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Research Article

On the Adjacency, Laplacian, and Signless Laplacian Spectrum of Coalescence of Complete Graphs

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Coalescence as one of the operations on a pair of graphs is significant due to its simple form of chromatic polynomial. The adjacency matrix, Laplacian matrix, and signless Laplacian matrix are common matrices usually considered for discussion under spectral graph theory. In this paper, we compute adjacency, Laplacian, and signless Laplacian energy (Q energy) of coalescence of pair of complete graphs. Also, as an application, we obtain the adjacency energy of subdivision graph and line graph of coalescence from its Q energy.

1. Introduction

Throughout the discussion by a graph we mean simple graph without self loops or multiple edges. Let G be a simple graph on n vertices with vertex set $[v_1, v_2, \ldots, v_n]$. The line graph of a graph G is the graph with vertex set as edge set of G with two vertices (edges of G) adjacent if and only if they are having a vertex in common. Similarly, the subdivision graph of a graph G is the graph $S(G)$ obtained by inserting a vertex of degree two in each edge of G. The adjacency matrix of G denoted by $A(G)$ is a matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if vertex v_i is adjacent to v_j and 0 otherwise. Clearly, $A(G)$ is real symmetric so that eigenvalues of $A(G)$ which are roots of its characteristic equation given by $P(G;\lambda) = |\lambda I - A(G)| = 0$ are real. They are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and can be arranged in descending order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The spectrum of G is collection of eigenvalues along with their multiplicity and energy of a graph is simply defined as $\sum |\lambda_i|$. For more details and rigorous treatment on adjacency spectra and energy, see [1–4]. Let d_i denote the degree of a vertex v_i which is the number of edges incident on it. The degree matrix D is a diagonal matrix having diagonal entry as the degree of the corresponding vertex. We denote the average degree of a graph G, as $\text{avd}(G) =$ twice the no edges of G/no. of vertices of G. The matrix $C(G) = D(G) - A(G)$ is called Laplacian matrix. The roots of the characteristic polynomial of the Laplacian matrix are called Laplacian eigenvalues denoted by $\mu_1, \mu_2, \ldots, \mu_n$. The matrix $C(G)$ is also real symmetric but singular so eigenvalues can be arranged as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The Laplacian spectrum of G is the collection of Laplacian eigenvalues along with their multiplicity and Laplacian energy is defined as $E_L(G) = |\sum \mu_i - \text{avd}(G)|$.

For an extensive literature on Laplacian spectra and energy, one can refer to [5–11]. On similar lines, the signless Laplacian matrix of a graph G is defined as $Q(G) = D(G) + A(G)$. The signless Laplacian eigenvalues are also real and can be denoted by $\nu_1, \nu_2, \ldots, \nu_n$. The signless Laplacian energy (or simply Q energy) is defined similar to Laplacian energy as $E_{SL}(G) = |\sum \nu_i - \text{avd}(G)|$.

Some results on signless Laplacian energy are available in [12–15]. The Laplacian and signless Laplacian eigenvalues for a connected graph are nonnegative.

Let H_1 and H_2 be graphs on disjoint sets of vertices, respectively. Suppose $U = \{u_1, u_2, \ldots, u_t\}$ is a clique in H_1 and $W = \{w_1, w_2, \ldots, w_t\}$ is a clique in H_2. Let G be a graph obtained from H_1 and H_2 by identifying (coalescing into a single vertex) u_i and w_i $1 \leq i \leq t$. Then, G is an overlap of H_1 and H_2 in K_t. It may be viewed as generalized coalescence denoted by $H_1 \circ H_2$.

