dual numbers, this translates to the existence of a map

$$\mathcal{B}_n(k[\varepsilon]_2)/\mathcal{B}_n(k) \to k \tag{1.1.3}$$

that, when composed with $K_{2n-1}(k[\varepsilon]_2, (\varepsilon))^{(n)} \to \mathcal{B}_n(k[\varepsilon]_2)/\mathcal{B}_n(k)$, gives an isomorphism. The map (1.1.3) is to be an analogue of both the volume map for euclidean scissor congruence groups and of polylogarithms.

In this paper we are interested in this question for weight two. Next we give details about this case.

**1.2.** Let $A$ be an artinian local ring and $I$ an ideal of $A$. In the rest of the paper, when we refer to weight two (rational) motivic cohomology of $A$ relative to $I$, what we mean are the groups $K_3(A, I)^{(2)}_{\mathbb{Q}}$ and $K_2(A, I)^{(2)}_{\mathbb{Q}}$ and not to the Voevodsky motivic cohomology groups in Section 1.1, which were there only to motivate the main results of this paper. This common abuse of notions is partly justified by the expected Chern character isomorphism (1.1.1), which is known to be true when $A$ is a field (1.1.2).

Let $k$ be an algebraically closed field of characteristic 0, let $S$ the semilocal ring of rational functions on $\mathbb{A}^1_k$ that are regular on $\{0, 1\}$, and let $J$ the Jacobson radical of $S$.

The first complex computing the weight two motivic cohomology is constructed by Bloch as follows. Localizing $\mathbb{A}^1_k$ away from 0 and 1 gives an exact sequence

$$0 \to K_3(k)^{(2)} \to K_2(S, J) \xrightarrow{\varphi} \bigoplus_{x \in k^\times \setminus \{1\}} k^\times \to K_2(k) \to 0$$

(see [Lichtenbaum 1987, proof of 7.1; Bloch 1977]), where $\varphi$ is the tame symbol map. Let

$$B(k) := K_2(S, J)/\,\mathrm{im}((1 + J) \otimes k^\times),$$

the part of $K_2(S, J)$ that does not come from the products of weight 1 terms. Then $(\bigoplus_{x \in k^\times \setminus \{1\}} k^\times)/\varphi((1 + J) \otimes k^\times) = k^\times \otimes k^\times$, and the sequence

$$0 \to K_3(k)^{(2)}_{\mathbb{Q}} \to B(k)_{\mathbb{Q}} \to (k^\times \otimes k^\times)_{\mathbb{Q}} \to K_2(k)_{\mathbb{Q}} \to 0,$$

remains exact (same references).

In complete analogy, Bloch and Esnault [2003] define a complex that computes the motivic cohomology of $k[t]_2$ relative to the ideal $(t)$ as follows. Let $R$ be the local ring of $\mathbb{A}^1_k$ at 0. Then localizing away from 0 on $\mathbb{A}^1$ gives the sequence

$$K_2(k[t], (t^2)) \to K_2(R, (t^2)) \xrightarrow{\varphi} \bigoplus_{x \in k^\times} k^\times \to K_1(k[t], (t^2)).$$

Let $\mathscr{C}$ denote the subgroup generated by the symbols $\langle a, b\rangle \in K_2(R, (t^2))$ with $a \in (t^2)$ and $b \in k$, and put $TB(k) := K_2(R, (t^2))/\mathscr{C}$. Then we have $k^\times \otimes k = (\bigoplus_{x \in k^\times} k^\times)/\varphi(\mathscr{C})$ and an exact sequence

$$0 \to K_2(k[t], (t^2))^{(2)} \to TB(k) \to k^\times \otimes k \to K_1(k[t], (t^2)) \to 0$$

[Bloch and Esnault 2003, Proposition 2.1 and Corollary 2.5]. Then we have

$$K_2(k[t], (t^2))^{(2)}_{\mathbb{Q}} \simeq K_3(k[t]_2, (t))^{(2)}_{\mathbb{Q}} \quad \text{and} \quad K_1(k[t], (t^2)) \simeq K_2(k[t]_2, (t))$$

(loc. cit.). Therefore the complex $TB(k) \to k^\times \otimes k$ (tensored with $\mathbb{Q}$), really computes the motivic cohomology of $k[t]_2$ relative to $(t)$. Moreover Bloch and Esnault define a dilogarithm map on $TB(k)$:

**Theorem 1.2.1** [Bloch and Esnault 2003, Corollary 2.5]. *Let $\mathfrak{m}$ be the maximal ideal of $R$. There is a well-defined map $\rho : TB(k) \to \mathfrak{m}^3/\mathfrak{m}^4$ such that*

$$\rho(\langle a, b\rangle) = -a \cdot db \quad \textit{for } \langle a, b\rangle \in K_2(R, (t^2)) \textit{ with } a \in \mathfrak{m}^2 \textit{ and } b \in R,$$

*and $\rho$ induces an isomorphism $K_3(k[t], (t^2))^{(2)} \to \mathfrak{m}^3/\mathfrak{m}^4$ of abelian groups.*

**1.3.** For $k$ a field of characteristic zero there is another natural complex, which is of more geometric origin and hence easier to relate to various definitions of categories of mixed Tate motives, that computes the weight two motivic cohomology groups of $k$.

Suppose $A$ is an artinian local ring with residue field $k$. The Bloch group $B_2(A)$ (denoted by $\mathfrak{p}(A)$ in [Suslin 1990]) is the free abelian group generated by the symbols $[x]$ such that $x(1-x) \in A^\times$, modulo the subgroup generated by elements of the form

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)],$$

for all $x, y \in A^\times$ such that $(1-x)(1-y)(1-x/y) \in A^\times$. The map that sends $[x]$ to $x \wedge (1-x) \in \bigwedge^2_{\mathbb{Z}} A^\times$ induces the two term complex $\gamma_A(2)$ that sits in $[1, 2]$:

$$\delta_A : B_2(A) \to \bigwedge\nolimits^2_{\mathbb{Z}} A^\times. \tag{1.3.1}$$

The complex $\gamma_k(2)$ can be thought of as a more explicit version of $\Gamma_k(2)$. In fact, there is a natural map $\gamma_k(2)_{\mathbb{Q}} \to \Gamma_k(2)_{\mathbb{Q}}$, which is expected to be an isomorphism

[Goncharov 1995, Conjecture 1.20], and there is an exact sequence [Suslin 1990]

$$0 \to K_3(k)^{(2)}_{\mathbb{Q}} \to B_2(k)_{\mathbb{Q}} \to (\bigwedge^2 k^\times)_{\mathbb{Q}} \to K_2(k)_{\mathbb{Q}} \to 0.$$

For $n \geq 2$, we are interested in the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$, where $\delta_{k[\varepsilon]_n}$ will be denoted by $\delta_n$. We show that it has the expected cohomology:

**Theorem 1.3.1.** *For $k$ a field of characteristic $0$, there is an exact sequence*

$$0 \to K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}} \longrightarrow B_2(k[\varepsilon]_n)_{\mathbb{Q}} \xrightarrow{\delta_n} (\bigwedge^2 k[\varepsilon]_n^\times)_{\mathbb{Q}} \longrightarrow K_2(k[\varepsilon]_n)_{\mathbb{Q}} \to 0.$$

For $n = 2$ this theorem gives a "yes" answer to [Goncharov 2004, Problem 2.3].

While proving the previous theorem we construct an additive dilogarithm map on $B_2(k[\varepsilon]_n)$:

**Theorem 1.3.2.** *For every $n \geq 2$, there is a natural map*

$$\mathrm{Li}_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$$

*that, when composed with $K_3(k[\varepsilon]_n, (\varepsilon))^{(2)} \hookrightarrow B_2(k[\varepsilon]_n)$, induces an isomorphism $K_3(k[\varepsilon]_n, (\varepsilon))^{(2)} \simeq k^{\oplus(n-1)}$ of abelian groups.*

The advantage of defining a dilogarithm map on $B_2(k[\varepsilon]_n)$ is that this group is closely related to the linear algebra-geometric complexes of mixed Tate motives. More precisely, $\mathrm{Li}_{2,n}$ immediately gives an analogue of the volume map for a pair of triangles over $k[\varepsilon]_n$, as in [Beĭlinson et al. 1990]: All one needs to do is to take the image of the pair of triangles in $B_2(k[\varepsilon]_n)$ under the map in [loc. cit., Proposition 3.7] and then apply $\mathrm{Li}_{2,n}$. In this context Theorems 1.3.1 and 1.3.2 imply that the class of a pair of triangles in $A_2(k[\varepsilon_n])/A_2(k)$ (loc. cit.) is determined by its image in $\bigwedge^2 k[\varepsilon_n]^\times / \bigwedge^2 k^\times$ and its image under $\mathrm{Li}_{2,n}$. This is a precise analogue of Sydler's theorem on Hilbert's 3rd problem that the scissors congruence class of a three-dimensional polyhedron is determined by its volume and its Dehn invariant [Goncharov 1999, Section 2.7]. We do not, however, pursue this application here.

**1.4.** In order to compare $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$ with the complex of Bloch and Esnault, we show that their argument extends to define a complex $T_n\mathbb{Q}(2)(k)$ by

$$T_n B(k) \to k^\times \otimes (\varepsilon \cdot k[\varepsilon]_n)$$

(for $n = 2$ this is the complex in Section 1.2). Let $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}} = \gamma_k(2)_{\mathbb{Q}} \oplus \gamma_{k[\varepsilon]_n}(2)^\circ_{\mathbb{Q}}$, and note that the cohomology groups of $\gamma_{k[\varepsilon]_n}(2)^\circ_{\mathbb{Q}}$ and $T_n\mathbb{Q}(2)(k)$ coincide. We define a subcomplex $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ of $\gamma_{k[\varepsilon]_n}(2)^\circ_{\mathbb{Q}}$ that has the same cohomology groups, and obtain a direct consequence of Theorems 1.2.1, 1.3.1, and 1.3.2:

**Corollary 1.4.1.** *For $k$ an algebraically closed field of characteristic $0$, the complexes $T_n\mathbb{Q}(2)(k)$ and $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ are isomorphic.*

**1.5.** The paper is organized as follows. In Section 2, we construct the additive dilogarithm, $\mathrm{Li}_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$. Two results in Section 2 are useful in studying $\mathrm{Li}_{2,n}$. On the one hand, $\mathrm{Li}_{2,n}$ is explicitly described in Proposition 2.2.3 and Definition 2.2.4. On the other hand, $\mathrm{Li}_{2,n}$ has a conceptual description: The image of an element in $B_2(k[\varepsilon]_n)$ under $\mathrm{Li}_{2,n}$ is obtained by lifting that element to an arbitrary element in $B_2(k[\varepsilon]_{2n-1})$ then taking its image in $\bigwedge^2 k[\varepsilon]^\times_{2n-1}$ under the map in (1.3.1), and finally choosing certain algebraic combinations of its coordinates in $\bigwedge^2 k[\varepsilon]^\times_{2n-1}$ as in Propositions 2.1.2, 2.2.1 and 2.2.2. It is this flexibility in the choice of the lifting that is used in the computations in Section 4.

In this paper, rather than working with K-theory we work with cyclic homology most of the time. This is possible since $K_*(k[\varepsilon]_n) = K_*(k[\varepsilon]_n, (\varepsilon)) \oplus K_*(k)$ and by the theorem of Goodwillie [1986], $\mathrm{HC}_{*-1}(k[\varepsilon]_n, (\varepsilon)) \simeq K_*(k[\varepsilon]_n, (\varepsilon))$, where HC denotes cyclic homology with respect to $\mathbb{Q}$. Note that since we are working with $\mathbb{Q}$-coefficients, K-theory is nothing other than the primitive part of the rational homology of GL [Loday 1992, Corollary 11.2.12].

In Sections 3.1 through 3.6 we make Goodwillie's theorem explicit, following [Loday 1992], by giving the description of a map from $\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))$ to $H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q})$. Then in Sections 3.7 and 3.8, Suslin and Guin's stability theorem and a construction of Bloch, Suslin and Goncharov is used to construct a map $H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q}) \to \ker(\delta_n)$. More details about Section 3 are given in Section 3.1. This explicit description will be needed in Section 4.

In Section 4, we prove Theorem 1.3.2. This is done by first using the description of $\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))$ given in [Cathelineau 1990/91] in Section 4.1.1, then constructing certain elements

$$\alpha_w \in \mathrm{HC}_2(k[\varepsilon]_n, \varepsilon)^{(1)} \quad \text{for } n+1 \le w \le 2n-1,$$

and chasing the images of these elements under the maps described in Sections 2 and 3. The proof also shows that $\{\alpha_w\}_{n+1 \le w \le 2n-1}$ form a basis for $\mathrm{HC}_2(k[\varepsilon]_n, \varepsilon)^{(1)}$.

In Section 5, using [Suslin 1990], [Guin 1989], and Section 4, we show that the infinitesimal part of $\ker(\delta)$ is canonically isomorphic to $\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$. From this Theorem 1.3.1 follows.

In Section 6, we first define a subcomplex $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ of $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. Then we extend the construction of Bloch and Esnault to higher moduli and finally prove Corollary 1.4.1, which compares the two constructions.

**Remarks.** First, we mention the work of J. Park [2007], which gives an additive Chow theoretic description of the additive dilogarithm of Bloch and Esnault, and the work of K. Rülling [2007], which proves that the complex of additive Chow groups with modulus (not necessarily of 2) has the expected cohomology groups on the level of zero cycles.

Second, there are many problems left unanswered in this note. The most important of these is the construction of additive polylogarithms for higher weights. We have made this construction, but we have yet to prove that the complex has the right cohomology groups. We will address in another paper the question of what happens in characteristic $p$, and we will also compare our construction to the work of Park and Rülling.

## 2. Additive dilogarithm

**Notation 2.0.1.** Let $k$ be a field of characteristic zero. An abelian group $A$ endowed with a group homomorphism $k^\times \to \mathrm{Aut}_{ab}(A)$ is said to be a $k^\times$-abelian group; we denote the action of $\lambda \in k^\times$ on $a \in A$ by $\lambda \times a$. If $f : A \to k$ is an additive map that satisfies $f(\lambda \times a) = \lambda^w \cdot f(a)$ for all $\lambda \in k^\times$ and $a \in A$, then we say that $f$ is of $k^\times$-weight $w$.

If $V$ is a $k$-module with a $k^\times$-action that is $k$-linear, that is, defined by a homomorphism $k^\times \to \mathrm{Aut}_{k\text{-mod}}(V)$, then we let

$$V_{\langle w \rangle} := \{v \in V \mid \lambda \times v = \lambda^w \cdot v \text{ for all } \lambda \in k^\times\}$$

be the subspace of elements of $V$ of weight $w$.

