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ON THE ADEQUACY OF PLANE-WAVE REFLECTION/TRANSMISSION COEFFICIENTS IN THE ANALYSIS OF SEISMIC BODY WAVES

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ABSTRACT

In order to estimate the effect (on body waves) of discontinuities within the Earth, it is common practice to use the theory for plane waves incident upon the plane boundary between two homogeneous half-spaces. The resulting reflection/*P–SV* conversion/transmission coefficients are shown here to be inaccurate for many problems of current interest. Corrected coefficients are needed, in particular, for cases where the discontinuity (upon which boundary conditions are to be applied) is near a turning point of the *P*- or *S*-wave rays, or if one of these rays intersects the discontinuity at a near-grazing angle.

Adequate corrections, based upon the Langer approximation to a full wave theory, are shown to be easily derived in practice. The method is first to write out the plane-wave coefficients as a rational polynomial, in sines and cosines of the angles of incidence upon the boundary, and second to introduce a multiplicative factor for each cosine. The new factors depart from unity only when the associated cosine tends to zero; i.e., when a turning point is approached. They incorporate all the corrections required for curvature of the boundary, frequency dependence, and Earth structure (velocity gradients) near the boundary.

INTRODUCTION

This paper is concerned with the theory of elastic body waves in the Earth which, near their turning point, are affected by some discontinuity such as the crust-mantle boundary (the "Moho") or the mantle-core boundary. A turning point is the position, along the propagation path taken by a body wave, at which a wave is traveling horizontally, so the general subject area of this paper includes problems of waves incident at near-grazing angles upon the boundary between two different media. There are many observed examples in the Earth of waves of this type: in particular, it appears in practice that every known core phase (*PKP*, *SKS*, *SKKP*, *PKKKKP*, *PKIKP*, etc.) is, at least in some range of epicentral distances, affected by the consequences of near-grazing incidence on either the core-mantle boundary, or the inner core/outer core boundary.

The practical interpretation of body waves (i.e., their travel times and amplitudes) has, in large part, simply been based upon the geometrical spreading of rays in inhomogeneous media, together with the use of plane-wave coefficients to allow for reflection and transmission at discontinuities (see, e.g., page 298 of Gutenberg and Richter, 1935). However, the justification for this procedure has come relatively recently (see, e.g., Scholte, 1956; Seckler and Keller, 1959). The justification involves development of a full-wave theory and asymptotic approximations to the body-wave amplitudes predicted by the full-wave theory. Unfortunately, ray theory is inadequate to explain many of the observed phenomena associated with body waves, such as caustics and various types of diffraction, so it must be acknowledged that some of the asymptotic approximations applied to the full-wave theory are inaccurate at seismic frequencies. In cases of body waves incident at near-grazing angles upon discontinuities in the Earth, the defective asymptotic approximation is the widely used WKBJ approximation to the function describing the radial dependence of waves propagating at fixed ray parameter. The WKBJ approximation fails in just these cases, because it is inaccurate near turning points. Since the justification for using plane-wave coefficients is based upon WKBJ approximations, it must be expected that such coefficients become suspect wherever near-grazing angles are involved. It is shown by a specific example that plane-wave coefficients can indeed be very inaccurate in such cases.

This paper is concerned with a uniformly asymptotic approximation, due principally to Langer, which is an improvement upon WKBJ approximations in precisely those cases involving waves interacting with a discontinuity at near-grazing angles. The suitability of Langer's methods in seismology has been pointed out previously by Alenitsyn (1967) and Chapman (1974). However, the matrix methods of these authors are avoided here, as they are not necessary to obtain the main conclusion of this paper: namely, that plane-wave coefficients can readily be modified in practice to obtain accurate frequency-dependent coefficients which describe the exchange from an incident wave to reflected and transmitted P and S waves, for the case of discontinuities in the Earth. The derived coefficients incorporate the effects of curvature of the discontinuity, and Earth structure (e.g., velocity gradients) on either side of the discontinuity.

The sections which follow give an introduction to the necessary parts of Langer's theory. Then it is shown how plane-wave coefficients can become inadequate, and how they can be generalized to give accurate results in practice.

Pragmatic Description of the Langer Approximation

Rather than repeat the original clear theoretical development (Langer, 1931, 1932, 1949, 1951), only an outline of the failure of other methods is given, and then it is shown why Langer's asymptotic formulas do indeed have the important properties which are claimed for them.

At periods shorter than about 1 min, Richards (1974) has shown that elastic displacement $\boldsymbol{u}$ in the inhomogeneous Earth can be meaningfully represented as the sum of three vector terms

$$\boldsymbol{u} = \rho^{-1/2}[\operatorname{grad} P + \operatorname{curl}\operatorname{curl}(rS, 0, 0)] + \mu^{-1/2}\operatorname{curl}(rH, 0, 0). \tag{1}$$

Here, the three right-hand-side vectors are, respectively, the P, SV, and SH components of displacement, with scalar potentials P, S, H satisfying (in the frequency domain) the decoupled wave equations

$$\nabla^2 P + \frac{\omega^2}{\alpha^2} P = 0; \nabla^2 S + \frac{\omega^2}{\beta^2} S = 0; \nabla^2 H + \frac{\omega^2}{\beta^2} H = 0. \tag{2}$$

In equations (1) and (2), the quantities ρ (density), μ (rigidity), α (P-wave speed) and β (S-wave speed) are each functions only of radius r in the spherical polar system (r, θ, ϕ). Thus, each of the wave equations in (2) has the form of a Helmholtz equation with radially varying wavenumber, and solutions are a sum over surface harmonics, each weighted by a radial factor. Efficient evaluation of the radial factor, appropriate to a given order n (say) of surface harmonic, is the key to providing practical solutions for a wide range of wave propagation problems. If this radial factor is, for P-waves, $R(r)$, then from (2) it follows that

$$\frac{d^2}{dr^2}(rR) + \omega^2\left[\frac{1}{\alpha^2} - \frac{n(n+1)}{\omega^2 r^2}\right] rR = 0. \tag{3}$$

This equation is central to the theory of wave propagation in spherically symmetric media. For body waves, the Legendre function phase permits identification of $(n+\frac{1}{2})/\omega$ as p, the seismic ray parameter describing horizontal slowness of the solution under study, and for this reason we often write $R(r, p)$ for the radial function. Occasionally, when the frequency-dependence is also under discussion, we use notation $R(r, p, \omega)$. The immediate objective now appears to be that of finding accurate solutions to (3), for fixed p (and ω), as r varies. However, solutions to (3) are usually required at fixed r, as p (and often ω) varies. This latter case is so common, because it arises whenever a boundary condition has to be taken into account. Solutions are therefore needed which remain uniformly valid as p varies, i.e., as the angle of incidence changes. It is this additional requirement which often vitiates the WKBJ solutions to (3), namely

$$R \sim U = Q^{-1/2} r^{-1} \exp(\pm i\omega\xi) \tag{4}$$

where

$$Q = [1/\alpha^2 - p^2/r^2]^{1/2},$$
$$\xi = \int_{r_p}^{r} Q\,dr,$$

and r_p is the "turning point" radius at which Q is zero. It should be noted that our discussion is restricted here to media for which a real ray bottoms at every depth.