The structure of $H_1 \circ H_2$ depends on vertices chosen for overlap. Its chromatic polynomial can be split into chromatic polynomials of H_1 and H_2 (see [16]). If $t = 1$, we call it vertex coalescence denoted by $H_1 \circ_v H_2$ and for $t = 2$ we call it edge coalescence denoted by $H_1 \circ_e H_2$. For vertex coalescence of two graphs H_1 and H_2, the adjacency matrix has the form

$$A(H_1 \circ_v H_2) = \begin{bmatrix} 0 & J & J \\ J^T & A(H_1 - u) & O \\ J^T & O & A(H_2 - v) \end{bmatrix}. \quad (1)$$

Similarly, the edge coalescence of two graphs G_1 and G_2 has the adjacency matrix structure

$$A(H_1 \circ_e H_2) = \begin{bmatrix} A(K_2) & J & J \\ J^T & A(H_1 - e) & O \\ J^T & O & A(H_2 - e) \end{bmatrix}, \quad (2)$$

where J is matrix of all 1's and O is the matrix of all zeros having appropriate order.

2. Results

2.1. Adjacency Energy

Theorem 1. *The energy (adjacency energy) of $K_m \circ_v K_n$ is given by $E[K_m \circ_v K_n] = (m + n - 4) + |\alpha| + |\beta| + |\delta|$, where α, β, and δ are the roots of the cubic:*

$$x^3 - (m+n-4)x^2 + (mn - 3m - 3n + 6)x + (2mn - 3m - 3n + 4) = 0. \quad (3)$$

Proof. The coalescence of the complete graphs K_m and K_n at a point results in a graph with $m+n-1$ vertices and $(mC_2 + nC_2)$ edges. The adjacency matrix takes the form

$$A(K_m \circ_v K_n) = \begin{bmatrix} 0 & J_{1\times m-1} & J_{1\times n-1} \\ J_{m-1\times 1} & A(K_{m-1}) & O \\ J_{n-1\times 1} & O & A(K_{n-1}) \end{bmatrix} \quad (4)$$

so that characteristic polynomial is

$$P[K_m \circ_v K_n : \lambda] = |\lambda I - A(K_m \circ_v K_n)| = \begin{vmatrix} \lambda & -J_{1\times m-1} & -J_{1\times n-1} \\ -J_{m-1\times 1} & \lambda I_{m-1} - A(K_{m-1}) & O \\ -J_{n-1\times 1} & O & \lambda I_{n-1} - A(K_{n-1}) \end{vmatrix}. \quad (5)$$

By performing $C_1 + \sum_{i=2}^{m-1} C_i/(\lambda - (m-2))$ and $C_1 + \sum_{j=m+1}^{m+n-1} C_j/(\lambda - (n-2))$ in succession, we have

$$\begin{aligned} P[K_m \circ_v K_n : \lambda] &= \left(\lambda - \frac{m-1}{\lambda - (m-2)} - \frac{n-1}{\lambda - (n-2)}\right) \cdot P[K_{m-1} : \lambda] P[K_{n-1} : \lambda] \\ &= \left(\lambda - \frac{m-1}{\lambda - (m-2)} - \frac{n-1}{\lambda - (n-2)}\right)[\lambda - (m-2)] \\ &\quad \cdot (\lambda+1)^{m-2}[\lambda - (n-2)](\lambda+1)^{n-2}. \end{aligned} \quad (6)$$

On simplifying, finally we get

$$P[K_m \circ_v K_n : \lambda] : (\lambda+1)^{m+n-4}\left[\lambda^3 - (m+n-4)\lambda^2 + (mn - 3m - 3n + 6)\lambda + (m-1)(n-2) + (n-1)(m-2)\right]. \quad (7)$$

From this equation, the theorem follows. □

Theorem 2. *The energy (adjacency energy) of edge coalescence $(K_m \circ_e K_n)$ of complete graphs is given by $E[K_m \circ_e K_n] = (m+n-5) + |\alpha| + |\beta| + |\delta|$, where α, β, and δ are the roots of the cubic equation:*

$$x^3 - (m+n-5)x^2 + (mn - 4m - 4n + 11)x + (3mn - 7m - 7n + 15) = 0. \quad (8)$$