Define $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$, $V_m := k[\varepsilon]_m^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and $B_2(k[\varepsilon]_m)$ as in Section 1.3.

For an object $A$ defined over $k[\varepsilon]_m$, we denote by $A^\circ$ its infinitesimal part, for example,

$$B_2(k[\varepsilon]_m) = B_2(k) \oplus B_2(k[\varepsilon]_m)^\circ, \quad k[\varepsilon]_m^\circ = \varepsilon \cdot k[\varepsilon]_m, \quad V_m^\circ = 1 + \varepsilon \cdot k[\varepsilon]_m.$$

When the context requires it we write (say) $K_*(k[\varepsilon]_m)^\circ$ instead of $K_*(k[\varepsilon]_m, (\varepsilon))$. Finally, since in what follows the infinitesimal part $A^\circ$ of an object $A$ is canonically a direct summand of $A$, we never mention the natural maps $A^\circ \to A$ and $A \to A^\circ$, and take other liberties of this kind.

The exponential map gives an isomorphism $k[\varepsilon]_m^\circ \simeq V_m^\circ$, which endows $V_m^\circ$ with a $k$-space structure. For $\lambda \in k^\times$, the $k$-algebra map that sends $\varepsilon$ to $\lambda \cdot \varepsilon$ defines an action of $k^\times$ on $k[\varepsilon]_m$ and $V_m^\circ$. Denote the weight $i$ subspace of $V_m^\circ$ under this action by $V_{m,\langle i \rangle}$, that is,

$$V_{m,\langle i \rangle} = \{v \in V_m^\circ \mid \lambda \times v = \lambda^i \cdot v \text{ for all } \lambda \in k^\times\} = \{\exp(a \cdot \varepsilon^i) \mid a \in k\}.$$

Then $V_m^\circ = \bigoplus_{1 \le i \le m-1} V_{m,\langle i \rangle}$. To simplify the notation we also put $V_{m,\langle 0 \rangle} := k^\times \otimes \mathbb{Q}$.

Let $k[\varepsilon]_m^{\times\times} \subseteq k[\varepsilon]_m$ denote the set of exceptional units, that is, those $a \in k[\varepsilon]_m^\times$ such that $1 - a \in k[\varepsilon]^\times$.

Let $\delta : \mathbb{Q}[k[\varepsilon]_m^{\times\times}] \to \bigwedge^2 V_m$ be the map that sends $x \in k[\varepsilon]_m^{\times\times}$ to $x \wedge (1-x)$. If we want to emphasize that we are working over $k[\varepsilon]_m$, we will use the notation $\delta_m$

instead of $\delta$. The map on $B_2(k[\varepsilon]_m)$ induced by $\delta_m$ is denoted by the same letter (see (1.3.1)).

**2.1. *Construction of* $\mathbf{li_2}$.** In this section we collect the combinatorial arguments in the construction of the additive dilogarithm over $k[\varepsilon]_n$. The crucial step is the statement that $S_k(m,n)_{\langle w\rangle}$ is one dimensional in Proposition 2.1.2. This implies that if one thinks that the additive dilogarithm on $k[\varepsilon]_n$ should be constructed by first lifting to $k[\varepsilon]_{2n-1}$ and then using $\delta$, then there is essentially one way to define it. That this is the right definition is justified in the next section.

**Definition 2.1.1.** Let $n, m \in \mathbb{N}$ such that $2 \le n \le m$. Let $\alpha_{m,n} : \mathbb{Q}[k[\varepsilon]_m^{\times\times}] \to \bigwedge^2 V_m$ denote the map that sends $\gamma \in k[\varepsilon]_m^{\times\times}$ to $\delta(\gamma) - \delta(\gamma\,|\,n)$, where $\gamma\,|\,n$ is the truncation of $\gamma$ to the sum of first $n$ powers of $\varepsilon$, that is, if $\gamma = a_0 + a_1 \cdot \varepsilon + \cdots + a_{m-1} \cdot \varepsilon^{m-1}$ then $\gamma\,|\,n = a_0 + a_1 \cdot \varepsilon + \cdots + a_{n-1} \cdot \varepsilon^{n-1}$.

Let $V(m,n)$ denote

$$\bigoplus_{\substack{0<i\le n-1\\ n\le j\le m-1}} (V_{m,\langle i\rangle} \otimes V_{m,\langle j\rangle}) \subseteq \bigwedge^2 V_m,$$

which we also consider as a quotient of $\bigwedge^2 V_m$ via the direct sum decomposition

$$\bigwedge^2 V_m = \bigoplus_{0\le i<j<m} (V_{m,\langle i\rangle} \otimes V_{m,\langle j\rangle}) \oplus \Big( \bigoplus_{0\le i<m} \bigwedge^2 V_{m,\langle i\rangle} \Big). \tag{2.1.1}$$

Finally denote by $V_k(m,n)$ the quotient

$$\bigoplus_{\substack{0<i\le n-1\\ n\le j\le m-1}} (V_{m,\langle i\rangle} \otimes_k V_{m,\langle j\rangle})$$

of $V(m,n)$, by $p(m,n) : \bigwedge^2 V_m \to V_k(m,n)$ the canonical projection, by $S_k(m,n)$ the $k^\times$-abelian group $V_k(m,n)/\operatorname{im}(p(m,n)\circ \alpha_{m,n})$ and by $S_k(m,n)_{\langle i\rangle}$ the image of $V_k(m,n)_{\langle i\rangle}$ in $S_k(m,n)$. This notation is justified by noting that $S_k(m,n)$ has a natural $k$-module structure induced from that of $V_k(m,n)$ such that its weight $i$ subspace is equal to $S_k(m,n)_{\langle i\rangle}$ and $S_k(m,n) = \bigoplus_{0<i} S_k(m,n)_{\langle i\rangle}$.

For $0 < i < j < m$, let $p_{i,j} : \bigwedge^2 V_m \to V_{m,\langle i\rangle} \otimes V_{m,\langle j\rangle}$ denote the projection determined by the decomposition (2.1.1). Then $l_{i,j} : \bigwedge^2 V_m \to k$ is defined by letting $(\log \otimes \log)(p_{i,j}(\alpha)) = l_{i,j}(\alpha) \cdot (\varepsilon^i \otimes \varepsilon^j)$ in $k[\varepsilon]_m \otimes_k k[\varepsilon]_m$ for any $\alpha \in \bigwedge^2 V_m$.

**Proposition 2.1.2.** *Let $n, m, w \in \mathbb{N}$ such that $2 \le n < w \le \min(2n-1, m)$. Then $S_k(m,n)_{\langle w\rangle}$ is a one-dimensional $k$-module. The unique linear functional*

$$\mathrm{li}_{2,(m,n),w} : S_k(m,n)_{\langle w\rangle} \to k$$

*such that* $\mathrm{li}_{2,(m,n),w}(\exp(\varepsilon)\otimes\exp(\varepsilon^{w-1}))=1$ *is given by*

$$\mathrm{li}_{2,(m,n),w}=\sum_{1\le j\le w-n} j\cdot l_{j,w-j}.$$

*Proof.* Let $\mathrm{li}_{2,(m,n),w}$ denote the map from $\bigwedge^2 V_m$ to $k$ given by the formula

$$\mathrm{li}_{2,(m,n),w}=\sum_{1\le j\le w-n} j\cdot l_{j,w-j}.$$

We would like to see that $\mathrm{li}_{2,(m,n),w}\circ\alpha_{m,n}=0$. Fix

$$x:=s+s(1-s)a_1\varepsilon+\cdots+s(1-s)a_{m-1}\varepsilon^{m-1}\in k[\varepsilon]_m^{\times\times}.$$

Let $A_m:=\{1,\dots,m-1\}$ and let $(A_m)^{\times\alpha}$ denote the cartesian product of $A_m$ with itself $\alpha$-times. Define $\mathfrak{p}:(A_m)^{\times\alpha}\to k$ by $\mathfrak{p}(i_1,\dots,i_\alpha):=a_{i_1}\cdot a_{i_2}\cdots a_{i_\alpha}$, and $\mathfrak{w}:(A_m)^{\times\alpha}\to\mathbb{N}$ by $\mathfrak{w}(i_1,\dots,i_\alpha):=i_1+i_2+\cdots+i_\alpha$. Note that even though $\mathfrak{p}$ depends on $x$, we suppress it from the notation. In order to simplify the notation let $A(\alpha):=(A_m)^{\times\alpha}$ and $B(\alpha):=(A_m)^{\times\alpha}\setminus(A_n)^{\times\alpha}$.

If $1\le\alpha,\beta\le w$, let

$$C(\alpha,\beta):=\{(a,b)\mid a\in A(\alpha),\ b\in B(\beta),\ \mathfrak{w}(a)+\mathfrak{w}(b)=w\}.$$

Let the permutation group $S_{\alpha+\beta}$ on $\alpha+\beta$ letters act on $A(\alpha)\times A(\beta)$ by permuting the coordinates. On $C(\alpha,\beta)\subseteq A(\alpha)\times A(\beta)$ consider the following equivalence relation. If $(a,b),(c,d)\in C(\alpha,\beta)$, then $(a,b)$ and $(c,d)$ are equivalent if there exists a permutation $\sigma\in S_{\alpha+\beta}$ such that $(a,b)^\sigma=(c,d)$. Denote the equivalence class of $(a,b)$ by $[(a,b)]$ and the set of all equivalence classes by $\mathcal{S}(\alpha,\beta)$. Let $\mathfrak{p}([a,b])=\mathfrak{p}(a)\cdot\mathfrak{p}(b)$.

Assume from now on that $\alpha+\beta\le w$. Note that since $w\le 2n-1$, any element $(a,b)\in C(\alpha,\beta)$ has a unique coordinate that is greater than or equal to $n$. Denote this coordinate by $e(a,b)$. Denote by $(a,b)_0$ the element of $C(\alpha,\beta)$ obtained by interchanging the last coordinate of $(a,b)$ with the coordinate containing $e(a,b)$.

Then we define a map $\iota:C(\alpha,\beta)\to C(\beta,\alpha)$ as follows. Let $(a,b)\in C(\alpha,\beta)$. Then $\iota(a,b)\in C(\beta,\alpha)$ is the element $(a,b)_0$, where we think of both $C(\alpha,\beta)$ and $C(\beta,\alpha)$ as subsets of $A(\alpha)\times A(\beta)\simeq A(\alpha+\beta)\simeq A(\beta)\times A(\alpha)$. This passes to equivalence classes and gives a map $\mathcal{S}(\alpha,\beta)\to\mathcal{S}(\beta,\alpha)$ that we continue to denote by $\iota$. Note that $\iota^2=1$, and if $G\in\mathcal{S}(\alpha,\beta)$, then $\mathfrak{p}(\iota(G))=\mathfrak{p}(G)$, and

$$\sum_{(a,b)\in G}\mathfrak{w}(a)=\sum_{(c,d)\in\iota(G)}\mathfrak{w}(c).$$

Letting $z=a_1\varepsilon+a_2\varepsilon^2+\cdots+a_{m-1}\varepsilon^{m-1}$, we have

$$x=s(1+(1-s)z)\quad\text{and}\quad 1-x=(1-s)(1-sz).$$

Computing in $k[\varepsilon]_m$, this gives

$$\log(x/s) = -\sum_{\ell=1}^{m-1} \frac{(s-1)^\ell z^\ell}{\ell} \quad \text{and} \quad \log((1-x)/(1-s)) = -\sum_{\ell=1}^{m-1} \frac{s^\ell z^\ell}{\ell}.$$

Since $z^\alpha = \sum_{u\in A(\alpha)} \mathfrak{p}(u)\varepsilon^{\mathfrak{w}(u)}$, we have

$$\log(x/s) = -\sum_{\ell=1}^{m-1} \frac{(s-1)^\ell}{\ell} \sum_{u\in A(\ell)} \mathfrak{p}(u)\varepsilon^{\mathfrak{w}(u)},$$

$$\log((1-x)/(1-s)) = -\sum_{\ell=1}^{m-1} \frac{s^\ell}{\ell} \sum_{u\in A(\ell)} \mathfrak{p}(u)\varepsilon^{\mathfrak{w}(u)}.$$

Then $\mathrm{li}_{2,(m,n),w}(\alpha_{m,n}(x))$ is equal to

$$\sum_{\substack{1\le\alpha\le w\\ 1\le\beta\le w}} \sum_{\substack{a\in A(\alpha)\\ b\in B(\beta)\\ \mathfrak{w}(a)+\mathfrak{w}(b)=w}} \frac{\mathfrak{w}(a)\cdot\mathfrak{p}(a)\cdot\mathfrak{p}(b)}{\alpha\cdot\beta}\cdot((s-1)^\alpha\cdot s^\beta - s^\alpha\cdot(s-1)^\beta)$$

$$= \sum_{\substack{1\le\alpha\le w\\ 1\le\beta\le w}} ((s-1)^\alpha\cdot s^\beta - s^\alpha\cdot(s-1)^\beta) \sum_{G\in\mathcal{G}(\alpha,\beta)} \left(\frac{\mathfrak{p}(G)}{\alpha\cdot\beta}\right) \sum_{(a,b)\in G} \mathfrak{w}(a).$$

On the other hand

$$\sum_{G\in\mathcal{G}(\alpha,\beta)} \left(\frac{\mathfrak{p}(G)}{\alpha\cdot\beta}\right) \sum_{(a,b)\in G} \mathfrak{w}(a) = \sum_{G\in\mathcal{G}(\alpha,\beta)} \left(\frac{\mathfrak{p}(\iota(G))}{\alpha\cdot\beta}\right) \sum_{(c,d)\in\iota(G)} \mathfrak{w}(c)$$
$$= \sum_{G\in\mathcal{G}(\beta,\alpha)} \left(\frac{\mathfrak{p}(G)}{\beta\cdot\alpha}\right) \sum_{(a,b)\in G} \mathfrak{w}(a).$$

Therefore $\mathrm{li}_{2,(m,n),w}(\alpha_{m,n}(x)) = 0$, and we have a linear functional

$$\mathrm{li}_{2,(m,n),w} : S_k(m,n)_{\langle w\rangle} \to k.$$

By the definition of $\mathrm{li}_{2,(m,n),w}$ it is clear that $\mathrm{li}_{2,(m,n),w}(\exp(\varepsilon)\otimes\exp(\varepsilon^{w-1})) = 1$.