The WKBJ solution has been rediscovered and fruitfully applied in many different fields, and its failure near turning points (i.e., for r varying near r_p, with p fixed) is well known. It is useful to see this failure in terms of the equation satisfied by U,

$$\frac{d^2}{dr^2}(rU) + \omega^2\left[\frac{1}{\alpha^2} - \frac{n(n+1)}{\omega^2 r^2}\right] rU + \left[\frac{1}{2Q}\frac{d^2Q}{dr^2} - \frac{3}{4Q^2}\left(\frac{dQ}{dr}\right)^2 - \frac{1}{4r^2}\right] rU = 0. \tag{5}$$

The difference between (5) and the equation (3) we seek to solve is a term which is relatively negligible at high enough frequencies, except near turning points (for which Q is near zero, giving a singularity in (5) in the term which elsewhere is relatively small).

Early attempts to provide a solution for (3) as r varies near r_p hinged on developing a Taylor series expansion for the coefficient of the rR term, i.e., expanding this term as a series in powers of $(r - r_p)$. See, for example, Rayleigh (1912) and Pekeris (1946). In the vicinity of the turning point, equation (3) is then seen to reduce essentially to an Airy equation, and solutions are given asymptotically by

$$R \sim V = r^{-1/2} Ai[y \exp(\pm 2i\pi/3)] \tag{6}$$

where

$$y = [2(1 - b_p)/p]^{1/3} \omega^{2/3} (p - r/\alpha_p),$$
$$b = (r/\alpha)(d\alpha/dr),$$

and a suffix p denotes evaluation at the turning point radius.

Equation (6) permits evaluation of the radial factor R in precisely the region for which the WKBJ approximation breaks down. However, since its applicability is restricted only to this turning point region, practical questions arise as to which approximation, (4) or (6), should be used for a particular choice of p, r, and ω. Is there an overlap in the regions for which (4) and (6) are accurate? Or (much worse) are these different regions disjoint? These are questions which apparently must be answered if the function R is required for a substantial range of values of p, with r fixed. Such is usually the case if the problem being studied involves waves observed at distances for which the ray path between source and receiver has near its turning point a discontinuity in the Earth's structure. Even though (4) may be accurate at the receiver, an evaluation of boundary conditions at the discontinuity requires that R be known accurately at the radius (depth) of the discontinuity itself, and the choice between (4) and (6) must apparently be made.

The necessity for making such a choice can, however, be evaded via Langer's uniformly asymptotic formulas, which provide the approximation

$$R \sim W = \xi^{1/2} Q^{-1/2} r^{-1} H^{(j)}_{1/3}(\omega\xi) \tag{7}$$

where $j=1$ or 2, $H^{(j)}_{1/3}(\omega\xi)$ represents a Hankel function of order 1/3, and ξ is the variable introduced in equation (4). The choice $j=1$ corresponds to an upcoming wave, and $j=2$ to a downgoing wave. This follows from the choice of sign convention in the Fourier transform underlying wave equations (2)—namely, that the time domain field $P(\mathbf{r}, t)$ is transformed to the frequency domain via $\int_{-\infty}^{\infty} P(\mathbf{r}, t)\exp(+i\omega t)dt$.)

The uniform validity of (7) can be appreciated directly from the equation satisfied by W; namely,

$$\frac{d^2 rW}{dr^2} + \omega^2\left[\frac{1}{\alpha^2} - \frac{n(n+1)}{\omega^2 r^2}\right] rW + \left[\frac{1}{2Q}\frac{d^2 Q}{dr^2} - \frac{3}{4Q^2}\left(\frac{dQ}{dr}\right)^2 + \frac{5}{36}\frac{Q^2}{\xi^2} - \frac{1}{4r^2}\right] rW = 0. \tag{8}$$

At distances far from the turning point, Q and ξ are not near zero, and there is a negligible difference between this equation and the equation satisfied by R, i.e., equation (3). The difference term is similar to that occurring in the equation satisfied by the WKBJ approximation, equation (5), but now there is an additional term $(5/36)Q^2\xi^{-2}$. The effect of this term is to remove the singularity at the turning point, as it may be shown that

$$\frac{1}{2Q}\frac{d^2 Q}{dr^2} - \frac{3}{4Q^2}\left(\frac{dQ}{dr}\right)^2 + \frac{5}{36}\frac{Q^2}{\xi^2} = E \text{ (say)}$$

is bounded, and tends to a constant as $r \to r_p$. To establish this last result, one first shows that $\xi = c_1(r-r_p)^{3/2} + c_2(r-r_p)^{5/2} + 0[(r-r_p)^{7/2}]$ as $r \to r_p$ (for some constants c_1 and c_2). Using $Q = d\xi/dr$, the power series for E can then be developed as $E = \text{constant} + 0(r-r_p)$ as $r \to r_p$. Since E is bounded everywhere, it follows that equation (8) is everywhere a close approximation to the equation for R, and, hence, that the explicit expression (7) is uniformly valid. Since the principal concern of this paper is an assessment of the usefulness of this expression in seismic body-wave problems, some of its general properties are given below.

(a) At fixed radius, the radial function $R(r, p)$ must everywhere be analytic in p, with no singularities at finite values (Friedman, 1951). This property is also true of W in equation (7). Although ξ, Q and $H^{(j)}_{1/3}$ individually have branch cuts, and $Q^{-1/2}$ and $H^{(j)}_{1/3}$ are unbounded (so that the product appears to be multivalued and singular), a choice for the branch cuts can be made so that W is indeed single-valued, analytic, and bounded for finite values of p.

(b) Using the asymptotic formulas for $H^{(1)/(2)}_{1/3}(\omega\xi)$ with large $\omega\xi$, $2^{1/2}\pi^{-1/2}(\omega\xi)^{-1/2}\exp[\pm i(\omega\xi - 5\pi/12)]$, it follows that the Langer approximation reduces to WKBJ formulas at distances away from the turning point. That is, W tends to a constant times U as $\omega\xi \to \infty$. Furthermore, W tends to a constant times V as $\omega\xi \to 0$. To see this, note that $\omega\xi \sim (2/3)(-y)^{3/2}$ as $r \to r_p$, and use the identity

$$H^{(1)/(2)}_{1/3}(z) = 2\exp(\mp i\pi/2)3^{1/6}(2/z)^{1/3} Ai[-(3/2z)^{2/3}\exp(\pm 2i\pi/3)]. \tag{9}$$

It follows that $W(r, p)$ retains the merits both of the WKBJ solution and of the non-uniformly asymptotic approximation (6), in regions where these are appropriate.

(c) It is often useful to establish a standard normalization for the radial functions, especially in cases where a particular source is to be represented by summation over surface harmonics. A convenient normalization is suggested by first considering the well-established theory for a homogeneous medium, $\alpha(r)=\alpha_s$ (say), since then the radial-wave equation (3) is solved exactly by spherical Hankel functions $h_n^{(j)}(\omega r/\alpha_s)$. To determine the Langer approximation for this function, it thus remains to find the $A^{(j)}$ such that

$$h^{(j)}_{\omega p-(1/2)}(\omega r/\alpha_s)\sim A^{(j)}\xi^{1/2}Q^{-1/2}r^{-1}H^{(j)}_{1/3}(\omega\xi). \tag{10}$$

Any choice of (r, p, ω) will suffice to evaluate the $A^{(j)}$, and taking r small and ωp large, one may use the Debye approximation for the left-hand side in (10), and the asymptotic form given above for $H^{(j)}_{1/3}(\omega\xi)$ at large $\omega\xi$, to conclude

$$A^{(1)/(2)}=(\pi\alpha_s/2)^{1/2}\exp(\pm i\pi/6)/\omega^{1/2}. \tag{11}$$

Note that the right-hand side of (10) already includes, via ξ, all the dependence of the radial function on (r, p). With (11), it is extended to give the dependence on (r, p, ω) which is needed to examine frequency effects and synthesis in the time domain.