Proof. The coalescence of the complete graphs K_m and K_n on an edge results in a graph with $m + n - 2$ vertices and $mC_2 + nC_2 - 1$ edges. The adjacency matrix takes the form

$$A(K_m \circ_e K_n) = \begin{bmatrix} A(K_2) & J_{2\times m-2} & J_{2\times n-2} \\ J_{m-2\times 2} & A(K_m - e) & O \\ J_{n-2\times 2} & O & A(K_n - e) \end{bmatrix} \quad (9)$$

so that the characteristic polynomial of the edge coalescence is

$$P[K_m \circ_e K_n : \lambda] = |\lambda I - A(K_m \circ_e K_n)| = \begin{vmatrix} \lambda I - A(K_2) & -J_{2\times m-2} & -J_{2\times n-2} \\ -J_{m-2\times 2} & \lambda I - A(K_m - e) & O \\ -J_{n-2\times 2} & O & \lambda I - A(K_n - e) \end{vmatrix}. \quad (10)$$

By performing $C_j + \sum_{i=3}^{m} C_i/(\lambda - m + 3)$ for $j = 1, 2$ we have

$$P[(K_m \circ_e K_n) : \lambda] = \begin{vmatrix} \lambda - \dfrac{m-2}{\lambda - m + 3} & -1 - \dfrac{m-2}{-m+3} & -J_{1\times m-2} & -J_{1\times n-2} \\ -1 - \dfrac{m-2}{-m+3} & \lambda - \dfrac{m-2}{\lambda - m + 3} & -J_{1\times m-2} & -J_{1\times n-2} \\ O & O & \lambda I - A(K_{m-2}) & O \\ -J_{n-2\times 1} & -J_{n-2\times 1} & O & \lambda I - A(K_{n-2}) \end{vmatrix}. \quad (11)$$

By performing $C_j + \sum_{i=m+1}^{m+n-2} C_i/(\lambda - n + 3)$ for $j = 1, 2$ we have

$$P\left[(K_m \circ_e K_n) : \lambda\right] = \begin{vmatrix} \lambda - \frac{m-2}{\lambda - m + 3} - \frac{n-2}{\lambda - n + 3} & -1 - \frac{m-2}{\lambda - m + 3} - \frac{n-2}{\lambda - n + 3} & -J_{1\times m-2} & -J_{1\times n-2} \\ -1 - \frac{m-2}{\lambda - m + 3} - \frac{n-2}{\lambda - n + 3} & \lambda - \frac{m-2}{\lambda - m + 3} - \frac{n-2}{\lambda - n + 3} & -J_{1\times n-2} & -J_{1\times n-2} \\ O & O & \lambda I - A\left(K_{m-2}\right) & O \\ O & O & O & \lambda I - A\left(K_{n-2}\right) \end{vmatrix}$$
$$= \left[\lambda - \frac{m-2}{\lambda - m + 3} - \frac{n-2}{\lambda - n + 3}\right]^2 - \left[-1 - \frac{m-2}{\lambda - m + 3} - \frac{n-2}{\lambda - n + 3}\right]^2 P\left[K_{m-2} : \lambda\right] P\left[K_{n-2} : \lambda\right]. \tag{12}$$

On expanding and simplifying, we get the required polynomial and hence the theorem. □

2.2. Laplacian Energy. Now we discuss the Laplacian energy of coalescence.

Lemma 3 (see [17]). *If G is any connected graph of order n with Laplacian eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ with $\mu_n = 0$, then, the number of spanning trees of G is given by*

$$\tau(G) = \frac{\mu_1, \mu_2, \dots, \mu_{n-1}}{n}. \tag{13}$$

Theorem 4. *The Laplacian energy of the vertex coalescence of complete graphs K_m and K_n is given by*

$$E_L\left(K_m \circ_v K_n\right) = \frac{\left(3m^2 + 3n^2 - 5m - 5n + 1\right)}{(m + n - 1)}. \tag{14}$$

Proof. The degree matrix of the vertex coalescence $K_m \circ_v K_n$ with suitable labeling has the form

$$D\left[K_m \circ_v K_n\right] = \begin{bmatrix} m+n-2 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & m-1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & m-1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n-1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & n-1 \end{bmatrix} \tag{15}$$