To finish the proof we only need to show that the space of linear functionals on $S_k(m,n)_{\langle w\rangle}$ is generated by $\mathrm{li}_{2,(m,n),w}$, or equivalently that if $l$ is a linear combination of $\{l_{2,w-2}, l_{3,w-3}, \dots, l_{w-n,n}\}$ such that $l(\alpha_{m,n}(x)) = 0$ for all $x\in k[\varepsilon]_m^{\times\times}$, then $l$ is zero. So let $l = \sum_{2\le i\le w-n} c_i\cdot l_{i,w-i}$ satisfy $l(\alpha_{m,n}(x)) = 0$ for all $x\in k[\varepsilon]_m^{\times\times}$. Assume that $l\neq 0$ and let $i_0$ be the smallest integer $i$ such that $c_i\neq 0$. For all $s\in k^{\times\times}$ and $a_1, a_{i_0-1}, a_{w-i_0}\in k$, we have

$$l(\alpha_{m,n}(s + s(1-s)\cdot a_1\cdot\varepsilon + s(1-s)\cdot a_{i_0-1}\cdot\varepsilon^{i_0-1} + s(1-s)\cdot a_{w-i_0}\cdot\varepsilon^{w-i_0})) = 0.$$

If we denote the left hand side of the above equation by $l(s, a_1, a_{i_0-1}, a_{w-i_0})$, then

$$c_{i_0} \cdot \tfrac{1}{2}((s-1)^2 s - s^2(s-1)) \cdot (a_1 \cdot a_{i_0-1} \cdot a_{w-i_0}) = l(s, a_1, a_{i_0-1}, a_{w-i_0}) - l(s, a_1, 0, a_{w-i_0}) = 0.$$

Therefore $c_{i_0} = 0$, contradicting the assumption. □

**2.2.** ***Construction of* Li.** Using the construction in the previous section, we show that $\mathrm{li}_{2,(2n-1,n),w}$ descends to a function on $B_2(k[\varepsilon]_n)$, as defined in Section 1.3.

**Proposition 2.2.1.** *For $n+1 \le w \le 2n-1$, the map*

$$\mathrm{li}_{2,(2n-1,n),w} \circ \delta : \mathbb{Q}[k[\varepsilon]_{2n-1}^{\times\times}] \to k$$

*factors through the canonical projection* $\mathbb{Q}[k[\varepsilon]_{2n-1}^{\times\times}] \to \mathbb{Q}[k[\varepsilon]_n^{\times\times}]$.

We denote the induced map from $\mathbb{Q}[k[\varepsilon]_n^{\times\times}]$ to $k$ by $\mathrm{Li}_{2,n,w}$.

*Proof.* This follows from the fact that $\mathrm{li}_{2,(2n-1,n),w} \circ \alpha_{2n-1,n} = 0$ by the construction in Proposition 2.1.2. □

**Proposition 2.2.2.** *The map* $\mathrm{Li}_{2,n,w} : \mathbb{Q}[k[\varepsilon]_n^{\times\times}] \to k$ *factors through the canonical projection* $\mathbb{Q}[k[\varepsilon]_n^{\times\times}] \to B_2(k[\varepsilon]_n)$.

We continue to denote the induced map by $\mathrm{Li}_{2,n,w}$.

*Proof.* We need to show that for $x, y \in k[\varepsilon]_n^{\times\times}$ such that $x/y \in k[\varepsilon]_n^{\times\times}$,

$$\mathrm{Li}_{2,n,w}([x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)]) = 0.$$

If $\tilde{x}$ and $\tilde{y}$ are arbitrary liftings of $x$ and $y$ to $k[\varepsilon]_{2n-1}^{\times\times}$, then Proposition 2.2.1 implies that the left side of the last equation is equal to

$$(\mathrm{li}_{2,(2n-1),w} \circ \delta)([\tilde{x}] - [\tilde{y}] + [\tilde{y}/\tilde{x}] - [(1-\tilde{x}^{-1})/(1-\tilde{y}^{-1})] + [(1-\tilde{x})/(1-\tilde{y})]).$$

The proposition then follows from the fact that

$$\delta([\tilde{x}] - [\tilde{y}] + [\tilde{y}/\tilde{x}] - [(1-\tilde{x}^{-1})/(1-\tilde{y}^{-1})] + [(1-\tilde{x})/(1-\tilde{y})]) = 0. \qquad \square$$

If $\underline{c} = (c_1, \ldots, c_r) \in \mathbb{N}^r$ and $x = s + s(1-s)a_1\varepsilon + \cdots + s(1-s)a_{n-1}\varepsilon^{n-1} \in k[\varepsilon]_n^{\times\times}$ then

$$\mathfrak{p}(x; \underline{c}) := a_{c_1} \cdot a_{c_2} \cdots a_{c_r} \quad \text{and} \quad \mathfrak{w}(\underline{c}) := c_1 + \cdots + c_r.$$

Let $C(\alpha) := \{1, 2, \ldots, n-1\}^{\times\alpha}$.

**Proposition 2.2.3.** *For $n+1 \le w \le 2n-1$, we have*

$$\mathrm{Li}_{2,n,w}([x]) = \sum_{1 \le \alpha, \beta \le w} \frac{(s-1)^\alpha \cdot s^\beta - s^\alpha \cdot (s-1)^\beta}{\alpha \cdot \beta} \sum_{\substack{(a,b) \in C(\alpha) \times C(\beta) \\ \mathfrak{w}(a) \le w-n \\ \mathfrak{w}(a,b) = w}} \mathfrak{w}(a) \cdot \mathfrak{p}(x; (a,b)).$$

*Proof.* Direct computation. □

**Definition 2.2.4.** Define the additive dilogarithm $\mathrm{Li}_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$ by

$$\mathrm{Li}_{2,n} := \bigoplus_{n+1 \le w \le 2n-1} \mathrm{Li}_{2,n,w}\,.$$

## 3. The map from cyclic homology to the Bloch group

**3.1.** This section is based on Goodwillie's theorem [1986] and the construction of Bloch [1977], Suslin [1990] and Goncharov [1995] of a map from the $K_3$ of a field to its Bloch group. Our main reference for cyclic homology and Goodwillie's theorem is [Loday 1992]. Here all cyclic homology groups are relative to $\mathbb{Q}$.

We will need the following to pass from cyclic homology to the rational homology of GL.

**Theorem 3.1.1** [Goodwillie 1986; Loday 1992, Theorem 11.3.1]. *Let $A$ be a $\mathbb{Q}$-algebra and $I$ a nilpotent ideal in $A$. Then there is a canonical isomorphism*

$$\mathrm{HC}_{n-1}(A, I) \simeq K_n(A, I)_{\mathbb{Q}} \quad \textit{for } n \ge 1.$$

**Remark 3.1.2.** This isomorphism is compatible with the $\lambda$-structures on both sides by [Cathelineau 1990/91, Theorem 1]. Hence, if $\mathrm{HC}_*(A, I)^{(i-1)}$ and $K_*(A, I)^{(i)}_{\mathbb{Q}}$ denote the $k^i$-eigenspace for the $k$-th Adams operator (for any $k$), then the above isomorphism induces an isomorphism $\mathrm{HC}_{*-1}(A, I)^{(i-1)} \simeq K_*(A, I)^{(i)}_{\mathbb{Q}}$ by [loc. cit., corollary in Section 1.3].

For a ring $A$, the Hurewicz map induces an isomorphism from $\oplus_{n>0} K_n(A)_{\mathbb{Q}}$ to the primitive part $\mathrm{Prim}\, H_*(\mathrm{GL}(A), \mathbb{Q})$ of the homology of GL [Loday 1992, 11.2.12 Corollary]. The map in Theorem 3.1.1 is constructed as the composition of a map from cyclic homology to the primitive part of the homology of GL and then using the inverse of the Hurewicz map. Since we will only need the map from cyclic homology to the homology of GL, we next describe the steps in its construction, following [Loday 1992].

In Section 3.2, cyclic homology of $A$ is computed as the homology of the Connes complex. This section also describes the natural map from the Connes complex to the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{gl}$. This map induces an isomorphism from cyclic homology to the primitive homology of $\mathfrak{gl}$. In Section 3.3, homology of $\mathfrak{gl}$ is replaced with the sum of the homology of its nilpotent parts $\mathfrak{t}_\sigma(A, I)$. In Section 3.4, homology of $\mathfrak{t}_\sigma(A, I)$ is replaced with that of the completion of its universal enveloping algebra, and in Section 3.5, the latter is replaced with the homology of the group algebra of $T_\sigma(A, I)$, via Malčev theory. We reach the group homology of $\mathrm{GL}(A)$ in Section 3.6.

In Section 3.7, this construction in combination with Suslin and Guin's stability theorem [Suslin 1984; Guin 1989] induces a map

$$\mathrm{HC}_{n-1}(A, I) \to H_n(\mathrm{GL}_n(A), \mathbb{Q}) \tag{3.1.1}$$

when $A$ is an artinian local algebra over $\mathbb{Q}$ and $I$ is a proper ideal of $A$. We will use this map for $n = 3$.

Finally we use a slight variation of the construction of Suslin and Goncharov in Section 3.8 to get a map $H_3(\mathrm{GL}_3(A), \mathbb{Q}) \to \ker(\delta)$.

The details can be found in [Loday 1992, Section 11.3] and the references therein. The main result of this section is Proposition 3.8.9.

## 3.2. *Map from cyclic homology to Lie algebra homology.*

**3.2.1.** For any associative $\mathbb{Q}$-algebra $A$, the Connes complex $C_*^\lambda(A)$ is defined as follows. Let $\mathbb{Z}/n\mathbb{Z}$ act on $A^{\otimes n}$ by

$$1 \times (a_1 \otimes a_2 \otimes \cdots \otimes a_n) = (-1)^{n-1} a_2 \otimes a_3 \otimes \cdots \otimes a_n \otimes a_1,$$

and let $C_{n-1}^\lambda(A)$ denote the coinvariants of $A^{\otimes n}$ under this action. Define

$$\begin{aligned} b : C_n^\lambda(A) &\to C_{n-1}^\lambda(A), \\ (a_0, a_1, \ldots, a_n) &\mapsto \sum_{0 \le i \le n-1} (-1)^i (a_0, \ldots, a_i \cdot a_{i+1}, \ldots, a_n) \\ &\qquad\qquad + (-1)^n (a_n \cdot a_0, a_1, \ldots, a_{n-1}). \end{aligned}$$

Then $C_*^\lambda(A)$ is the complex

$$\cdots \xrightarrow{b} C_{n+1}^\lambda(A) \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} \cdots \longrightarrow C_0^\lambda(A) \longrightarrow 0,$$

and $\mathrm{HC}_*(A) = H_*(C_*^\lambda(A))$ [Loday 1992, Theorem 2.1.5]: The cyclic homology relative to $\mathbb{Q}$ can be computed as the homology of the Connes complex.

**3.2.2.** For $\mathfrak{g}$ a Lie algebra over $\mathbb{Q}$, the Chevalley–Eilenberg complex $C_*(\mathfrak{g}, \mathbb{Q})$ of $\mathfrak{g}$ with coefficients in $\mathbb{Q}$ is defined by

$$\cdots \xrightarrow{d} \bigwedge^n \mathfrak{g} \xrightarrow{d} \bigwedge^{n-1} \mathfrak{g} \longrightarrow \cdots \xrightarrow{d} \mathfrak{g} \xrightarrow{d} \mathbb{Q} \longrightarrow 0,$$

where $d : \bigwedge^n \mathfrak{g} \to \bigwedge^{n-1} \mathfrak{g}$ is given by

$$d(a_1 \wedge a_2 \wedge \cdots \wedge a_n) = \sum_{1 \le i < j \le n} (-1)^{i+j-1} [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n.$$

The Lie algebra homology $H_*(\mathfrak{g}, \mathbb{Q})$ of $\mathfrak{g}$ with coefficients in $\mathbb{Q}$ is the homology of the complex $C_*(\mathfrak{g}, \mathbb{Q})$. The diagonal map $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ induces a map

$$\Delta : C_*(\mathfrak{g}, \mathbb{Q}) \to C_*(\mathfrak{g}, \mathbb{Q}) \otimes C_*(\mathfrak{g}, \mathbb{Q}),$$

which makes $(C_*(\mathfrak{g}, \mathbb{Q}), d)$ a DG-coalgebra. Let Prim $H_*(\mathfrak{g}, \mathbb{Q})$ denote the primitive elements in $H_*(\mathfrak{g}, \mathbb{Q})$, that is, those $\alpha$ such that $\Delta(\alpha) = 1 \otimes \alpha + \alpha \otimes 1$.

Let $\mathfrak{gl}_n(A)$ denote the Lie algebra of $n \times n$ matrices, and let $\mathfrak{gl}(A)$ denote the direct limit $\lim_{n\to\infty} \mathfrak{gl}_n(A)$ with respect to the natural inclusions $\mathfrak{gl}_n(A) \subseteq \mathfrak{gl}_m(A)$ for $n \leq m$. Then $\mathfrak{gl}(\mathbb{Q})$ acts on $C_*(\mathfrak{gl}(A), \mathbb{Q})$ by

$$[h, g_1 \wedge \cdots \wedge g_n] := \sum_{1 \leq i \leq n} g_1 \wedge \cdots \wedge [h, g_i] \wedge \cdots \wedge g_n.$$

Let $C_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ denote the complex of coinvariants with respect to this action, and let $H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ and Prim $H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ denote respectively the homology and the primitive part of the homology of the complex $C_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{g}l(\mathbb{Q})}$. Then the theorem of Loday, Quillen, and Tsygan says this:

**Theorem 3.2.1** [Loday 1992, Theorem 10.2.4]. *If $A$ is an algebra over $\mathbb{Q}$, then there is a natural isomorphism*

$$\mathrm{HC}_{*-1}(A) \simeq \mathrm{Prim}\, H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{g}l(\mathbb{Q})} \simeq \mathrm{Prim}\, H_*(\mathfrak{g}l(A), \mathbb{Q}).$$

Explicitly, the first isomorphism above is induced by the chain map that sends the class of $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $C^{\lambda}_{n-1}(A)$ to the class of $a_1 e_{12} \wedge a_2 e_{23} \wedge \cdots \wedge a_n e_{n1}$ in $C_n(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$. Here $e_{ij}$ denotes the matrix whose only nonzero entry is the one in the $i$-th row and the $j$-th column, which is 1.

**3.3. *Volodin's construction in the Lie algebra case.*** Assume that $I$ is a nilpotent ideal of $A$, and let $\mathrm{HC}_*(A, I)$ denote the cyclic homology of $A$ relative to $I$, the homology of the complex $C^{\lambda}_*(A, I)$ that is the kernel of the natural surjection $C^{\lambda}_*(A) \to C^{\lambda}_*(A/I)$.