A useful normalization can now be stated for the radial function in media with varying velocity $\alpha(r)$. We use (10) and (11) to give

$$R(r,p,\omega)=g^{(1)/(2)}\sim(\pi\alpha_s/2)^{1/2}\exp(\pm i\pi/6)(\omega\xi/Q)^{1/2}(\omega r)^{-1}H^{(1)/(2)}_{1/3}(\omega\xi). \tag{12}$$

Again, note that $j=1$ corresponds to an upcoming wave (and $j=2$ to a downgoing wave) at radii above the turning point. Each of $g^{(1)}$ and $g^{(2)}$ grows exponentially with depth below the turning point, and the linear combination which there decays with depth is simply $g^{(1)}+g^{(2)}$. The Wronskian of $g^{(1)}$ and $g^{(2)}$ is often needed, and, with our normalization,

$$g^{(1)}\frac{\partial g^{(2)}}{\partial r}-g^{(2)}\frac{\partial g^{(1)}}{\partial r}=-\frac{2i\alpha_s}{r^2\omega}.$$

The explicit formula (12) still contains a constant velocity value, α_s, and Richards (1973) has shown how the waves radiated from an explosive point source, situated at the radius r_s (say) such that $\alpha_s=\alpha(r_s)$, can be represented as an integral over $g^{(j)}$ functions normalized as in (12). The integral is taken with respect to ray parameter p, and by taking integration paths into the complex p plane, a rapid and numerically stable method becomes available for the examination of a wide variety of body-wave phenomena associated (for example) with caustics and diffraction.

(d) The radial derivative, $\partial R(r,p,\omega)/\partial r$, is often needed to evaluate boundary conditions. From (12) and the property

$$\frac{d}{d\xi}H^{(1)/(2)}_{1/3}(\omega\xi)=\omega\exp(\pm 2i\pi/3)H^{(1)/(2)}_{2/3}(\omega\xi),$$

we find

$$\frac{\partial}{\partial r}g^{(1)/(2)}\sim\left(\frac{\pi\alpha_s Q\omega\xi}{2}\right)^{1/2}\frac{\exp(\pm 5i\pi/6)}{r}H^{(1)/(2)}_{2/3}(\omega\xi). \tag{13}$$

The explicit formulas (12) and (13) are the main subject of discussion in the remainder of this paper. Since they are not exact, it is well to be aware at the outset of the order of error they introduce. If $|\omega\xi|$ is large, and ω large, then $g^{(j)}$ in (12) is $0(\omega^{-1})$, and terms $0(\omega^{-2})$ are neglected in this Langer formula for $g^{(j)}$; in (13), $\partial g^{(j)}/2r$ is $0(1)$ and terms $0(\omega^{-1})$ are neglected. However, if ξ is very small, so that $|\omega\xi|$ is small even though ω is large, then $g^{(j)}$ is $0(\omega^{-5/6})$ and terms $0(\omega^{-3/2})$ are neglected; $\partial g^{(j)}/\partial r$ is $0(\omega^{-1/6})$ and terms $0(\omega^{-5/6})$ are neglected. The size of these neglected terms can be estimated by using the dimensionless ratio $\alpha/(\omega r)$. At worst, the Langer approximation can introduce errors of order $[\alpha/(\omega r)]^{2/3}$, and this remains below 5 per cent for almost all body waves with period less than 40 sec. Where this error is unacceptable, it is possible to develop explicitly the next term in the asymptotic series (Langer, 1951; Chapman, 1974).

(e) Two specific stages may be identified in practical computations involving the Langer approximation. The first is to obtain ξ for a given radius, ray parameter, and Earth model; and the second is to evaluate a low-order (1/3 or 2/3) Hankel function, possibly with complex argument.

The integral for ξ can be made a sum of FORTRAN-supplied functions, if the interpolation law $\alpha(r) = ar^b$ is used between radii on which α is specified. This interpolation law is widely used in travel-time computations (Julian and Anderson, 1968), and we can thus use available programs for travel-time T and distance Δ, as a function of ray parameter, together with the result

$$\xi(r, p) = T(p) - p\Delta(p)$$

(see Bullen, 1963, page 112). Here, T and Δ are the time and distance taken to travel down the ray from level r to the turning point r_p. A problem can arise in that the velocity profile, as interpolated between shells on which $\alpha(r)$ is given, is not analytic. Then r_p, and hence ξ and the Langer approximation, do not turn out to be analytic functions of p, although their variation with p does turn out to be sufficiently smooth in practice.

Program packages to evaluate $H_\nu^{(j)}(z)$ for small ν and complex z are widely available: SHARE program 1489 has been found completely adequate.

(f) The theory outlined above is for media in which precisely one turning point r_p exists for each p. However, for a medium with a low-velocity zone, the value of r in $R(r, p, \omega)$ may be such that r_p is a multivalued function of p. Formulas (12) and (13) are no longer uniformly asymptotic in p, since they fail at p values such that two different solutions r_p are nearly equal. The theory must then be re-cast with confluent hypergeometric functions, rather than $H_{1/3}^{(j)}$. Langer (1951) has given a specific example of this type, in the context of microwave propagation.

Transmission and Reflection Coefficients for a Discontinuity: Generalization of Results for Plane Waves/Plane Boundaries

At the heart of any derivation of transmission/reflection coefficients, for a horizontal discontinuity, lies the constraint that certain linear combinations of the vertical-wave function and its vertical derivative are continuous throughout the medium. In this section, it is shown that this constraint can be used to derive transmission/reflection coefficients for a discontinuity between two inhomogeneous media (e.g., the core-mantle boundary) in terms of the corresponding coefficients for a simple plane-wave problem, with a plane boundary between two homogeneous media. The general problem (for inhomogeneous media, with a discontinuity on a spherical surface such as the Moho, or inner core/outer core boundary) would appear to involve coefficients which are very different from the plane wave/plane boundary problem, since phenomena associated with grazing incidence are, in general, likely to be frequency-dependent. However, the complications of the more general problem are found to be resolved by generalizing, in the plane-wave problem, the concept of "cosine of angle of incidence". Fortunately, the generalization which results is well suited to evaluation via the Langer approximation, since it is independent of the normalization used for the radial functions $R(r, p, \omega)$, and many of the factors in formulas (12) and (13) are not required.

These several results are developed below by first considering in some detail the elementary case of sound waves in a fluid, and subsequently outlining a P-SV problem for the core mantle boundary.