The adjacency matrix is

$$A\left(K_m \circ_v K_n\right) = \begin{bmatrix} O & J & J \\ J^T & A\left[K_{m-1}\right] & O \\ J^T & O & A\left[K_{n-1}\right] \end{bmatrix}. \tag{16}$$

The Laplacian matrix now becomes

$$C\left(K_m \circ_v K_n\right) = \begin{bmatrix} m+n-2 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & m-1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & m-1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & m-1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & n-1 & -1 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & \cdots & n-1 \end{bmatrix} \tag{17}$$

so that the Laplacian polynomial is

$$\begin{aligned} &\left|\mu I - C\left(K_m \circ_v K_n\right)\right| \\ &= \begin{vmatrix} \mu-(m+n-2) & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & \mu-(m-1) & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \mu-(m-1) & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \mu-(m-1) & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu-(m-1) & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \mu-(m-1) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \mu-(m-1) \end{vmatrix}. \end{aligned} \tag{18}$$

Performing $C_1 - (1/(\mu-1)) \sum_{i=2}^{n} C_i$ we get

$$\begin{aligned} &\left|\mu I - C\left(K_m \circ_v K_n\right)\right| \\ &= \begin{vmatrix} \mu-(m+n-2)-\dfrac{m-1}{\lambda-1} & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & \mu-(m-1) & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \mu-(m-1) & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \mu-(m-1) & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu-(m-1) & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \mu-(m-1) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \mu-(m-1) \end{vmatrix}. \end{aligned} \tag{19}$$

Again performing $C_1 - (1/(\mu-1))\sum_{i=m+1}^{m+n-1} C_i$ and directly expanding along first column, we get

$$\begin{aligned}&|\mu I - C(K_m \circ_v K_n)|\\ &\quad = \left[\mu - (m+n-2) - \frac{n-1}{\mu-1} - \frac{m-1}{\mu-1}\right]\\ &\quad \cdot L[K_{m-1}:\lambda]\, L[K_{n-1}:\lambda] = \mu[\mu-(m+n-1)]\\ &\quad \cdot (\mu-1)(\mu-m)^{m-2}(\mu-n)^{n-2}.\end{aligned} \tag{20}$$

So that the Laplacian spectrum is $S_L(K_m \circ_v K_n) = m, m-2$ times, $n, n-2$ times, $m+n-1$, 1, and 0.

Now the $\mathrm{avd}(K_m \circ_v K_n) = (m(m-1)+n(n-1))/(m+n-1)$; hence, the Laplacian energy becomes

$$\begin{aligned}E_L(K_m \circ_v K_n) &= \left|\sum_{i=1}^{m+n-1} \mu_i - \frac{m(m-1)+n(n-1)}{m+n-1}\right|\\ &= \left|m - \frac{m(m-1)+n(n-1)}{m+n-1}\right|(m-2)\\ &\quad + \left|n - \frac{m(m-1)+n(n-1)}{m+n-1}\right|(n-2)\\ &\quad + \left|m+n-1 - \frac{m(m-1)+n(n-1)}{m+n-1}\right|\\ &\quad + \left|1 - \frac{m(m-1)+n(n-1)}{m+n-1}\right|\\ &\quad + \left|0 - \frac{m(m-1)+n(n-1)}{m+n-1}\right|\\ &= \frac{3m^2+3n^2-5m-5n+1}{m+n-1}.\end{aligned} \tag{21}$$

□

Corollary 5. *The number of spanning trees of $K_m \circ_v K_n$ according to Lemma 3 is $\tau(K_m \circ_v K_n) = m(m-2)n(n-2)(m+n-1)/(m+n-1) = m(m-2)n(n-2)$ as expected since $K_m \circ_v K_n$ has a cut point with number of spanning trees m^{m-2} and n^{n-2} in each block (complete graph) separately.*