For any permutation $\sigma \in S_n$, let $\mathfrak{t}_\sigma(A, I)$ denote the Lie subalgebra of $\mathfrak{gl}(A)$ given by $\mathfrak{t}_\sigma(A, I) := \{(a_{ij}) \in \mathfrak{gl}(A) : a_{ij} \in I \text{ if } \sigma(j) \leq \sigma(i)\}$. Let $x(A, I) := \sum_\sigma C_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q})$ denote the sum of the subcomplexes

$$C_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \subseteq C_*(\mathfrak{gl}(A), \mathbb{Q}),$$

over all $n$ and $\sigma \in S_n$, and let $H_*(\mathfrak{gl}(A, I), \mathbb{Q})$ denote the homology of $x(A, I)$. Then the map in Theorem 3.2.1 induces an isomorphism

$$\mathrm{HC}_{*-1}(A, I) \simeq \mathrm{Prim}\, H_*(\mathfrak{g}l(A, I), \mathbb{Q}) \simeq \sum_\sigma \mathrm{Prim}\, H_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \tag{3.3.1}$$

[Loday 1992, Proposition 11.3.12].

**3.4. *From the Lie algebra to the universal enveloping algebra.*** For a Lie algebra $\mathfrak{g}$ over $\mathbb{Q}$, let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra and $\hat{\mathcal{U}}(\mathfrak{g})$ its completion with respect to its augmentation ideal. We will next express the homology of $\mathfrak{g}$ in terms of the homology of $\mathcal{U}(\mathfrak{g})$.

Let $B$ be an associative algebra over $\mathbb{Q}$ endowed with an augmentation map $\varepsilon : B \to \mathbb{Q}$. Let $C_*(B, \mathbb{Q})$ denote the complex

$$\cdots \xrightarrow{\ b\ } B^{\otimes n} \xrightarrow{\ b\ } B^{\otimes(n-1)} \xrightarrow{\ b\ } \cdots \xrightarrow{\ b\ } \mathbb{Q} \longrightarrow 0,$$

where $b : B^{\otimes n} \to B^{\otimes(n-1)}$ is the map that sends $b_1 \otimes \cdots \otimes b_n$ to

$$\varepsilon(b_1) \cdot b_2 \otimes \cdots \otimes b_n + \sum_{1 \le i \le n-1} (-1)^i b_1 \otimes \cdots \otimes b_i \cdot b_{i+1} \otimes \cdots \otimes b_n + (-1)^n \varepsilon(b_n) \cdot b_1 \otimes \cdots \otimes b_{n-1}.$$

Let $H_*(B, \mathbb{Q})$ denote the homology of this complex.

Then the natural maps

$$H_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \simeq H_*(\mathcal{U}(\mathfrak{t}_\sigma(A, I)), \mathbb{Q}) \simeq H_*(\hat{\mathcal{U}}(\mathfrak{t}_\sigma(A, I)), \mathbb{Q}) \tag{3.4.1}$$

are isomorphisms [Loday 1992, Theorem 3.3.2]. Here the first map is induced by the chain map $\alpha_{\mathrm{as}}$, where "as" stands for antisymmetrization, defined by

$$\alpha_{\mathrm{as}}(t_1 \wedge \cdots \wedge t_n) = \sum_{\tau \in S_n} \operatorname{sign}(\tau) \cdot t_{\tau(1)} \otimes \cdots \otimes t_{\tau(n)}.$$

**3.5.** ***Malčev theory.*** For $\sigma \in S_n$, let $T_\sigma(A, I) \subseteq \mathrm{GL}_n(A)$ denote the group

$$\{1 + (a_{ij}) \in \mathrm{GL}_n(A) \mid a_{ij} \in I \text{ if } \sigma(j) \le \sigma(i)\}.$$

For a discrete group $G$, denote by $U(G)$ its group ring over $\mathbb{Q}$, and by $\hat{U}(G)$ its completion with respect to the augmentation ideal.

Since $T_\sigma(A, I)$ is a unipotent group with Lie algebra $\mathfrak{t}_\sigma(A, I)$, the natural maps

$$H_*(\hat{\mathcal{U}}(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) = H_*(\hat{U}(T_\sigma(A, I), \mathbb{Q}) \simeq H_*(U(T_\sigma(A, I)), \mathbb{Q}). \tag{3.5.1}$$

are isomorphisms [Loday 1992, Section 11.3.13].

Combining (3.3.1), (3.4.1) and (3.5.1) we get a map

$$\mathrm{HC}_{*-1}(A, I) \to \sum_{\sigma} H_*(U(T_\sigma(A, I)), \mathbb{Q}) \to H_*(U(\mathrm{GL}(A)), \mathbb{Q}). \tag{3.5.2}$$

**3.6.** ***Group homology.*** Let $G$ be any (discrete) group and $C_*(G, \mathbb{Q})$ the complex

$$\cdots \xrightarrow{\ d\ } \mathbb{Q}[G^{n+1}] \xrightarrow{\ d\ } \mathbb{Q}[G^n] \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \mathbb{Q}[G] \longrightarrow 0,$$

where $C_n(G, \mathbb{Q}) = \mathbb{Q}[G^{n+1}]$ and $d$ is the map that sends $(g_0, g_1, \ldots, g_n)$ to

$$\sum_{0 \le i \le n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n).$$

Let $G$ act on this complex by multiplication on the left, that is, $g \times (g_0, \ldots, g_n) := (g \cdot g_0, \ldots, g \cdot g_n)$, and let $H_*(G, \mathbb{Q}) := H_*(C_*(G, \mathbb{Q})_G)$ where the subscript $G$ denotes the space of coinvariants.

The natural map $C_*(U(G), \mathbb{Q}) \to C_*(G, \mathbb{Q})$ that sends $g_1 \otimes g_2 \otimes \cdots \otimes g_n$ to

$$(1, g_1, g_1 \cdot g_2, \ldots, g_1 \cdot g_2 \cdots g_n)$$

induces an isomorphism $H_*(U(G), \mathbb{Q}) \to H_*(G, \mathbb{Q})$ [Loday 1992, Appendix C.3]. Applying this to $\mathrm{GL}(A)$ and combining with (3.5.2) we obtain the map

$$\mathrm{HC}_{*-1}(A, I) \to H_*(\mathrm{GL}(A), \mathbb{Q}). \tag{3.6.1}$$

**3.7. *Suslin's stability theorem.*** Suslin's stability theorem [1990] was generalized by Guin:

**Theorem 3.7.1** [Guin 1989, Section 2]. *For any $1 \le n$ and any artinian local algebra $A$ over $\mathbb{Q}$, the map $H_n(\mathrm{GL}_n(A), \mathbb{Q}) \to H_n(\mathrm{GL}(A), \mathbb{Q})$ induced by the inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}$ is an isomorphism.*

Therefore if $A$ is an artinian local algebra over $\mathbb{Q}$ and $I$ is a proper ideal, we have a map $\rho_1 : \mathrm{HC}_{n-1}(A, I) \to H_n(\mathrm{GL}_n(A), \mathbb{Q})$.

**3.8. *Bloch–Suslin map.*** Let $A$ be an artinian local algebra over $\mathbb{Q}$ with residue field $k$. We now describe the Bloch–Suslin map [Goncharov 1995, Section 2.6]

$$\rho_2 : H_3(\mathrm{GL}_3(A), \mathbb{Q}) \to \ker(\delta_A),$$

where $\delta_A : B_2(A)_{\mathbb{Q}} \to \bigwedge^2 A^\times_{\mathbb{Q}}$ is the differential in the Bloch complex.

**Definition 3.8.1.** Let $V$ be a finite free module over $A$, and denote $\tilde{C}_m(V)$ by the $\mathbb{Q}$-vector space with basis consisting of $m$-tuples $(x_0, \ldots, x_{m-1})$ of elements of $V$ that are in general position, that is, for any $I \subseteq \{0, 1, \ldots, m-1\}$ with $|I| \le \mathrm{rank}(V)$, the set $\{x_i \mid i \in I\}$ can be extended to a basis of $V$. Let $C_m(V)$ denote the coinvariants of this space under the natural action of $\mathrm{GL}(V)$. Finally, let $\tilde{C}_m(p) := \tilde{C}_m(A^{\oplus p})$ and $C_m(p) := C_m(A^{\oplus p})$.

**Remark 3.8.2.** Let $\tilde{C}_m(\mathbb{P}(V))$ denote the $\mathbb{Q}$-space with basis $(v_0, \ldots, v_{m-1})$ of $m$-tuples of points in $\mathbb{P}(V)$ that are in general position, and define

$$d : \tilde{C}_{m+1}(\mathbb{P}(V)) \to \tilde{C}_m(\mathbb{P}(V)), \quad (v_0, \ldots, v_m) \mapsto \sum_{0 \le i \le m} (-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_m).$$

Let $C_m(\mathbb{P}(V))$ denote the space of coinvariants of $\tilde{C}_m(\mathbb{P}(V))$ under the natural action of $\mathrm{GL}(V)$. Then by identifying $[x]$ with $(0, x, 1, \infty) \in C_4(\mathbb{P}(A^{\oplus 2}))$ and by comparing the dilogarithm relation in the definition of $B_2(A)$ to $d(0, x, y, 1, \infty)$ in $C_4(\mathbb{P}(A^{\oplus 2}))$, we see that

$$B_2(A)_{\mathbb{Q}} = C_4(\mathbb{P}(A^{\oplus 2}))/d(C_5(\mathbb{P}(A^{\oplus 2}))).$$

For $(x_1, \ldots, x_4)$ a quadruple of points in $\mathbb{P}^1_A$, we denote the corresponding element in $B_2(A)_{\mathbb{Q}}$ by $[x_1, \ldots, x_4]$.

Since $A$ is a local ring, a subset of $V$ is in general position if its reduction modulo the maximal ideal is in general position in the $k$-space $V \otimes_A k$.

Define two maps $d, d' : \tilde{C}_{m+1}(p) \to \tilde{C}_m(p)$ by

$$d(x_0, x_1, \ldots, x_m) = \sum_{0 \le i \le m} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_m),$$
$$d'(x_0, x_1, \ldots, x_m) = \sum_{1 \le i \le m} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_m).$$

Let $\varepsilon : \tilde{C}_1(p) \to \mathbb{Q}$ be the map that sends each term to the sum of its coefficients.

**Lemma 3.8.3.** *The following complexes are acyclic.*

$$\cdots \xrightarrow{d} \tilde{C}_2(p) \xrightarrow{d} \tilde{C}_1(p) \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0,$$
$$\cdots \xrightarrow{d'} \tilde{C}_2(p) \xrightarrow{d'} \tilde{C}_1(p) \longrightarrow 0.$$

*Proof.* Let $\sum_{j \in J} a_j \cdot (x_0(j), \ldots, x_{m-1}(j))$ be an $m$-cycle in the first or the second complex. Since the reductions modulo the maximal ideal $\{\bar{x}_0(j), \ldots, \bar{x}_{m-1}(j)\}$ are in general position in $k^{\oplus p}$ and $k$ is an infinite field, we can choose $\alpha \in A$ such that all $\{\bar{x}_0(j), \ldots, \bar{x}_{m-1}(j), \bar{\alpha}\}$ are in general position. Note that if $W_i$ for $1 \le i \le r$ are proper subspaces of a vector space $W$ over an infinite field, then $\bigcup_{1 \le i \le r} W_i \ne W$. If $m \ge 2$ and $d \sum_{j \in J} a_j \cdot (x_0(j), \ldots, x_{m-1}(j)) = 0$, or if $m = 1$ and $\sum_{j \in J} a_j = 0$, we have

$$(-1)^m d\Big(\sum_{j \in J} a_j \cdot (x_0(j), \ldots, x_{m-1}(j), \alpha)\Big) = \sum_{j \in J} a_j \cdot (x_0(j), \ldots, x_{m-1}(j)).$$

Similarly, if $m \ge 2$ and $d' \sum_{j \in J} a_j \cdot (x_0(j), \ldots, x_{m-1}(j)) = 0$, or if $m = 1$, we have

$$(-1)^m d'\Big(\sum_{j \in J} a_j \cdot (x_0(j), \ldots, x_{m-1}(j), \alpha)\Big) = \sum_{j \in J} a_j \cdot (x_0(j), \ldots, x_{m-1}(j)). \quad \square$$

Define maps $\lambda : \tilde{C}_m(p) \to \tilde{C}_m(p)$ by

$$\lambda(x_0, \ldots, x_{m-1}) = \sum_{0 \le i \le m-1} \operatorname{sign}(\sigma(m)^i)(x_{\sigma(m)^i(0)}, \ldots, x_{\sigma(m)^i(m-1)}),$$

where $\sigma(m) := (0\ 1\ \cdots\ m-1)$ is the standard $m$-cyclic permutation.

Then $\lambda \circ d = d' \circ \lambda$, and we have a double complex

$$\begin{array}{ccccc}
\cdots \xrightarrow{d} & \tilde{C}_3(3) & \xrightarrow{d} & \tilde{C}_2(3) \\
& \downarrow \lambda & & \downarrow \lambda \\
\cdots \xrightarrow{d'} & \tilde{C}_3(3) & \xrightarrow{d'} & \tilde{C}_2(3).
\end{array}$$

**Definition 3.8.4.** Let $\tilde{D}$ be the complex associated to the double complex above. That is, $\tilde{D}_0 = \tilde{C}_2(3)$ and $\tilde{D}_i = \tilde{C}_{i+2}(3) \oplus \tilde{C}_{i+1}(3)$ and the maps are given by $(x, y) \to (d'(x) + \lambda(y), -d(y))$.

Let $\varepsilon : \tilde{D}_0 \to \mathbb{Q}$ be the map that sends each term to the sum of its coefficients. Then by Lemma 3.8.3 the complex $\cdots \longrightarrow \tilde{D}_1 \longrightarrow \tilde{D}_0 \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$ is acyclic.

If we endow $\tilde{D}$ with its natural $\mathrm{GL}_3(A)$ action and $\mathbb{Q}$ with the trivial action, then the complex above is an acyclic complex of $\mathrm{GL}_3(A)$-modules. Therefore we get a canonical map

$$H_3(\mathrm{GL}_3(A), \mathbb{Q}) \to H_3(D), \tag{3.8.1}$$

where $D := \tilde{D}_{\mathrm{GL}_3(A)}$ is the complex of coinvariants of $\tilde{D}$.

Next we define a map from $H_3(D)$ to $B_2(A)_{\mathbb{Q}}$. This will be a slight modification of Goncharov's map [1995, Section 2.6].