Sound waves in a transversely homogeneous fluid, with discontinuity at $z = 0$

The plane wave/plane boundary problem. This elementary problem serves to introduce notation, and also to provide reflection/transmission coefficients in a form which subsequently are generalized. In Figure 1 is shown the coordinate system, for a plane wave incident from above upon the boundary between two homogeneous fluids. Using pressure P as the dependent variable, not to be confused with the P-wave potential of the previous section, the incident plane wave with frequency ω, ray parameter q $(= \sin i/\alpha)$, and amplitude A is given by

$$P^{\text{inc}} = A \exp(-i\omega z \cos i_1/\alpha_1) \exp[i\omega(qx - t)] \quad \text{(downgoing).} \tag{14}$$

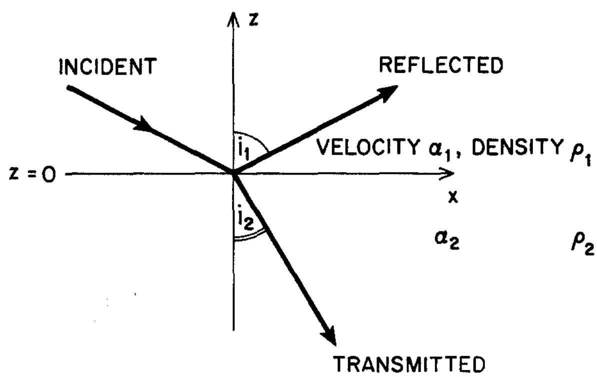

FIG. 1. Parameters for the elementary problem of an incident pressure wave, reflected and transmitted from the boundary between two homogeneous fluids. Recall that the ray parameter for plane layered media is $(\sin i)/\alpha$.

Here, $\cos i_1 = (1 - \alpha_1{}^2 q^2)^{1/2}$, and the factor $\exp[i\omega(qx - t)]$ is common to reflected and transmitted waves

$$P^{\text{ref}} = B \exp(+i\omega z \cos i_1/\alpha_1) \exp[i\omega(qx - t)] \quad \text{(upcoming)} \tag{15}$$

$$P^{\text{trans}} = C \exp(-i\omega z \cos i_2/\alpha_2) \exp[i\omega(qx - t)] \quad \text{(downgoing).} \tag{16}$$

Since P and $(1/\rho)(\partial P/\partial z)$ are continuous across $z = 0$, the reflection coefficient is found to be

$$\frac{B}{A} = \frac{\rho_2 \alpha_2 \cos i_1 - \rho_1 \alpha_1 \cos i_2}{\rho_2 \alpha_2 \cos i_1 + \rho_1 \alpha_1 \cos i_2} \tag{17}$$

and the transmission coefficient is

$$\frac{C}{A} = \frac{2\rho_2 \alpha_2 \cos i_1}{\rho_2 \alpha_2 \cos i_1 + \rho_1 \alpha_1 \cos i_2}. \tag{18}$$

At this stage, it is instructive to take a close look at the reasons why cosines appear in the familiar formulas (17) and (18). We note that in either medium the upcoming vertical wave function is $V^{(1)}(z, q, \omega)$, say, with

$$V^{(1)}(z, q, \omega) = \exp(i\omega z \cos i/\alpha)$$

and the downgoing factor is

$$V^{(2)}(z, q, \omega) = \exp(-i\omega z \cos i/\alpha).$$

For homogeneous media, then, in terms of the vertical wave functions, the cosines in coefficient formulas (17) and (18) arise from the relations

$$\cos i = \frac{\alpha}{i\omega}\frac{\partial V^{(1)}}{\partial z}\bigg/ V^{(1)} \equiv C^{(1)} \text{ (say)} = C^{(1)}(z, q, \omega)$$

and also (19)

$$\cos i = \frac{-\alpha \partial V^{(2)}}{i\omega \quad \partial z}\bigg/ V^{(2)} \equiv C^{(2)} \text{ (say)} = C^{(2)}(z, q, \omega).$$

The main points to be made in the generalization to inhomogeneous media are that (a) the three quantities, $\cos i$, $C^{(1)}$, and $C^{(2)}$ are in general unequal (although they *are* equal at the WKBJ level of approximation to the vertical-wave functions $V^{(1)}$ and $V^{(2)}$); (b) it is $C^{(1)}$ and $C^{(2)}$ which should appear, in place of $\cos i$, in the formulas for reflection/transmission coefficients; and that (c) major differences due to this replacement will show up where $\cos i$ is small (i.e., near grazing), for this is where the WKBJ approximation breaks down, and $\cos i$ departs most from $C^{(1)}$ and $C^{(2)}$.

In fact, cosines appear in four places in the reflection coefficient (17). Taking account of whether an upcoming or downgoing function is differentiated, in imposing the constraint of continuity on $(1/\rho)(\partial P/\partial z)$, one finds the reflection coefficient is

$$\frac{B}{A} = \frac{\rho_2\alpha_2 C_1^{(2)} - \rho_1\alpha_1 C_2^{(2)}}{\rho_2\alpha_2 C_1^{(1)} + \rho_1\alpha_1 C_2^{(2)}} \tag{20}$$

and the transmission coefficient is

$$\frac{C}{A} = \frac{\rho_2\alpha_2[C_1^{(1)} + C_2^{(2)}]}{\rho_2\alpha_2 C_1^{(1)} + \rho_1\alpha_1 C_2^{(2)}}. \tag{21}$$

A suffix on the $C^{(j)}$ and $V^{(j)}$ functions is used wherever it is necessary to indicate the half-space for which identities (19) are applied.

These coefficients, (20) and (21), do give the required generalization for inhomogeneous media, as discussed next.

The problem of two inhomogeneous fluid half-spaces, in contact at a plane boundary. The equation of motion is now $\rho\ddot{\boldsymbol{u}} = -\operatorname{grad} P$ and the equation of state (for small motions) is $P = -k \operatorname{div} \boldsymbol{u}$, where k is the bulk modulus. Although we seek reflection/transmission coefficients for pressure, we shall work with the dependent variable

$$X = [\rho(z)]^{-1/2} P,$$

since X satisfies an equation which, in the frequency domain, is effectively a Helmholtz equation with variable wavenumber. This equation (Brekhovskikh, 1960, p. 171) is

$$\nabla^2 X + \frac{\omega^2}{[\alpha(z)]^2} X + \left[\frac{\rho''}{2\rho} - \frac{3}{4}\left(\frac{\rho'}{\rho}\right)^2\right] X = 0$$

where $\alpha \equiv (k/\rho)^{1/2}$, and a prime denotes the derivative $\partial/\partial z$. The terms in ρ'' and $(\rho')^2$ can be ignored, since they do not affect the Langer approximation for X.

Figure 2 shows an example of two inhomogeneous half-spaces, with the medium immediately above the boundary being faster than that below, so that near-grazing incidence is possible from above, with a real ray being transmitted. If the ray parameter were somewhat greater than that illustrated in Figure 2b, then total internal reflection occurs in the upper medium, as shown in Figure 2c. A turning point is present, and angle i_1 is complex at $z=0$. However, energy can still leak through into the lower-velocity region below $z=0$, and can there propagate again along a direction which departs from the boundary with a real value for i_2. We shall find this effect is successfully quantified by the transmission coefficient (21). In Figure 3 is shown an example where the faster medium lies below the boundary. Figure 3b shows the case of a transmitted wave departing almost at the grazing angle (90°), and we shall discuss in a later section an example of this type, in

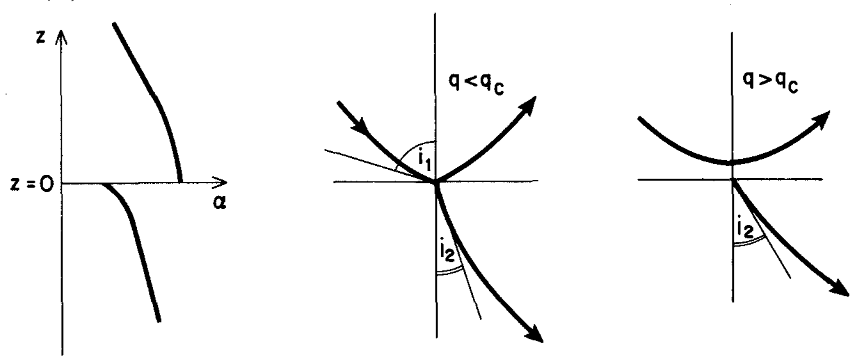

FIG. 2. Illustrations for a pressure wave incident within an inhomogeneous fluid upon the boundary of another inhomogeneous fluid having lower velocities: (a) the velocity profile near the boundary $z=0$; (b) ray trajectories for ray parameter q less than the boundary-grazing value q_c; and (c) ray trajectories for $q>q_c$.

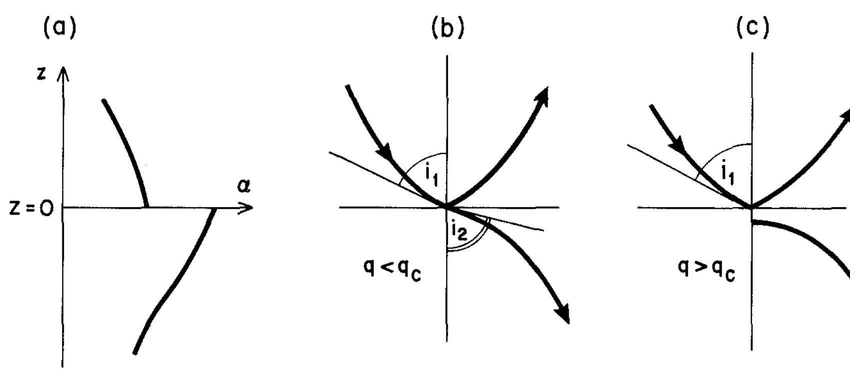

FIG. 3. As for Figure 2, but with the lower velocities on the side of the incident wave: (a) the velocity profile; (b) the case of ray parameter q less than q_c, the critical value at which the transmitted ray departs at the grazing angle ($i_2=90°$); and (c) the case $q>q_c$.

which there is a substantial effect on the amplitude and phase of the *reflected* wave, even though this does not itself approach grazing.

In any of the inhomogeneous regions depicted in Figures 2 and 3, the separated solution for X which propagates to the right, with ray parameter q and frequency ω, is

$$X=V(z,q,\omega)\exp[i\omega(qx-t)]$$

where V satisfies

$$\frac{\partial^2 V}{\partial z^2}+\omega^2\left(\frac{1}{\alpha^2}-q^2\right)V=0 \tag{22}$$

(cf equation 3). Uniformly asymptotic solutions are

$$V^{(1)} \sim \exp(i\pi/6)\xi^{1/2}Q^{-1/2}H^{(1)}_{1/3}(\omega\xi) \quad \text{(upcoming) and}$$
$$V^{(2)} \sim \exp(-i\pi/6)\xi^{1/2}Q^{-1/2}H^{(2)}_{1/3}(\omega\xi) \quad \text{(downgoing),} \tag{23}$$

with $Q^2=(\alpha^{-2}-q^2)$ and $\xi=\int_{z_q}^{z} Q\,dz$, where z_q is the unique depth at which $Q=0$.

The phase factors in (23) have been chosen to make $V^{(1)}+V^{(2)}$ the solution of (22) which is exponentially decaying below the turning point. In practice, when (23) is to be computed, the depth z_q often has to be found by extrapolation (analytic continuation) of $\alpha(z)$ outside the depth range in which the velocity profile is physically defined. This presents no difficulty when, for example, the profile is given by some power law $\alpha(z)\propto z^{\mu}$, or by a polynomial. However, when the profile is given merely as velocity values at a finite number of depths, the solution z_q of equation $\alpha(z)=q^{-1}$ requires some prior curve fitting.

The incident, reflected, and transmitted waves of Figures 2 and 3 are

$$X^{\text{inc}}=AV_1^{(2)}(z,q,\omega)\exp[i\omega(qx-t)]$$
$$X^{\text{ref}}=BV_1^{(1)} \qquad \exp[i\omega(qx-t)]$$
$$X^{\text{trans}}=CV_2^{(2)} \qquad \exp[i\omega(qx-t)]$$

and the ratios B/A, C/A are determined from requiring continuity of $\rho^{1/2}X$ and $\rho^{-1/2}\partial X/\partial z$ across $z=0$. [N.B. $\rho^{-1}\partial P/\partial z=\rho^{-1/2}\partial X/\partial z-\frac{1}{2}\rho^{-3/2}(\partial\rho/\partial z)X$, but the last term here is of the order of terms we neglect in the Langer approximation for $\partial X/\partial z$, and hence is itself negligible. This is a characteristic feature of the asymptotic method we are developing: once the correct dependent variable is clearly identified, only the leading term need be retained in most of the operations which have to be carried out on the variable.] These constraints give two equations relating A, B, and C, with solutions

$$\frac{B}{A}=\frac{V_1^{(2)}(0,q,\omega)}{V_1^{(1)}(0,q,\omega)}\cdot\frac{\rho_2\alpha_2C_1^{(2)}-\rho_1\alpha_1C_2^{(2)}}{\rho_2\alpha_2C_1^{(1)}+\rho_1\alpha_1C_2^{(2)}}$$
$$\frac{C}{A}=\left[\frac{\rho_1(0)}{\rho_2(0)}\right]^{1/2}\cdot\frac{V_1^{(2)}(0,q,\omega)}{V_2^{(2)}(0,q,\omega)}\cdot\frac{\rho_2\alpha_2[C_1^{(1)}+C_1^{(2)}]}{\rho_2\alpha_2C_1^{(1)}+\rho_1\alpha_1C_2^{(2)}}. \tag{24}$$

Recalling that the incident pressure at $z=0$ is

$$P^{\text{inc}}=[\rho_1(0)]^{1/2}AV_1^{(2)}(0,q,\omega)\exp[i\omega(qx-t)],$$

with similar results for reflected and transmitted pressures at $z=0$, it follows that the reflection coefficient is

$$\frac{P^{\text{ref}}}{P^{\text{inc}}}=\frac{B}{A}\left(\frac{\rho_1}{\rho_1}\right)^{1/2}\frac{V_1^{(1)}}{V_1^{(2)}}=\frac{\rho_2\alpha_2C_1^{(2)}-\rho_1\alpha_1C_2^{(2)}}{\rho_2\alpha_2C_1^{(1)}+\rho_1\alpha_1C_2^{(2)}} \tag{25}$$

and the transmission coefficient is

$$\frac{P^{\text{trans}}}{P^{\text{inc}}}=\frac{C}{A}\left(\frac{\rho_2}{\rho_1}\right)^{1/2}\frac{V_2^{(2)}}{V_1^{(2)}}=\frac{\rho_2\alpha_2[C_1^{(1)}+C_1^{(2)}]}{\rho_2\alpha_2C_1^{(1)}+\rho_1\alpha_1C_2^{(2)}} \tag{26}$$

which indeed are the results anticipated in equations (20) and (21). Given the intricacy of theories of wave propagation in inhomogeneous media, the seismologist who is concerned with applications can justifiably feel relieved that familiar plane-wave formula (e.g., equations 17 and 18) require only minor modification, involving a generalization only of the cosine terms. Furthermore, the replacements $C^{(j)}$ are simple to compute, since from (19), (13) and (12) we find the explicit result

$$C^{(1)/(2)}(z,q,\omega) \sim (1-\alpha^2 q^2)^{1/2} \exp(\pm i\pi/6) \frac{H_{2/3}^{(1)/(2)}(\omega\xi)}{H_{1/3}^{(1)/(2)}(\omega\xi)}. \tag{27}$$