From the double complex above, we are interested in the part

$$\begin{array}{ccccc} C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ C_6(3) & \xrightarrow{d'} & C_5(3) & \xrightarrow{d'} & C_4(3). \end{array}$$

We define a map $\phi$ from this double complex to the double complex

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_2(A)_{\mathbb{Q}} & \longrightarrow & \bigwedge^2 A^{\times}_{\mathbb{Q}}. \end{array}$$

In $\phi$, the only nontrivial map

$$\begin{array}{ccc} C_5(3) & \xrightarrow{d'} & C_4(3) \\ \downarrow & & \downarrow \\ B_2(A)_{\mathbb{Q}} & \xrightarrow{\delta} & \bigwedge^2 A^{\times}_{\mathbb{Q}} \end{array} \tag{3.8.2}$$

is a composition of the following two maps:

The first map is

$$\begin{array}{ccc} C_5(3) & \xrightarrow{d'} & C_4(3) \\ \downarrow{\scriptstyle -p} & & \downarrow{\scriptstyle p} \\ C_4(2) & \xrightarrow{d} & C_3(2), \end{array}$$

where $p : C_{m+1}(3) \to C_m(2)$ sends $(v_0, v_1, \ldots, v_{m-1})$ to $(\tilde{v}_1, \ldots, \tilde{v}_{m-1})$. Here $\tilde{v}_i$ denotes the image of $v_i$ in $A^{\oplus 3}/\langle v_0 \rangle$, and the term $(\tilde{v}_1, \ldots, \tilde{v}_{m-1})$ is identified with an element of $C_m(2)$ by choosing any isomorphism between $A^{\oplus 3}/\langle v_0 \rangle$ and $A^{\oplus 2}$.

The second map is

$$\begin{array}{ccc} C_4(2) & \xrightarrow{d} & C_3(2) \\ \alpha\downarrow & & \downarrow\beta \\ B_2(A)_{\mathbb{Q}} & \xrightarrow{\delta} & \bigwedge^2 A^{\times}_{\mathbb{Q}}, \end{array} \tag{3.8.3}$$

where $\alpha$ is the map that sends $(v_0, v_1, v_2, v_3)$ to $[\underline{v}_0, \underline{v}_1, \underline{v}_2, \underline{v}_3]$. Here $\underline{v}_i$ denotes the image of $v_i$ in $\mathbb{P}(A^{\oplus 2})$, and $[\underline{v}_0, \underline{v}_1, \underline{v}_2, \underline{v}_3]$ denotes the image of $(\underline{v}_0, \underline{v}_1, \underline{v}_2, \underline{v}_3)$ under the map $C_4(\mathbb{P}(A^{\oplus 2})) \to B_2(A)_{\mathbb{Q}}$, as in Remark 3.8.2. And $\beta$ is the map that sends $(v_0, v_1, v_2)$ to

$$\Big(\frac{v_0 \wedge v_1}{v_1 \wedge v_2}\Big) \wedge \Big(\frac{v_0 \wedge v_2}{v_1 \wedge v_2}\Big).$$

The next three lemmas imply that the maps defined so far can be extended to a map $\phi$ of the double complexes.

**Lemma 3.8.5.** *The map* $C_6(3) \xrightarrow{d'} C_5(3) \xrightarrow{-p} C_4(2) \xrightarrow{\alpha} B_2(A)_{\mathbb{Q}}$ *is zero.*

*Proof.* This follows from that $-p \circ d'(v_0, v_1, v_2, v_3, v_4, v_5) = d(v_1, v_2, v_3, v_4, v_5)$, and that this maps to zero in $B_2(A)_{\mathbb{Q}}$, by Remark 3.8.2. □

**Lemma 3.8.6.** *The map* $C_5(3) \xrightarrow{\lambda} C_5(3) \xrightarrow{-p} C_4(2) \xrightarrow{\alpha} B_2(A)_{\mathbb{Q}}$ *is zero.*

*Proof.* (See [Goncharov 1995, Lemma 2.18].) Let $(v_0, \dots, c_4) \in C_5(3)$. Then there is a conic $Q$ passing through the images of the five points $v_0, v_1, v_2, v_3, v_4$ in the projective plane. Choosing any isomorphism, we identify $Q$ with $\mathbb{P}^1_A$. Let the images of $v_i$ be $x_i \in \mathbb{P}^1_A$ under this isomorphism. The composition of the maps in the statement of the lemma then maps $(v_0, \dots, v_4)$ in $C_5(3)$ to

$$-\sum_{0 \le i \le 4} [x_i, x_{i+1}, \dots, x_{i+3}]$$

in $B_2(A)_{\mathbb{Q}}$, where the indices are modulo 5.

**Claim 3.8.7.** *In* $B_2(A)_{\mathbb{Q}}$ *we have*

$$[x_1, x_2, x_3, x_4] = \operatorname{sign}(\sigma) \cdot [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] \quad \textit{for any } \sigma \in S_4.$$

*Proof of the claim.* Note that since we are working with $\mathbb{Q}$-modules we have $[0, x, 1, \infty] = -[0, x/(x-1), 1, \infty]$ by [Suslin 1990, Lemmas 1.2 and 1.5]. Hence

$$[0, x, 1, \infty] = -[x, 0, 1, \infty] \quad \text{and} \quad [0, x, 1, \infty] = -[0, 1/x, 1, \infty]$$

[loc. cit., Lemma 1.2]. Hence $[0, x, 1, \infty] = -[0, 1, x, \infty]$, and using again that $[0, x, 1, \infty] = -[0, x/(x-1), 1, \infty]$, we have $[0, x, 1, \infty] = -[0, x, \infty, 1]$.

Therefore the formula in the statement holds for the transpositions (1 2), (2 3), and (3 4). Since these generate $S_4$, the statement follows. □

Finally by the claim,

$$\sum_{0\le i\le 4}[x_i, x_{i+1},\ldots,x_{i+3}] = \sum_{0\le i\le 4}(-1)^i[x_0,\ldots,\hat{x}_i,\ldots,x_4]$$

and the right side is zero in $B_2(A)_{\mathbb{Q}}$ by Remark 3.8.2. □

**Lemma 3.8.8.** *The map* $C_4(3) \xrightarrow{\lambda} C_4(3) \xrightarrow{p} C_3(2) \xrightarrow{\beta} \bigwedge^2 A^{\times}_{\mathbb{Q}}$ *is zero.*

*Proof.* First note that $\beta$ sends $(v_0, v_1, v_2)$ to

$$\Big(\frac{v_0\wedge v_1}{v_1\wedge v_2}\Big)\wedge\Big(\frac{v_0\wedge v_2}{v_1\wedge v_2}\Big) = \Big(\frac{\ell(v_0\wedge v_1)}{\ell(v_1\wedge v_2)}\Big)\wedge\Big(\frac{\ell(v_0\wedge v_2)}{\ell(v_1\wedge v_2)}\Big),$$

where $\ell : \det_A(A^{\oplus 2}) \to A$ is any surjective $A$-linear map. Therefore since we are looking at configurations in general position, the composition $\beta\circ p$ maps $(y_0, y_1, y_2, y_3)\in C_4(3)$ to

$$\Big(\frac{y_0\wedge y_1\wedge y_2}{y_0\wedge y_2\wedge y_3}\Big)\wedge\Big(\frac{y_0\wedge y_1\wedge y_3}{y_0\wedge y_2\wedge y_3}\Big).$$

This implies the statement by direct computation. □

Therefore $\phi$ is a map of double complexes that induces a map $H_3(D)\to\ker(\delta)$ of the homology of the associated complexes. Combining this with the map

$$H_3(\mathrm{GL}_3(A),\mathbb{Q})\to H_3(D)$$

in (3.8.1), we obtain a map $\rho_2 : H_3(\mathrm{GL}_3(A),\mathbb{Q})\to\ker(\delta)$.

Therefore applying Sections 3.1–3.7 to $(A, I) = (k[\varepsilon]_n, (\varepsilon))$ proves this:

**Proposition 3.8.9.** *The composition* $T := \rho_2\circ\rho_1$ *defines a natural map*

$$T : \mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon))^{(1)} \hookrightarrow \mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon)) \to B_2(k[\varepsilon]_n)_{\mathbb{Q}},$$

*whose image lies in* $\ker(\delta_n)$.

## 4. Nonvanishing of $\mathbf{Li}_{2,n}$ on $\mathbf{HC_2}(k[\varepsilon]_n,(\varepsilon))^{(1)}$

**4.1.** This section shows that $\mathrm{Li}_{2,n}$ is the correct map, as we show that it does not vanish on $\mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon))^{(1)}$. First we describe $\mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon))^{(1)}$ and define some elements $\alpha_w$ in it on which we will evaluate the additive dilogarithm.

**4.1.1.** Note that $\mathrm{HC}_*(k[\varepsilon]_n,(\varepsilon))$ is a $k^{\times}$-abelian group, where $\lambda\in k^{\times}$ acts as the map induced by the $k$-algebra automorphism of $k[\varepsilon]_n$ that sends $\varepsilon$ to $\lambda\cdot\varepsilon$. This action is compatible with the decomposition (Remark 3.1.2) of

$$\mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon)) = \mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon))^{(1)}\oplus\mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon))^{(2)}$$

[Cathelineau 1990/91, pages 593–594];

$$\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)} = \bigoplus_{n+1\le w\le 2n-1} \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}_{\langle w\rangle},$$

where each summand is isomorphic to $k$ (loc. cit.); and

$$\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(2)} = \bigoplus_{1\le w\le n-1} \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(2)}_{\langle w\rangle},$$

where each summand is isomorphic to $\Omega^2_{k/\mathbb{Q}}$ (loc. cit.).

**4.1.2.** $\chi(n) = 0$ if $n$ is even and $\chi(n) = 1$ if $n$ is odd. For $n+1 \le w \le 2n-1$, let

$$\alpha_w := \sum_{0\le j<(2n-1-w)/2} (\varepsilon^{n-1-j}, \varepsilon^{w-n+j}, \varepsilon) + \tfrac{1}{2}\cdot\chi(w)\cdot(\varepsilon^{(w-1)/2}, \varepsilon^{(w-1)/2}, \varepsilon)$$

in $C_2^{\lambda}(k[\varepsilon]_n)$. Since $\alpha_w$ is a cycle, as can be checked by direct computation, with $k^{\times}$-weight $w$, it defines an element $\alpha_w \in \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}_{\langle w\rangle}$ by Section 4.1.1.

**4.2. *Computation of* $\mathbf{Li_{2,n}}$ *on* $\mathbf{HC_2}$.** In this section, we compute $\mathrm{Li}_{2,n}(T(\alpha_w))$ (which is the same as $\mathrm{Li}_{2,n,w}(T(\alpha_w))$). This we will do in several steps.

**4.2.1.** *From* $\mathfrak{gl}_3(k[\varepsilon]_n)$ *to* $\mathfrak{gl}_2(k[\varepsilon]_n)$. Consider the 2-chain $(\varepsilon^a, \varepsilon^b, \varepsilon) \in C_2^{\lambda}(k[\varepsilon]_n)$ in the Connes complex, where $a+b \ge n$. By the map in Section 3.2, at the chain complex level, $(\varepsilon^a, \varepsilon^b, \varepsilon)$ goes to $\beta_{a,b} := \varepsilon^a e_{12}\wedge\varepsilon^b e_{23}\wedge\varepsilon e_{31} \in C_3(\mathfrak{gl}_3(k[\varepsilon]_n))_{\mathfrak{gl}_3(\mathbb{Q})}$. Therefore we need to compute the image of

$$\beta_w := \sum_{0\le j<(2n-1-w)/2} \beta_{n-1-j,w-n+j} + \tfrac{1}{2}\chi(w)\beta_{(w-1)/2,(w-1)/2}$$

in $k$. Let $\gamma_{a,b} := \varepsilon^a e_{12}\wedge\varepsilon^b e_{21}\wedge\varepsilon e_{11}$, and

$$\gamma_w := \sum_{0\le j<(2n-1-w)/2} \gamma_{n-1-j,w-n+j} + \tfrac{1}{2}\chi(w)\gamma_{(w-1)/2,(w-1)/2}.$$

We defined $T$ as the composition

$$\begin{aligned}\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)} &\to \mathrm{Prim}\, H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \simeq \mathrm{Prim}\, H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})\\ &\to H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \to H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q}) \to \ker(\delta).\end{aligned}$$

Let $T' : \mathrm{Prim}\, H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \to \ker(\delta)$ and $T'' : H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \to \ker(\delta)$ be the obvious compositions.

The following lemma enables us to work in the homology of $\mathfrak{gl}_2(k[\varepsilon]_n)$ rather than that of $\mathfrak{gl}_3(k[\varepsilon]_n)$.

**Lemma 4.2.1.** *We have* $(\mathrm{Li}_{2,n,w}\circ T')(\beta_w) = (\mathrm{Li}_{2,n,w}\circ T')(\gamma_w)$.

*Proof.* First note that

$$d(e_{13}\wedge\varepsilon^a e_{12}\wedge\varepsilon^b e_{21}\wedge\varepsilon e_{31}) = -\beta_{a,b}+\gamma_{a,b} \\ -\varepsilon^a e_{12}\wedge\varepsilon^b e_{21}\wedge\varepsilon e_{33} - e_{13}\wedge\varepsilon^{a+1}e_{32}\wedge\varepsilon^b e_{21},$$

that $\varepsilon^a e_{12}\wedge\varepsilon^b e_{21}\wedge\varepsilon e_{33}$ is a cycle; and that $e_{13}\wedge\varepsilon^{a+1}e_{32}\wedge\varepsilon^b e_{21}$ is a boundary in $C_*(\mathfrak{gl}(k[\varepsilon]_n))_{\mathfrak{gl}(\mathbb{Q})}$, since this element corresponds to the element $(1,\varepsilon^{a+1},\varepsilon^b)$ in the Connes complex and $d(1,\varepsilon^{a+1},\varepsilon^b,1)=(1,\varepsilon^{a+1},\varepsilon^b)$.

Therefore since $\beta_w$ is a cycle, so is $\gamma_w$, and to prove the lemma it suffices to show that $(\mathrm{Li}_{2,n,w}\circ T')(\varepsilon^a e_{12}\wedge\varepsilon^b e_{21}\wedge\varepsilon e_{33})=0$ for $a+b\geq n$.