Noting that $(1-\alpha^2 q^2)^{1/2} = \cos i$, equation (27) can be seen as providing a necessary correction factor in cases where z, q, ω are such that $\omega\xi$ is small. (When $\omega\xi$ is large, this factor approaches unity.) For real values of the ray parameter, $C^{(1)}$ and $C^{(2)}$ are complex

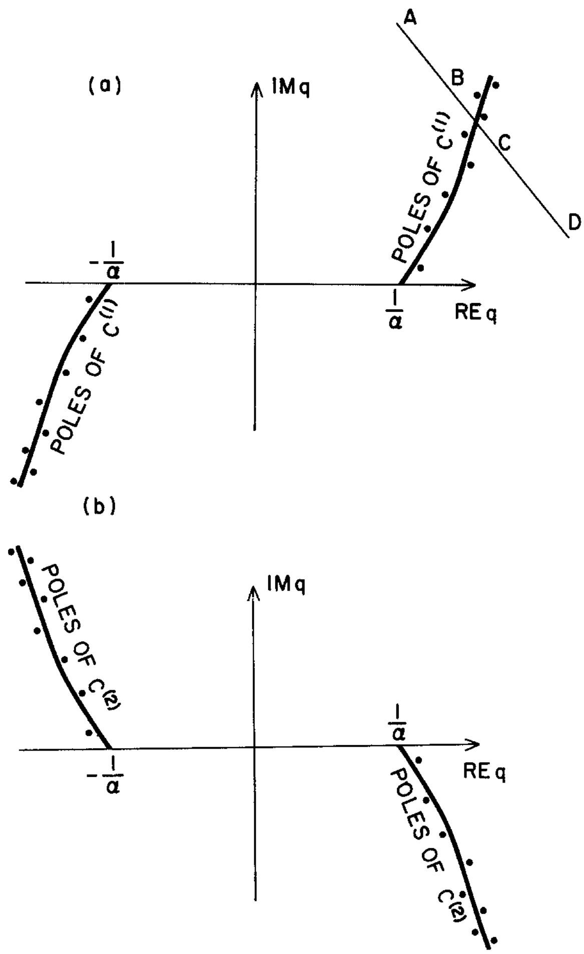

FIG. 4. Diagrams of the complex ray parameter plane, showing (a) poles of $C^{(1)}$, and (b) poles of $C^{(2)}$. All these poles lie close to Stokes' lines (on which ξ is real).

conjugates. If $q > 1/\alpha(z)$, then the branches chosen to make $V^{(1)}$ and $V^{(2)}$ analytic in q are such that

$$C^{(1)} \sim i(\alpha^2 q^2 - 1)^{1/2} \sim -C^{(2)} \quad \text{as} \quad \omega \to \infty. \tag{28}$$

More generally, it may be shown (Choy and Cormier, personal communication) that $C^{(1)}$ and $C^{(2)}$ are analytic functions of q, with singularities consisting of strings of poles which have properties similar to branch cuts. Thus, the string of poles of $C^{(1)}$ are the zeros of $H_{1/3}^{(1)}(\omega\xi)$, and are shown in Figure 4a. The two strings depart from the real axis at values near $\pm 1/\alpha$, and lie close to Stokes' lines given in the first and third quadrants by requiring ξ

to be real. As shown in Figure 4b, the strings of poles for $C^{(2)}$ lie close to Stokes' lines in the second and fourth quadrants. Away from the singularities, $C^{(j)} \sim (1-\alpha^2 q^2)^{1/2}$. For example, this approximation would be accurate along lines such as AB in Figure 4a. As a line of singularities is approached, the approximation fails, and fails radically if the path BC lies on or very close to one of the poles in the string. After crossing the line of singularities, then again $C^{(1)} \sim (1-\alpha^2 q^2)^{1/2}$, but a sign change is found to have occurred, giving the appearance of having crossed a branch cut in the square root. Since $ABCD$ does not cross a line of singularities for $C^{(2)}$, $C^{(2)} \sim (1-\alpha^2 q^2)^{1/2}$ is accurate for the whole path $ABCD$, and does not undergo a jump to the other root.

This discussion of acoustic waves is concluded with a brief description of reflection/transmission coefficients for the problem shown in Figure 2. The critical ray parameter is for grazing incidence ($i_1 = 90°$) and has value q_c (say) $= [\alpha_1(0)]^{-1}$.

For values $q < q_c$, the ray picture (Figure 2b) is similar to Figure 1 in that three real rays are interacting with the boundary. Nothing new is needed to interpret the reflection/transmission formulas (25) and (26), although, if q is only slightly less than q_c, it may be necessary to use Hankel functions to evaluate $C_1{}^{(1)}$ and $C_1{}^{(2)}$. $C_2{}^{(2)}$ is probably given sufficiently accurately by $\cos i_2$. However, if $q > q_c$, then the ray picture of Figure 2c results. The transmission coefficient is likely to be small, since energy must leak (or, "tunnel") from level $z = z_q$ to $z = 0$, and the reflection coefficient should tend to unit amplitude. Both these results can be inferred from the coefficient formulas (25) and (26), using the asymptotic property (28). However, this inference is misleading, because it is based on the use of $z = 0$ as the reference level for the incident wave, and, in the ray picture, the incident wave never appears to reach this level, nor does the reflected ray depart from it. A natural reference level to use is the turning point level $z = z_q$, but this has the disadvantage of varying with ray parameter and of being the level at which ray theory completely fails. Therefore, a reference level $z = z_0$ well above $z = z_q$ is taken, so that reflection/transmission coefficients include a phase factor due to vertical propagation. The ratios of interest are then taken between

$$P^{\text{inc}}(z_0, q, \omega) = [\rho_1(z_0)]^{1/2} A V_1{}^{(2)}(z_0, q, \omega) \exp[i\omega(qx - t)] \tag{29}$$

$$P^{\text{ref}}(z_0, q, \omega) = [\rho_1(z_0)]^{1/2} B V_1{}^{(1)}(z_0, q, \omega) \exp[i\omega(qx - t)] \tag{30}$$

and

$$P^{\text{trans}}(0, q, \omega) = [\rho_2(0)]^{1/2} C V_2{}^{(2)}(0, q, \omega) \exp[i\omega(qx - t)]. \tag{31}$$

The ratios B/A, C/A are given in equation (24), so that now the reflection coefficient is

$$\frac{P^{\text{ref}}(z_0, q, \omega)}{P^{\text{inc}}(z_0, q, \omega)} = \frac{V_1{}^{(1)}(z_0)}{V_1{}^{(2)}(z_0)} \cdot \frac{V_1{}^{(2)}(0)}{V_1{}^{(1)}(0)} \cdot \frac{\rho_2 \alpha_2 C_1{}^{(2)} - \rho_1 \alpha_1 \cos i_2}{\rho_2 \alpha_2 C_1{}^{(1)} + \rho_1 \alpha_1 \cos i_2}. \tag{32}$$

If the reference level z_0 is far above the turning point, then the first factor here is given by

$$\frac{V_1{}^{(1)}(z_0)}{V_1{}^{(2)}(z_0)} = -i \exp(2i\omega\xi)$$

in which $\xi = \int_{z_q}^{z_0} (\cos i/\alpha) dz$, and $-i$ is the familiar phase shift at a turning point. If also z_q is sufficiently far above the boundary $z = 0$, then each of the remaining two factors in (32) is -1, and $P^{\text{inc}} + P^{\text{ref}}$ could have been analyzed purely in terms of the standing wave $V_1{}^{(1)} + V_1{}^{(2)}$.