Note that since

$$d(e_{12}\wedge\varepsilon^a e_{11}\wedge\varepsilon^b e_{21}\wedge\varepsilon e_{33}) = -\varepsilon^a e_{12}\wedge\varepsilon^b e_{21}\wedge\varepsilon e_{33} \\ +\varepsilon^a e_{11}\wedge\varepsilon^b e_{11}\wedge\varepsilon e_{33} - \varepsilon^a e_{11}\wedge\varepsilon^b e_{22}\wedge\varepsilon e_{33},$$

it is sufficient to show the vanishing of both

$$(\mathrm{Li}_{2,n,w}\circ T')(\varepsilon^a e_{11}\wedge\varepsilon^b e_{11}\wedge\varepsilon e_{33}) = (\mathrm{Li}_{2,n,w}\circ T'')(\varepsilon^a e_{11}\wedge\varepsilon^b e_{11}\wedge\varepsilon e_{33}),$$
$$(\mathrm{Li}_{2,n,w}\circ T')(\varepsilon^a e_{11}\wedge\varepsilon^b e_{22}\wedge\varepsilon e_{33}) = (\mathrm{Li}_{2,n,w}\circ T'')(\varepsilon^a e_{11}\wedge\varepsilon^b e_{22}\wedge\varepsilon e_{33}).$$

The equalities above follow immediately from the fact that $\varepsilon^a e_{11}\wedge\varepsilon^b e_{11}\wedge\varepsilon e_{33}$ and $\varepsilon^a e_{11}\wedge\varepsilon^b e_{22}\wedge\varepsilon e_{33}$ are cycles not only in $C_*(\mathfrak{gl}_3(k[\varepsilon]_n),\mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})}$ but also in $C_*(\mathfrak{gl}_3(k[\varepsilon]_n),\mathbb{Q})$.

Propositions 2.2.1 and 2.2.2 give that $\mathrm{Li}_{2,n,w}(x)=(\mathrm{li}_{2,(2n-1,n),w}\circ\delta_{2n-1})(\tilde{x})$ for $x\in B_2(k[\varepsilon]_n)$, where $\tilde{x}\in B_2(k[\varepsilon]_{2n-1})$ is any lift of $x$.

Let $\tilde{\alpha}\in\{\varepsilon^a e_{11}\wedge\varepsilon^b e_{11}\wedge\varepsilon e_{33},\ \varepsilon^a e_{11}\wedge\varepsilon^b e_{22}\wedge\varepsilon e_{33}\}\subseteq C_3(\mathfrak{gl}(k[\varepsilon]_{2n-1}),\mathbb{Q})$, and let $\alpha$ the reduction of $\tilde{\alpha}$ to $C_3(\mathfrak{gl}(k[\varepsilon]_n),\mathbb{Q})$. Then

$$\mathrm{Li}_{2,n,w}(T''(\alpha)) = (\mathrm{li}_{2,(2n-1,n),w}\circ\delta_{2n-1})(\underline{T}''(\tilde{\alpha})).$$

Here $\underline{T}''$ denotes the chain map, mapping $C_3(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}),\mathbb{Q})$ to $B_2(k[\varepsilon]_{2n-1})_{\mathbb{Q}}$ and $C_2(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}),\mathbb{Q})$ to $\bigwedge^2 V_{2n-1}$, that induces $T''$. The map $\underline{T}''$ depends on certain choices (see the next paragraph).

Recall how $\underline{T}''(\tilde{\alpha})$ is defined in Section 3: Through the antisymmetrization map $\alpha_{\mathrm{as}}$ (Section 3.4) and the exponential map [Loday 1992, Sections 3.5 and 11.3.13], we get a chain map $C_*(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}),\mathbb{Q})^\circ\to C_*(\hat{U}(\mathrm{GL}_3(k[\varepsilon]_{2n-1})),\mathbb{Q})$. In fact, it is immediately seen that the image of $\tilde{\alpha}$ under these maps lies inside the image of $C_*(U(\mathrm{GL}_3(k[\varepsilon]_{2n-1})),\mathbb{Q})$ in $C_*(\hat{U}(\mathrm{GL}_3(k[\varepsilon]_{2n-1})),\mathbb{Q})$. To get from $C_*(U(\mathrm{GL}_3(k[\varepsilon]_{2n-1})),\mathbb{Q})$ to $C_*(\mathrm{GL}_3(k[\varepsilon]_{2n-1}),\mathbb{Q})$ we pass via the map described in Section 3.6. Bypassing the need for stabilization since we are already in $\mathrm{GL}_3$, and using that $\tilde{D}$ is an acyclic complex of $\mathrm{GL}_3(k[\varepsilon]_{2n-1})$ modules, we get a (non-canonical) map from $C_*(\mathrm{GL}_3(k[\varepsilon]_{2n-1}),\mathbb{Q})$ to $\tilde{D}$. Finally taking $\mathrm{GL}_3(k[\varepsilon]_{2n-1})$

coinvariants, we end up in the complex $D$, and using the map of double complexes (induced by (3.8.2)), we pass from $D$ to the complex

$$\gamma_{k[\varepsilon]_{2n-1}}(2)_{\mathbb{Q}} : B_2(k[\varepsilon]_{2n-1})_{\mathbb{Q}} \xrightarrow{\delta_{2n-1}} \bigwedge^2 V_{2n-1}.$$

Since $\underline{T}''$ is a map of complexes, $\delta_{2n-1}(\underline{T}''(\tilde{\alpha})) = \underline{T}''(d(\tilde{\alpha})) = 0$, as $d(\tilde{\alpha}) = 0$ in $C_*(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$. This implies that

$$\mathrm{Li}_{2,n,w}(T''(\alpha)) = \mathrm{li}_{2,(2n-1,n),w}(\delta_{2n-1}(T''(\tilde{\alpha}))) = 0. \qquad \square$$

The next lemma will help us to reduce the computation to $\mathfrak{gl}_2$:

**Lemma 4.2.2.** *The chain $\gamma_w$ as defined above is a cycle in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ and hence defines an element in $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$.*

*Proof.* We already know that $\gamma_w$ defines a cycle in $C_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})}$. Since $C_i(\mathfrak{gl}_m(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_m(\mathbb{Q})} = (\bigwedge^i \mathfrak{gl}_m(k[\varepsilon]_n))_{\mathfrak{gl}_m(\mathbb{Q})}$ (see Section 3.2.2) and

$$(\bigwedge^i \mathfrak{gl}_m(k[\varepsilon]_n))_{\mathfrak{gl}_m(\mathbb{Q})} = (\bigwedge^i \mathfrak{gl}_i(k[\varepsilon]_n))_{\mathfrak{gl}_i(\mathbb{Q})} \quad \text{for } m \geq i$$

[Loday 1992, Corollary 9.2.8 and (10.2.10.1)], we have

$$d(\gamma_w) = 0 \in C_2(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})} = C_2(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}. \qquad \square$$

**4.2.2.** *From $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ to $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$.* In order to continue with the computation of $\mathrm{Li}_{2,n,w}(T'(\gamma_w))$, we need to compute the image of $\gamma_w$ in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n, \mathbb{Q})$. This would be a very long computation, but in fact we will see in this section that we can get away with much less. The following proposition will be crucial.

**Proposition 4.2.3.** *For any $\mathbb{Q}$-algebra $A$, let $\mathfrak{gl}_n(\mathbb{Q})$ act on $\mathfrak{gl}_n(A)$ by its adjoint action. Let $C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})}$ be the subcomplex of $C_*(\mathfrak{gl}_n(A), \mathbb{Q})$ on which $\mathfrak{gl}_n(\mathbb{Q})$ acts trivially. Then the canonical map*

$$C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})} \to C_*(\mathfrak{gl}_n(A), \mathbb{Q}) \to C_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})}$$

*is an isomorphism and there is a canonical direct sum of complexes*

$$C_*(\mathfrak{gl}_n(A), \mathbb{Q}) = C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})} \oplus L_*, \tag{4.2.1}$$

*with $\mathfrak{gl}_n(\mathbb{Q})$-action, such that $L_*$ is acyclic.*

*Proof.* This is [Loday 1992, Proposition 10.1.8], taking $\mathfrak{g} = \mathfrak{gl}_n(A)$ and $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{Q})$, and noting the reductivity of $\mathfrak{gl}_n(\mathbb{Q})$ [loc. cit., 10.2.9]. $\square$

To continue, we need to compute the image $\gamma'_w$ of $\gamma_w$ in $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})$. Then we should lift $\gamma'_w$ to a chain $\tilde{\gamma}'_w$ in $C'_3(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ and continue just as in the last part of the proof of Lemma 4.2.1. Namely,

$$\begin{aligned}\mathrm{Li}_{2,n,w}(T'(\gamma_w)) &= \mathrm{Li}_{2,n,w}(\underline{T}''(\tilde{\gamma}'_w)) = \mathrm{li}_{2,(2n-1,n),w}(\delta_{2n-1}(\underline{T}''(\tilde{\gamma}'_w))) \\ &= \mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(d(\tilde{\gamma}'_w))).\end{aligned}$$

Let $\tilde{\gamma}^*_w$ be any chain in $C_3(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$ that has a cycle for its image in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ (under the canonical maps) and that lifts $\gamma_w$. Then, by Proposition 4.2.3, the first component $\tilde{\gamma}^{*(1)}_w$ of $\tilde{\gamma}^*_w$ under the decomposition in (4.2.1) is a lift of the element $\gamma'_w$. Therefore we can choose $\tilde{\gamma}'_w := \tilde{\gamma}^{*(1)}_w$, and to continue we need to compute $d(\tilde{\gamma}^{*(1)}_w) = d(\tilde{\gamma}^*_w)^{(1)}$.

For the rest of the computation, we will let $\tilde{\gamma}^*_w := \tilde{\gamma}_w$, where

$$\begin{aligned}\tilde{\gamma}_w &:= \sum_{0\le j<(2n-1-w)/2} \tilde{\gamma}_{n-1-j,w-n+j} + \tfrac{1}{2}\chi(w)\tilde{\gamma}_{(w-1)/2,(w-1)/2}, \\ \tilde{\gamma}_{a,b} &:= \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{11}.\end{aligned}$$

Combining the above we have

$$\mathrm{Li}_{2,n,w}(T'(\gamma_w)) = \mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(d(\tilde{\gamma}_w)^{(1)})). \tag{4.2.2}$$

Next we will compute $d(\tilde{\gamma}_{a,b})^{(1)}$. For any $\mathbb{Q}$-algebra $A$ there is a canonical isomorphism for $i \ge n$ given by

$$C_n(\mathfrak{gl}_i(A), \mathbb{Q})_{\mathfrak{gl}_i(\mathbb{Q})} = \left(\textstyle\bigwedge^n \mathfrak{gl}_i(A)\right)_{\mathfrak{gl}_i(\mathbb{Q})} \to (\mathbb{Q}[S_n] \otimes A^{\otimes n})_{S_n}, \tag{4.2.3}$$

where $S_n$ acts on $\mathbb{Q}[S_n]$ by conjugation and on $A^{\otimes n}$ by permuting the factors and multiplying with sign [Loday 1992, 10.2.10.1].

Letting

$$\Gamma_{x,y} := xe_{12} \wedge ye_{21} + xe_{21} \wedge ye_{12} + \tfrac{1}{2}x(e_{22} - e_{11}) \wedge y(e_{22} - e_{11}) \quad \text{for } x, y \in A,$$

we see by direct computation that $\Gamma_{x,y} \in C'_2(\mathfrak{gl}_2(A), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$.

Under the map (4.2.3),

$$\begin{aligned}\Gamma_{x,y} \mapsto (3\cdot\tau)\otimes(x\otimes y), \qquad x(e_{11}-e_{22}) \wedge ye_{11} &\mapsto (1\cdot\tau)\otimes(x\otimes y), \\ xe_{21} \wedge ye_{12} &\mapsto (1\cdot\tau)\otimes(x\otimes y), \\ xe_{12} \wedge ye_{21} &\mapsto (1\cdot\tau)\otimes(x\otimes y),\end{aligned}$$

where $S_2 = \{\mathrm{id}, \tau\}$. Therefore, using Proposition 4.2.3, we have

$$(x(e_{11} - e_{22}) \wedge ye_{11})^{(1)} = (xe_{21} \wedge ye_{12})^{(1)} = (xe_{12} \wedge ye_{21})^{(1)} = \tfrac{1}{3}\Gamma_{x,y}.$$

Since $d(\tilde{\gamma}_{a,b}) = \varepsilon^{a+b}(e_{11} - e_{22}) \wedge \varepsilon e_{11} - \varepsilon^b e_{21} \wedge \varepsilon^{a+1} e_{12} - \varepsilon^a e_{12} \wedge \varepsilon^{b+1} e_{21}$, we have

$$d(\tilde{\gamma}_{a,b})^{(1)} = \tfrac{1}{3}(\Gamma_{\varepsilon^{a+b},\varepsilon} - \Gamma_{\varepsilon^b,\varepsilon^{a+1}} - \Gamma_{\varepsilon^a,\varepsilon^{b+1}}). \tag{4.2.4}$$

**4.2.3.** *Fixing a choice for* $\underline{T}''$. We need to fix a choice for the restriction of $\underline{T}''$ to $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$. So, recalling the last part of the proof of Lemma 4.2.1, we need to fix the map from $C_*(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q}) \to \tilde{D} \to D \to \gamma_{k[\varepsilon]_{2n-1}}(2)_{\mathbb{Q}}$, in degree 2.

Fixing $v_1, v_2, v_3$ any three vectors in $k[\varepsilon]_{2n-1}^{\oplus 2}$ in general position, we define a map that sends $(g_1, g_2, g_3) \in C_2(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to

$$(w, g_1v_1, g_2v_2, g_3v_3) - (w, g_1v_1, g_2v_2, g_2v_3) \\ - (w, g_1v_1, g_1v_2, g_3v_3) + (w, g_1v_1, g_1v_2, g_2v_3)$$

in $\tilde{C}_4(k[\varepsilon]_{2n-1}^{\oplus 3}) = \tilde{C}_4(3) \subseteq \tilde{C}_4(3) \oplus \tilde{C}_3(3)$, where we view

$$k[\varepsilon]_{2n-1}^{\oplus 2} = \{(a_1, a_2, a_3) \in k[\varepsilon]_{2n-1}^{\oplus 3} \mid a_3 = 0\},$$

and we let $w = (0, 0, 1)$. It is seen without difficulty that this map can be extended to a map of complexes $C_*(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q}) \to \tilde{D}$.

Composing with the remaining map given in (3.8.2) this gives a map that sends $(g_1, g_2, g_3)$ to

$$\beta((g_1v_1, g_2v_2, g_3v_3) - (g_1v_1, g_2v_2, g_2v_3) - (g_1v_1, g_1v_2, g_3v_3) + (g_1v_1, g_1v_2, g_2v_3))$$

in $\bigwedge^2 V_{2n-1}$, where $\beta$ is the map in (3.8.3). From now on we fix $v_1 := (1, 1)$, $v_2 := (0, 1)$ and $v_3 := (1, 0)$ and denote the resulting map by $\underline{T}''$.

**4.2.4.** *Computing* $\mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(\Gamma_{\varepsilon^p,\varepsilon^q}))$. From (4.2.2) and (4.2.4) we realize that we need to compute $\mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(\Gamma_{\varepsilon^p,\varepsilon^q}))$ for $p + q = w$. We will do this in a few steps.

**Lemma 4.2.4.** *For* $i = 1, 2$ *and* $p + q = w$, *with* $p, q \geq 1$, *we have*

$$\mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{ii} \wedge \varepsilon^q e_{ii})) = 0.$$

*Proof.* The element $\varepsilon^p e_{ii} \wedge \varepsilon^q e_{ii}$ maps to

$$\varepsilon^p e_{ii} \otimes \varepsilon^q e_{ii} - \varepsilon^q e_{ii} \otimes \varepsilon^p e_{ii} \in C_2(\mathcal{U}(\mathfrak{gl}_2(k[\varepsilon]_{2n-1})), \mathbb{Q}).$$

Since $\varepsilon^x e_{ii} = \log(1 - (1 - \exp(\varepsilon^x e_{ii}))) = -\sum_{1\leq k}(1 - \exp(\varepsilon^x e_{ii}))^k/k$ for $x \geq 1$, we see that $\varepsilon^p e_{ii} \otimes \varepsilon^q e_{ii}$ is a $\mathbb{Q}$-linear combination of terms of the form

$$\exp(\varepsilon^s e_{ii})^u \otimes \exp(\varepsilon^t e_{ii})^v.$$

Let $g_1 := \exp(\varepsilon^s e_{ii})^u$ and $g_2 := \exp(\varepsilon^t e_{ii})^v$. Then $g_1 \otimes g_2$ maps to $(1, g_1, g_1g_2)$, which maps to

$$(v_1, g_1v_2, g_1g_2v_3) - (v_1, g_1v_2, g_1v_3) - (v_1, v_2, g_1g_2v_3) + (v_1, v_2, g_1v_3). \quad (4.2.5)$$

Since, depending on $i$, $g_1(v_2) = v_2$ or $g_1(v_3) = g_1g_2(v_3) = v_3$, we see that the last expression is 0. □

**Lemma 4.2.5.** *The value of* $\mathrm{li}_{2,(2n-1,n),w}$ *on the image of the element* $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ *in* $\bigwedge^2 V_{2n-1}$*, under the chain map that we fixed in Section 4.2.3, is* 0 *if* $p+q \neq w$ *and* $p, q \geq 1$.

*Proof.* By Proposition 2.1.2 to compute the value of $\mathrm{li}_{2,(2n-1,n),w}$ on the image of $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ in $\bigwedge^2 V_{2n-1}$, we first need to project that image to $S_k(2n-1,n)_{\langle w \rangle}$. But for $\lambda \in \mathbb{Q}$, replacing $\varepsilon$ with $\lambda\varepsilon$ multiplies $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ by $\lambda^{p+q}$, whereas the projection of its image to $S_k(2n-1,n)_{\langle w \rangle}$ gets multiplied by $\lambda^w$. Therefore this projection is 0. Hence the statement in the lemma. □

**Lemma 4.2.6.** *For* $p+q=w$ *with* $p, q \geq 1$*, we have*

$$\mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{22} \wedge \varepsilon^q e_{11})) = \mathrm{li}_{2,(2n-1,n),w}((1+\varepsilon^q) \wedge (1+\varepsilon^p)).$$

*Proof.* The expression $\varepsilon^p e_{22} \wedge \varepsilon^q e_{11}$ maps to

$$\varepsilon^p e_{22} \otimes \varepsilon^q e_{11} - \varepsilon^q e_{11} \otimes \varepsilon^p e_{22}. \tag{4.2.6}$$

Both $\varepsilon^p e_{ii} \otimes \varepsilon^q e_{jj}$ and $\exp(\varepsilon^p e_{ii}) \otimes \exp(\varepsilon^q e_{jj}) - \exp(\varepsilon^p e_{ii}) \otimes 1 - 1 \otimes \exp(\varepsilon^q e_{jj})$ have the same $k^\times$-weight $w$ component, and therefore by Lemma 4.2.5 have the same image. Note that terms of the form $1 \otimes g$ and $g \otimes 1$ map to 0, because of the computation in (4.2.5). Hence the left side of the expression in the lemma is equal to the image of $\exp(\varepsilon^p e_{22}) \otimes \exp(\varepsilon^q e_{11}) - \exp(\varepsilon^q e_{11}) \otimes \exp(\varepsilon^p e_{22})$. Since $\exp(\varepsilon^q e_{11})v_2 = v_2$, using the expression (4.2.5) we see that $\exp(\varepsilon^q e_{11}) \otimes \exp(\varepsilon^p e_{22})$ maps to 0. Again using (4.2.5) and the definition of $\beta$ and $\mathrm{li}_{2,(2n-1,n),w}$, we see that $\exp(\varepsilon^p e_{22}) \otimes \exp(\varepsilon^q e_{11})$ maps to $\mathrm{li}_{2,(2n-1,n),w}((1+\varepsilon^q) \wedge (1+\varepsilon^p))$. □

**Lemma 4.2.7.** *For* $p+q=w$ *with* $p, q \geq 1$*, we have*

$$\mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{12} \wedge \varepsilon^q e_{21})) = \mathrm{li}_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q)).$$

*Proof.* Exactly as in the proof of Lemma 4.2.6, we see that the left side of the expression above is equal to the image of

$$\exp(\varepsilon^p e_{12}) \otimes \exp(\varepsilon^q e_{21}) - \exp(\varepsilon^q e_{21}) \otimes \exp(\varepsilon^p e_{12}).$$

As $\exp(\varepsilon^q e_{21})(v_2) = v_2$, we see, using (4.2.5), that $\exp(\varepsilon^q e_{21}) \otimes \exp(\varepsilon^p e_{12})$ maps to 0. Finally using (4.2.5), and the definition of $\beta$ and $\mathrm{li}_{2,(2n-1,n),w}$ we see that $\exp(\varepsilon^p e_{12}) \otimes \exp(\varepsilon^q e_{21})$ maps to $\mathrm{li}_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q))$. □

**Lemma 4.2.8.** *For* $p+q=w$ *with* $p, q \geq 1$*,*

$$\mathrm{li}_{2,(2n-1,n),w}(\underline{T}''(\Gamma_{\varepsilon^p,\varepsilon^q})) = 3\,\mathrm{li}_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q)).$$

*Proof.* This follows from Lemmas 4.2.4, 4.2.6, and 4.2.7, together with the fact, which is immediate from the definition of $\mathrm{li}_{2,(2n-1,n),w}$, that

$$\mathrm{li}_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q)) = \mathrm{li}_{2,(2n-1,n),w}((1+\varepsilon^p) \wedge (1+\varepsilon^q)). \qquad \square$$

Let $[|\cdot|]$ denote the greatest integer function.

**Theorem 4.2.9.** *With the notation as in Section 4.1.2,*

$$\mathrm{Li}_{2,n}(T(\alpha_w)) = \begin{cases} -([|\tfrac{1}{2}(2n-1-w)|] + w - n + 1 + \tfrac{1}{2}\chi(w)) & \text{if } w \neq 2n-1, \\ -\tfrac{1}{2}(2n-1) & \text{if } w = 2n-1. \end{cases}$$

*Proof.* Since $\mathrm{Li}_{2,n}(T(\alpha_w)) = \mathrm{Li}_{2,n,w}(T'(\beta_w))$, using Lemma 4.2.1, (4.2.2), (4.2.4) we see that $\mathrm{Li}_{2,n}(T(\alpha_w)) = \frac{1}{2}\,\mathrm{li}_{2,(2n-1,n),w} \circ \underline{T}''$ evaluated on

$$\sum_{0\le j<(2n-1-w)/2} (\Gamma_{\varepsilon^{w-1},\varepsilon} - \Gamma_{\varepsilon^{w-n+j},\varepsilon^{n-j}} - \Gamma_{\varepsilon^{n-1-j},\varepsilon^{w-n+j+1}}) \\ + \tfrac{1}{2}\chi(w)(\Gamma_{\varepsilon^{w-1},\varepsilon} - 2\Gamma_{\varepsilon^{(w-1)/2},\varepsilon^{(w+1)/2}}).$$

Using Lemma 4.2.8 and the definition of $\mathrm{li}_{2,(2n-1,n),w}$ we see that if $w \neq 2n-1$, then the contribution from $j = 0$ is $-(w-n+1)$; the contribution from each of the terms where $0 < j$ is $-1$; the last term contributes $-\frac{1}{2}\chi(w)$.

In the case $w = 2n-1$, there is only one contribution, coming from the last term, and this is $\frac{1}{2}\chi(2n-1)(-1-2(n-1)) = -\frac{1}{2}(2n-1)$. □

**4.3. *Proof of Theorem 1.3.2.*** In order to prove this, by Goodwillie's theorem (Theorem 3.1.1), Remark 3.1.2 and Sections 4.1.1 and 4.1.2, we need only show

$$\mathrm{Li}_{2,n,w} : (k \simeq)\, \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}_{\langle w\rangle} \to k$$

is an isomorphism. We know that this map is nonzero by Theorem 4.2.9, and replacing $\varepsilon$ by $\lambda\varepsilon$ has the effect of multiplication by $\lambda^w$, using the vector space structures on both sides [Hesselholt 2005, Proposition 8.1]. This immediately implies the theorem when $k$ is algebraically closed. In the general case, we just need to use Theorem 1.3.2 for $\bar{k}$, and the equivariance of $\mathrm{Li}_{2,n,w}$ with respect to $\mathrm{Gal}(\bar{k}/k)$ and take galois invariants on both sides.

## 5. The complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$

**5.1.** To compute the kernel of $\delta_n$ in Theorem 1.3.1, we will need the following proposition. Following Suslin's notation, let $T_m(A) \subseteq \mathrm{GL}_m(A)$ denote the subgroup of diagonal matrices.

**Proposition 5.1.1.** *The map $\rho_2 : H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to \ker(\delta_n)$ from Section 3.8 has the property that*

$$\rho_2(H_3(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q}) = \ker(\delta_n) \quad \text{and} \quad H_3(T_3(k[\varepsilon]_n), \mathbb{Q}) \subseteq \ker(\rho_2).$$

*Proof.* The first statement is proved in the case of fields in [Suslin 1990, Section 2]. The same proof works for $k[\varepsilon]_n$, if in the first line of [Suslin 1990, page 222], we use [Guin 1989, Theorem 2.2.2] to show that

$$H_*(T_2(k[\varepsilon]_n), \mathbb{Q}) = H_*(UT_2(k[\varepsilon]_n), \mathbb{Q}),$$

where $UT_2(A)$ denotes upper triangular matrices in $\mathrm{GL}_2(A)$ (this is denoted by $B_2(A)$ in [Suslin 1990]). We note that there is a slight difference between the construction of our map $\rho_2$ and the corresponding map of Suslin. Namely, Suslin uses configurations in the projective space rather than the affine space, but the resulting maps $H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to \ker(\delta_n)$ are the same.

The proof of [Suslin 1990, Proposition 3.1] works for $k[\varepsilon]_n$ as well, proving the second statement. □

**Proposition 5.1.2.** *The map* $T : \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \to \ker(\delta_n)^{\circ}$ (*see Notation 2.0.1*) *defined in Proposition 3.8.9 is surjective.*

*Proof.* Because of Proposition 5.1.1, Theorem 3.1.1 and Remark 3.1.2, it suffices to show that the image of $K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}}$ in $H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}/H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}$, under the composition of the maps

$$K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}} \to K_3(k[\varepsilon]_n)_{\mathbb{Q}} \to H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q}) \simeq H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \\ \to H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}/H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^{\circ},$$

contains that of $H_3(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q})$ in $H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}/H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}$.

Let $\bigwedge V$ denote the graded symmetric algebra over a graded vector space $V$. By the Milnor–Moore theorem, $H_*(\mathrm{GL}(A), \mathbb{Q}) \simeq \bigwedge((K_*(A)_{\mathbb{Q}})_{>0})$ [Loday 1992, Corollary 11.2.12]; by the stability theorem,

$$H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) = H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q})$$

[Guin 1989, Section 2]. Combining these, we obtain

$$H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) = \bigwedge^3 K_1(k[\varepsilon]_n)_{\mathbb{Q}} \oplus (K_1(k[\varepsilon]_n)_{\mathbb{Q}} \otimes K_2(k[\varepsilon]_n)_{\mathbb{Q}}) \oplus K_3(k[\varepsilon]_n)_{\mathbb{Q}}.$$

The first two components of the decomposition lie inside

$$H_1(\mathrm{GL}_1(k[\varepsilon]_n), \mathbb{Q}) \otimes H_2(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q}) \subseteq H_3(T_3(k[\varepsilon]_n), \mathbb{Q}),$$

(by the proof of [Suslin 1990, Lemma 4.2]; [Guin 1989]). Therefore it suffices to prove that the image of $K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}}$ under the canonical projection

$$H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to \mathrm{Prim}\, H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to (\mathrm{Prim}\, H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}))^{\circ}$$

contains the image of $H_3(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q})$.

By the construction of $\rho_1$ in Sections 3.2–3.7 and Remark 3.1.2, the last translates to showing that the image $\mathrm{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n),\mathbb{Q}))$ of $H_3(\mathfrak{gl}_2(k[\varepsilon]_n),\mathbb{Q})$ in

$$(\mathrm{Prim}\, H_3(\mathfrak{gl}_3(k[\varepsilon]_n),\mathbb{Q}))^\circ = \mathrm{HC}_2(k[\varepsilon]_n)^\circ = \mathrm{HC}_2(k[\varepsilon]_n)^{\circ(1)} \oplus \mathrm{HC}_2(k[\varepsilon]_n)^{\circ(2)}$$

is contained in $\mathrm{HC}_2(k[\varepsilon]_n)^{\circ(1)}$.

First note that $\alpha_w$ for $n+1 \le w \le 2n-1$ form a basis for $\mathrm{HC}_2(k[\varepsilon]_n)^{\circ(1)}$ by Theorem 4.2.9 and Section 4.1.1. By Lemmas 4.2.1 and 4.2.2 and Proposition 4.2.3 and the discussion following it, the image of $\alpha_w$ in $H_3(\mathfrak{gl}_3(k[\varepsilon]_n))^\circ$ is equal to that of an element $\gamma'_w \in H_3(\mathfrak{gl}_2(k[\varepsilon]_n))^\circ$. This implies immediately that $\mathrm{HC}_2(k[\varepsilon]_n)^{\circ(1)} \subseteq \mathrm{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n),\mathbb{Q}))$.