The transmission coefficient is

$$\frac{P^{\text{trans}}(0, q, \omega)}{P^{\text{inc}}(z_0, q, \omega)} = \left[\frac{\rho_1(0)}{\rho_1(z_0)}\right]^{1/2} \cdot \frac{1}{V_1{}^{(2)}(z_0)} \cdot \frac{\rho_2 \alpha_2 V_1{}^{(2)}(0)[C_1{}^{(1)} + C_1{}^{(2)}]}{\rho_2 \alpha_2 C_1{}^{(1)} + \rho_1 \alpha_1 \cos i_2}. \tag{33}$$

The factor $1/V_1^{(2)}(z_0)$ here carries the incident wave down from level z_0 to the turning point, and our interest is principally in the factor $V_1^{(2)}(0)[C_1^{(1)}+C_1^{(2)}]$, which must describe the tunnelling effect, quantifying the amount of energy which can leak from the level $z=z_q$ to $z=0$ in Figure 2c. The form of this factor is initially somewhat surprising, as $V_1^{(2)}(0)$ is an exponentially *growing* term as frequency increases, of order $\exp(\omega\eta)$ where $\eta = \int_0^{\bar{z}_q}(\alpha^2 q^2-1)^{1/2}dz$. However, we can use the Wronskian

$$V^{(2)}\frac{\partial V^{(1)}}{\partial z}-V^{(1)}\frac{\partial V^{(2)}}{\partial z}=\frac{4i}{\pi}$$

[appropriate for the normalization used in (23)] to find that $C_1^{(1)}+C_1^{(2)}$ is of order $\exp(-2\omega\eta)$. In fact,

$$V^{(2)}(0)[C_1^{(1)}+C_1^{(2)}]=\frac{4\alpha_1(0)}{\pi\omega V_1^{(1)}(0)}$$

and the exponential decay is of the order expected.

The above interpretation of reflection/transmission coefficients has merely demonstrated that our formulas do have the correct properties as frequency is increased at fixed ray parameter. This discussion separated the cases of $q \lesssim q_c$, and it extended the work of Gans (1915) and Brekhovskikh (1960, pages 206–215) in that properties of the wave function itself were emphasized, rather than properties of non-uniformly asymptotic formulas such as (4) and (6). Let us now turn to a numerical study, in which the different ranges of ray parameter are treated with the same uniformly asymptotic theory.

A numerical example: P waves within the core, incident upon the mantle from below.

The Langer approximation was introduced above for the radial wave function $R(r,p,\omega)$ in spherical polar coordinates (r,θ,ϕ), and was subsequently used for vertical wave functions in cartesian coordinates. In this section, the quantities $C^{(1)}$ and $C^{(2)}$ are obtained for a specific problem in spherical geometry. These functions are compared with the corresponding cosine of plane-wave theory and are shown to be significantly different in a manner which can influence even short-period seismic body waves.

The "cosine" to be studied here is that for P waves which impinge upon the core-mantle boundary from above. When such waves are incident at near-grazing angles, they can generate a variety of effects which plane-wave theory would not predict. These waves can be diffracted around the base of the mantle (Phinney and Alexander, 1966), and can also tunnel efficiently into the core when the ray parameter is greater than that for the core-grazing ray (Richards, 1973). The angle of incidence, i_m (say), with suffix m denoting the base of the mantle, is shown at the top left of Figure 5, and $\cos i_m$ is simply $(1-\alpha_m^2 p^2 r_{cm}^{-2})^{1/2}$, where p is the ray parameter and r_{cm} is the radius of the core-mantle boundary. The radial wave functions, and their radial derivatives, can be calculated for the lower mantle using a specific Earth model, and the B1 model of Jordan and Anderson (1974) is used here. Then $C^{(1)}=C^{(1)}(r_{cm},p,\omega)=\alpha_m(dR/dr)/(i\omega R)$ can be found from (27). In this model, the core-grazing ray parameter has value 254.9 sec/radian, and $C^{(1)}$ and $\cos i_m$ are shown in Figure 5. At infinite frequency, $C^{(1)}$ and $\cos i_m$ are identical, being real for $p<254.9$ and positive imaginary for $p>254.9$. At frequency 1 Hz, Figure 5 shows that the use of $\cos i_m$ for $C^{(1)}$ might lead to substantial errors throughout the range $253<p<257$. At frequency 0.1 Hz, $\cos i_m$ is a poor approximation to $C^{(1)}$ throughout the plotted range $248 \leq p \leq 262$.

To assess the significance of differences between values of $\cos i_m$ and $C^{(1)}$, the consequences for the reflection coefficient for P waves within the core, incident upon the core-mantle boundary from below are shown in Figure 6. This is a case similar to that shown in Figure 3, in that the incident wave is traveling within a relatively low-velocity medium and impinges upon the boundary of a higher-velocity medium. However, transmitted SV is now also possible (see the upper part of Figure 6), and the plane-wave reflection coefficient is RKK (say) where

$$RKK = \frac{-\rho_c\alpha_c \cos i_m + \rho_m \cos i_c\{\alpha_m \cos^2 2j_m + 4p^2 r_{cm}^{-2}\beta_m{}^3 \cos j_m \cos i_m\}}{\rho_c\alpha_c \cos i_m + \rho_m \cos i_c\{\alpha_m \cos^2 2j_m + 4p^2 r_{cm}^{-2}\beta_m{}^3 \cos j_m \cos i_m\}}. \tag{34}$$

Suffixes c and m, respectively, denote the top of the core and the base of the mantle.

In order to generalize this coefficient to account for frequency-dependence, curvature of the core-mantle boundary, and velocity gradients near the boundary, we have seen it is necessary to replace cosine terms in formulas such as (34). Only $\cos i_m, \cos i_c$, and $\cos j_m$

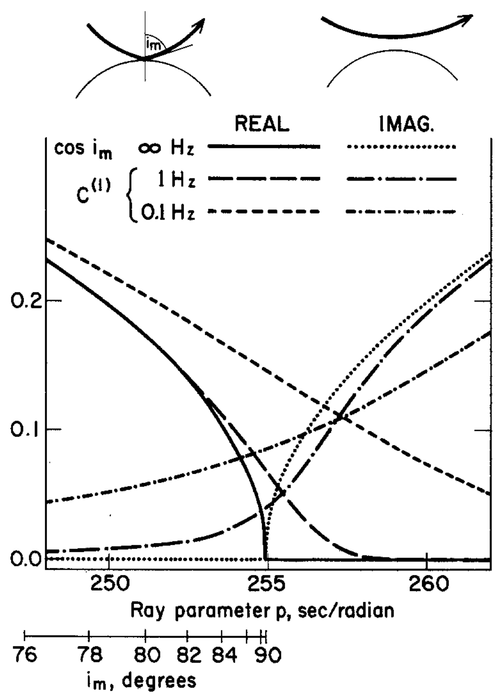

FIG. 5. Real and imaginary parts are plotted (against ray parameter) for the cosine and generalized cosine $C^{(1)}$ (equation 27) appropriate to P waves at the base of the mantle in Earth model B1 (Jordan and Anderson, 1974). At the *top left* and *right* are shown ray positions for, respectively, ray parameter less than and greater than the core-grazing value of 254.9 s/rad. As frequency increases from 0.1 Hz to 1 Hz, one sees that curves for $C^{(1)}$ tend to the values at infinite frequency, given by $\cos i_m$. At the *bottom* of the figure are shown values of angle i_m, for ray parameter values which permit the angle to be real.