On the other hand [Loday 1992, Theorems 10.3.4 and 4.6.8] and [Loday and Quillen 1984, Remark 6.10] imply that there is a natural map

$$(\mathrm{Prim}\, H_3(\mathfrak{gl}_3(k[\varepsilon]_n),\mathbb{Q}))^\circ / \mathrm{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n),\mathbb{Q})) \to \mathrm{HC}_2(k[\varepsilon]_n)^{\circ(2)}$$

which induces an automorphism of $\mathrm{HC}_2(k[\varepsilon]_n)^{\circ(2)}$ when precomposed with

$$\mathrm{HC}_2(k[\varepsilon]_n)^{\circ(2)} \to (\mathrm{Prim}\, H_3(\mathfrak{gl}_3(k[\varepsilon]_n),\mathbb{Q}))^\circ / \mathrm{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n),\mathbb{Q})).$$

These imply that $\mathrm{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n),\mathbb{Q})) = \mathrm{HC}_2(k[\varepsilon]_n)^{\circ(1)}$. □

The corollary below computes the infinitesimal part of the first cohomology of the complex $\gamma_{k[\varepsilon]_n}(2)_\mathbb{Q}$. Note that $H^1(\gamma_{k[\varepsilon]_n}(2)_\mathbb{Q})^\circ = \ker(\delta_n)^\circ$.

**Corollary 5.1.3.** *The maps*

$$T : \mathrm{HC}_2(k[\varepsilon]_n,(\varepsilon))^{(1)} \to \ker(\delta_n)^\circ \quad \textit{and} \quad \mathrm{Li}_{2,n} : \ker(\delta_n)^\circ \to k^{\oplus n-1}$$

*are isomorphisms.*

*Proof.* This follows from the fact that $T$ is surjective (Proposition 5.1.2) and that $\mathrm{Li}_{2,n} \circ T$ is an isomorphism (Theorem 1.3.2). □

**Proposition 5.1.4.** *There are natural isomorphisms*

$$H^2(\gamma_{k[\varepsilon]_n}(2)_\mathbb{Q})^\circ \simeq \mathrm{HC}_1(k[\varepsilon]_n)^\circ = \mathrm{HC}_1(k[\varepsilon]_n))^{\circ(1)} \simeq \bigoplus_{1\le i\le n-1} \Omega^1_k.$$

*Proof.* Note that by the definition of Milnor K-theory [Loday 1992, 11.1.16]

$$H^2(\gamma_{k[\varepsilon]_n}(2)_\mathbb{Q}) = K_2^M(k[\varepsilon]_n). \tag{5.1.1}$$

Since

$$K_2^M(k[\varepsilon]_n) = K_2(k[\varepsilon]_n) \tag{5.1.2}$$

[Guin 1989, Section 4.2], we have, by [Loday 1992, Proposition 2.1.14],

$$K_2(k[\varepsilon]_n)^\circ = \mathrm{HC}_1(k[\varepsilon]_n)^\circ = \Omega^1_{k[\varepsilon]_n}/(\Omega^1_k + d(k[\varepsilon]_n)) \simeq \bigoplus_{1\le i\le n-1} \Omega^1_k.$$

Finally $\mathrm{HC}_1(k[\varepsilon]_n) = \mathrm{HC}_1(k[\varepsilon]_n)^{(1)}$ follows from [loc. cit., Theorem 4.6.7]. □

**5.2.** ***Proof of Theorem 1.3.1.*** Over $k$ this is the main theorem in [Suslin 1990]. However, note that there the indecomposable quotient $K_3(k)_{\mathrm{ind},\mathbb{Q}}$ of $K_3(k)_{\mathbb{Q}}$ appears instead of $K_3(k)^{(2)}_{\mathbb{Q}}$. To see that these two groups are canonically isomorphic, see [Lichtenbaum 1987, page 207]. Therefore we only need to compute the cohomology of the infinitesimal part of the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. And this is done in Corollary 5.1.3 and (5.1.1) and (5.1.2).

## 6. Comparison with the additive dilogarithm of Bloch and Esnault

In this section we compare the complex $\gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$ to the complex $T_n\mathbb{Q}(2)(k)$ of Bloch and Esnault.

**6.1.** ***The reduced complex.*** To make the comparison we first define a subcomplex of $\gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$: Define $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ to be the subcomplex of $\gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$ whose degree 2 term is

$$k^{\times} \otimes V_n^{\circ} \subseteq \left(\bigwedge^2 V_n\right)^{\circ}$$

and whose degree 1 term is the inverse image $\delta_n^{-1}(k^{\times} \otimes V_n^{\circ}) \subseteq B_2(k[\varepsilon]_n)^{\circ}_{\mathbb{Q}}$. Denote this last group by $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$. Then we have

$$\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}} : B_2(k[\varepsilon]_n)'_{\mathbb{Q}} \to k^{\times} \otimes V_n^{\circ}.$$

We need a lemma to compute the cohomology of this reduced complex.

**Lemma 6.1.1.** *The natural map $(k^{\times})^{\otimes(i-1)} \otimes k[\varepsilon]_n^{\times} \to K_i^M(k[\varepsilon]_n)$ is a surjection.*

*Proof.* By the definition of Milnor K-theory, it is clear that it suffices to prove the lemma for $i = 2$. In this case the lemma follows from the isomorphism

$$K_2(k[\varepsilon]_n) \simeq K_2(k) \oplus \frac{\Omega^1_{k[\varepsilon]_n}}{\Omega^1_k + d(k[\varepsilon]_n)},$$

[Graham 1973, Theorem 3] and the observation that $k^{\times} \otimes k[\varepsilon]_n^{\times}$ surjects onto the expression on the right, under this isomorphism. Note that $K_2^M(k[\varepsilon]_n) = K_2(k[\varepsilon]_n)$ [Guin 1989]. □

**Proposition 6.1.2.** *The inclusion $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}} \to \gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$ is a quasiisomorphism.*

*Proof.* The only thing that needs justification is the surjectivity of the induced map on the degree 2 cohomology groups or equivalently the surjectivity of the composition

$$k^{\times}_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_n^{\circ} \to \left(\bigwedge^2 V_n\right)^{\circ} \to \Omega^1_{k[\varepsilon]_n}/(\Omega^1_k + d(k[\varepsilon]_n)),$$

where the last map is the one in the proof of Proposition 5.1.4. But this is exactly Lemma 6.1.1. □

**6.2.** ***The construction of Bloch and Esnault with higher modulus.*** For the rest of the section we assume that $k$ is algebraically closed. In [2003], Bloch and Esnault construct the additive weight 2 complex with modulus 2; their proof goes through to give a construction for all moduli $n \geq 2$. We describe the properties of this complex below. The proofs and the details of the construction can be found in [Bloch and Esnault 2003, Section 2].

Following their notation, we let $R$ be the local ring of 0 in $\mathbb{A}^1_k$. The localization (away from 0) sequence for the pair $(k[t], (t^n))$ splits into the exact sequences

$$K_2(k[t], (t^n)) \to K_2(R, (t^n)) \xrightarrow{\partial} \bigoplus_{x \in k^\times} K_1(k) \to K_1(k[t], (t^n)) \to 0$$

and

$$0 \to K_1(R, (t^n)) \xrightarrow{\partial} \bigoplus_{x \in k^\times} K_0(k) \to K_0(k[t], (t^n)) \to 0,$$

since $K_0(R, (t^n)) = 0$ and the map $K_1(R, (t^n)) \to \bigoplus_{x \in k^\times} K_0(k)$ is injective, as $K_1(R, (t^n)) = 1 + (t^n)$ and the map is given by the divisor of the function [Lichtenbaum 1987, Appendix]. This description also gives a canonical identification

$$K_0(k[t], (t^n)) = (k[t]_n^\times)^\circ.$$

Using the product structure on K-theory, let

$$T_n B_2(k) := (K_2(R, (t^n))/\operatorname{im}(K_1(k) \cdot K_1(R, (t^n)))_{\mathbb{Q}},$$

and let $T_n H^1_M(k, 2)$ be the image of $K_2(k[t], (t^n))_{\mathbb{Q}}$ in $T_n B_2(k)$. Then the above exact sequences give the exact sequence

$$0 \to T_n H^1_M(k, 2) \to T_n B_2(k) \to k^\times \otimes V_n^\circ \to K_1(k[t], (t^n))_{\mathbb{Q}} \to 0. \qquad (6.2.1)$$

We let $T_n\mathbb{Q}(2)(k) : T_n B_2(k) \to k^\times \otimes V_n^\circ$ denote the middle part of this sequence. This is the exact generalization to higher moduli of the complex considered by Bloch and Esnault [2003] (the complex described in Section 1.2).

We will try to express the cohomology groups of $T_n\mathbb{Q}(2)(k)$ in terms of the groups $K_*(k[t]_n, (t))_{\mathbb{Q}}$.

First note that the long exact sequence for the pair $(k[t], (t^n))$, together with the homotopy invariance of K-theory, gives canonical isomorphisms

$$K_{*+1}(k[t]_n, (t)) \simeq K_*(k[t], (t^n)),$$

and therefore there is a surjection

$$\left(\frac{K_3(k[t]_n, (t))}{K_1(k) \cdot K_2(k[t]_n, (t))}\right)_{\mathbb{Q}} \simeq \left(\frac{K_2(k[t], (t))}{K_1(k) \cdot K_1(k[t], (t))}\right)_{\mathbb{Q}} \to T_n H^1_M(k, 2). \qquad (6.2.2)$$

**Lemma 6.2.1.** *There is a canonical surjection $K_3(k[t]_n, (t))^{(2)}_{\mathbb{Q}} \to T_n H^1_M(k, 2)$.*

*Proof.* By [Lichtenbaum 1987, page 191],

$$K_3(k[t]_n)_{\mathbb{Q}} = K_3(k[t]_n)^{(2)}_{\mathbb{Q}} \oplus K^M_3(k[t]_n)_{\mathbb{Q}},$$

and by Lemma 6.1.1, the image of $K_1(k) \otimes K_2(k[t]_n)$ in $K_3(k[t]_n)$ is $K^M_3(k[t]_n)$. Hence that (6.2.2) is a surjection proves the lemma. □

Let

$$\rho : T_n B_2(k) = \Big(\frac{K_2(R, (t^n))}{K_1(k) \cdot K_1(R, (t^n))}\Big)_{\mathbb{Q}} \to \Big(\frac{K_2(k[t]_{2n-1}, (t^n))}{K_1(k) \cdot K_1(k[t]_{2n-1}, (t^n))}\Big)_{\mathbb{Q}} =: N$$

denote the map induced by reduction modulo $(t^{2n-1})$. We will prove that $\rho$ behaves like an additive dilogarithm in this setting.

**Proposition 6.2.2.** *The composition $K_3(k[t]_n, (t))^{(2)}_{\mathbb{Q}} \to T_n H^1_M(k, 2) \to N$ induced by the inclusion*

$$K_3(k[t]_n, (t))^{(2)}_{\mathbb{Q}} \to K_3(k[t]_n, (t))_{\mathbb{Q}},$$

(6.2.2), *and $\rho$ is an isomorphism.*

*Proof.* This map is induced by the long exact sequence of the pair $(k[t]_{2n-1}, (t^n))$:

$$\cdots \to K_3(k[t]_n, (t)) \to K_2(k[t]_{2n-1}, (t^n)) \to K_2(k[t]_{2n-1}, (t)) \to \cdots.$$

By Goodwillie's theorem, Remark 3.1.2 and Section 4.1.1, the map

$$K_3(k[t]_{2n-1}, (t))^{(2)}_{\mathbb{Q}} \to K_3(k[t]_n, (t))^{(2)}_{\mathbb{Q}}$$

is equivalent to a map $k^{\oplus(2n-2)} \to k^{\oplus(n-1)}$, where the $k^\times$-weights in the source range in $[2n, 4n-3]$, whereas in the target they range in $[n, 2n-1]$. Therefore this last map is zero and hence $K_3(k[t]_n, (t))^{(2)}_{\mathbb{Q}} \to K_2(k[t]_{2n-1}, (t^n))_{\mathbb{Q}}$ is injective.

By [Stienstra 1981, Theorem 1.11], $K_2(k[t]_{2n-1}, (t^n))_{\mathbb{Q}} \simeq k^{\oplus(n-1)} \oplus (\Omega^1_k)^{\oplus(n-1)}$, and $K_1(k) \otimes K_1(k[t]_{2n-1}, (t^n)) \to K_2(k[t]_{2n-1}, (t^n))_{\mathbb{Q}}$ has image $(\Omega^1_k)^{\oplus(n-1)}$. □

**Corollary 6.2.3.** *There are canonical isomorphisms*

$$H^1(T_n\mathbb{Q}(2)(k)) \simeq K_3(k[t]_n, (t))^{(2)}_{\mathbb{Q}} \simeq \mathrm{HC}_2(k[t]_n, (t))^{(1)},$$
$$H^2(T_n\mathbb{Q}(2)(k)) \simeq K_2(k[t]_n, (t))_{\mathbb{Q}} \simeq \mathrm{HC}_1(k[t]_n, (t)).$$

*Proof.* The first isomorphism is an immediate consequence of Lemma 6.2.1 and Proposition 6.2.2, and the second is a consequence of the isomorphism

$$K_2(k[t]_n, (t)) \simeq K_1(k[t], (t^n)),$$

which follows from the long exact sequence for $(k[t], (t^n))$ and the homotopy invariance of $K$-theory. □

*Proof of Corollary 1.4.1.* First we note that the degree 2 terms of $T_n\mathbb{Q}(2)(k)$ and $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ are both equal to $k^\times \otimes V_n^\circ$ and that the cohomology groups of the two complexes are canonically isomorphic (Theorem 1.3.1, Proposition 6.2.2, and Corollary 6.2.3). In both cases the projection from $k^\times \otimes V_n^\circ$ to the degree 2 cohomology is induced by the composition

$$k^\times \otimes V_n^\circ \to K_2^M(k[\varepsilon]_n) \to \Omega^1_{k[\varepsilon]_n}/(\Omega^1_k + d(k[\varepsilon]_n))$$

(see the proof of Lemma 6.1.1). Therefore the images of $T_nB_2(k)$ and of $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$ in $k^\times \otimes V_n^\circ$ are the same. The exact sequence (6.2.1) and Proposition 6.2.2 give a splitting of $T_nB_2(k)$; and Theorems 1.3.1 and 1.3.2 give a splitting of $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$. This proves the corollary. □

We would like to emphasize that the isomorphism given in the statement of the corollary uses the additive dilogarithm in both constructions and thus should not be considered as natural.

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sunver@ku.edu.tr *Koç University, Department of Mathematics, Rumelifeneri Yolu, 34450 Sarıyer-İstanbul, Turkey*