need be considered, as $\cos 2j_m = 1 - 2\sin^2 j_m = 1 - 2p^2\beta_m{}^2 r_{cm}^{-2}$. [Using various trigonometrical identities, RKK can be presented in many different ways. The form of the right-hand side in (34) has been chosen to bring out the explicit dependence on cosines of i_m, i_c, and j_m. Since cosines are to be replaced by the ratios $C^{(1)}$ and $C^{(2)}$ given in (27), it might be thought that the identity $\cos^2 i = 1 - \sin^2 i = 1 - p^2 r^{-2}\alpha^2$ is to be avoided in prior manipulation of the plane-wave coefficients. However, full advantage *can* still be taken of this identity, since it is effectively equivalent to the radial wave equation

$$\frac{d^2}{dr^2}(rR) = -\omega^2\left(\frac{1}{\alpha^2} - \frac{p^2}{r^2}\right) rR.$$

That is, "$\cos^2 i$" arises from a double derivative of the radial wave function, but this is still equivalent to scalar multiplication, even in the radially heterogeneous medium.]

Our present concern is with evaluation of *RKK* in model B1, for ray parameter values such that the transmitted *P* wave into the base of the mantle departs from the core-mantle boundary at near-grazing angles, and in fact tunnels out, if $p > 254.9$ sec/radian (see the upper right of Figure 6). The angles i_c and j_m are nowhere near 90° for the range $248 \leqq p \leqq 262$, and so $\cos i_c$ and $\cos j_m$ may be retained in (34). To determine where $\cos i_m$ should be replaced by $C^{(1)}$, and where by $C^{(2)}$, we note that the problem has no downgoing *P* wave in the mantle, so $C^{(2)}$ does not appear, and we work with

$$RKK = \frac{-\rho_c\alpha_c C^{(1)} + \rho_m \cos i_c\{\alpha_m \cos^2 2j_m + 4p^2 r_{cm}^{-2}\beta_m 3 \cos j_m C^{(1)}\}}{\rho_c\alpha_c C^{(1)} + \rho_m \cos i_c\{\alpha_m \cos^2 2j_m + 4p^2 r_{cm}^{-2}\beta_m{}^3 \cos j_m C^{(1)}\}}. \tag{35}$$

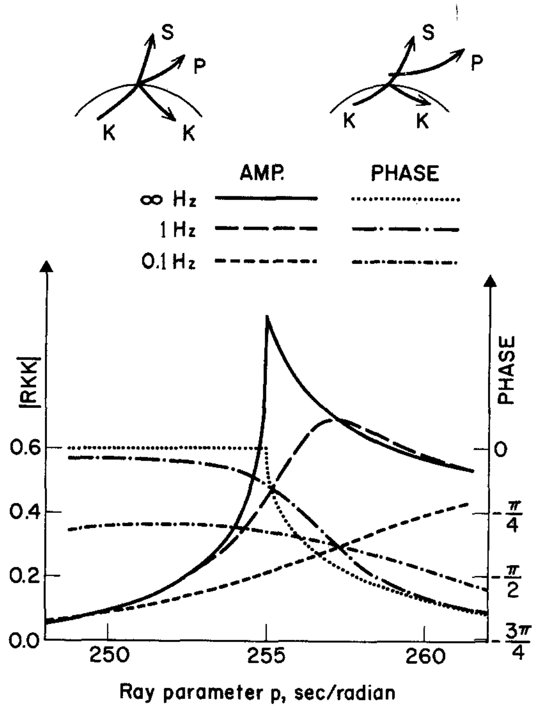

FIG. 6. Amplitude and phase for the reflection coefficient *RKK*, i.e., for the internally reflected core wave. Values are plotted against ray parameter, and the computation is based on Earth model B1 of Jordan and Anderson (1974). Solid line and dotted line give amplitude and phase for the plane-wave reflection coefficient (equation 34). The frequency-dependent coefficient is also plotted for 1 Hz and 0.1 Hz (values from equation 35), and substantial departures from the plane-wave coefficient (which is correct in the limit of high frequencies) are apparent.

At infinite frequency, formulas (34) and (35) yield the same value, with amplitude and phase as shown in Figure 6. At finite frequencies, (35) should be used, and the figure shows substantial frequency-dependent effects in both the amplitude and phase. Even at 1 Hz, the amplitude of *RKK* is substantially different from the plane-wave value throughout the range $254 < p < 257$. It is within this range that core phases such as *PKKKKP* are observed, and in this case of multiple internal reflections, *RKK* is raised to the third power. At 0.1 Hz, *RKK* is very poorly represented by the plane-wave formula.

CONCLUSIONS

The Langer approximation is found to have significant practical applications in the analysis of seismic body waves. It is a little more cumbersome to work with than the

WKBJ approximation (which it replaces), since both require prior evaluation of travel-time T and epicentral distance Δ as a function of ray parameter to construct the frequency-independent quantity $\xi = T - p\Delta$. Whereas the Langer approximation involves computation of $H_{1/3}^{(1)/(2)}(\omega\xi)$, and the WKBJ method requires only $\exp(\pm i\omega\xi)$, the former has merits of providing a uniformly asymptotic approximation to vertical wave functions. It can therefore be applied to analyze body waves near their turning point.

Such applications commonly arise when the body wave under study has, somewhere along its propagation path through the Earth, interacted with a discontinuity (such as the core-mantle boundary) in a fashion which generates waves propagating nearly parallel to the discontinuity. Examples include waves incident at near-grazing angles upon the discontinuity. Further examples arise when the angle of incidence is much less than 90°, but is near critical in the sense that a transmitted wave or mode-converted wave (e.g., S to P) travels nearly parallel to the discontinuity.

Plane-wave coefficients are found to be inadequate for such waves, but it is found that such coefficients can readily be adapted to the required form. The technique involves recognition that cosines, appearing in plane-wave coefficients, arise from vertical differentiation of the vertical wave functions in the vicinity of the discontinuity. This vertical differentiation is simple to carry out in radially inhomogeneous media, using the Langer approximation, and the resulting generalized "cosine" incorporates the effects of frequency-dependence, curvature of the discontinuity, and Earth structure on either side of the discontinuity.

Practical examples of the failure of plane-wave coefficients abound in seismic core phases. A numerical study is given for the cosine associated with P waves at the base of the mantle (see Figure 5). Further numerical results show that even for fairly short periods ($\sim$1 Hz), plane-wave reflection and transmission coefficients are often inaccurate for the core-mantle boundary. Accurate coefficients can readily be obtained (see Figure 6), and have significant frequency-dependence.

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