Theorem 6. *The Laplacian energy of the edge coalescence of complete graphs K_m and K_n is given by*

$$\begin{aligned}E_L(K_m \circ_e K_n) &= \left|\frac{(m-n)(m-1)+2}{m+n-2}\right|(m-3)\\ &\quad + \left|\frac{(n-m)(m-1)+2}{m+n-2}\right|(n-3)\\ &\quad + \left|\alpha - \frac{m^2+n^2-m-n-2}{m+n-2}\right|\\ &\quad + \left|\beta - \frac{m^2+n^2-m-n-2}{m+n-2}\right|\\ &\quad + \left|\delta - \frac{m^2+n^2-m-n-2}{m+n-2}\right|,\end{aligned} \tag{22}$$

where α, β, and δ are the roots of the cubic equation:

$$\begin{aligned}&x^3 - 2(m+n-1)x^2 + (m^2+n^2+2mn-4)x\\ &\quad - 2(m+n-2)^2 = 0.\end{aligned} \tag{23}$$

Proof. The degree matrix of the edge coalescence $(K_m \circ_e K_n)$ with suitable labeling has the form

$$D(K_m \circ_e K_n) = \begin{bmatrix} m+n-3 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & m+n-3 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & m-1 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & m-1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & n-1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & n-1 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & n-1\end{bmatrix}. \tag{24}$$

The adjacency matrix is

$$A(K_m \circ_e K_n) = \begin{bmatrix} O & 1 & J_{1\times m-2} & J_{1\times n-2}\\ 1 & 0 & J_{1\times m-2} & J_{1\times n-2}\\ J_{m-2\times 1} & J_{m-2\times 1} & A(K_{m-2}) & O\\ J_{n-2\times 1} & J_{n-2\times 1} & O & A(K_{n-2})\end{bmatrix}.$$

$$C\left(K_m \circ_v K_n\right) = \begin{bmatrix} m+n-3 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & m+n-3 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & m-1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & m-1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & 0 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & 0 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & \cdots & 0 & -1 & -1 & \cdots & n-1 \end{bmatrix} \quad (25)$$

The Laplacian polynomial is then given by

$$\begin{aligned} &\left|\mu I - C\left(K_m \circ_e K_n\right)\right| \\ &= \begin{vmatrix} \mu-(m+n-3) & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & \mu-(m+n-3) & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \mu-(m-1) & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \mu-(m-1) & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & \mu-(n-1) & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 & 1 & \mu-(n-1) & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & \mu-(n-1) \end{vmatrix}. \end{aligned} \quad (26)$$

Performing $C_1 - (1/(\mu - 2)) \sum_{i=3}^{m} C_i$ and $C_2 - (1/(\mu - 2)) \sum_{i=3}^{m} C_i$,

$$\begin{aligned} &\left|\mu I - C\left(K_m \circ_e K_n\right)\right| \\ &= \begin{vmatrix} \mu-(m+n-3)-\dfrac{m-2}{\mu-2} & 1-\dfrac{m-2}{\mu-2} & \cdots & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1-\dfrac{m-2}{\mu-2} & \mu-(m+n-3)-\dfrac{m-2}{\mu-2} & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \mu-(m-1) & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \mu-(m-1) & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & \mu-(n-1) & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 & 1 & \mu-(n-1) & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & \mu-(n-1) \end{vmatrix} \end{aligned} \quad (27)$$

Again performing $C_1 - (1/(\mu - 2))\sum_{i=m+1}^{m+n-2} C_i$ and $C_2 - (1/(\mu - 2))\sum_{i=m+1}^{m+n-2} C_i$ and directly expanding give

$$\left|\mu I - C\left(K_m \circ_e K_n\right)\right| = \begin{vmatrix} \mu - (m+n-3) - \dfrac{m+n-4}{\mu-2} & 1 - \dfrac{m+n-4}{\mu-2} \\ 1 - \dfrac{m+n-4}{\mu-2} & \mu - (m+n-3) - \dfrac{m+n-4}{\mu-2} \end{vmatrix} L\left[K_{m-2} : \mu\right] L\left[K_{n-2} : \mu\right]. \quad (28)$$

On simplifying, we get the Laplacian polynomial as

$$\begin{aligned} \left|\mu I - C\left(K_m \circ_e K_n\right)\right| &= \mu (\mu - m)^{m-3} (\mu - n)^{n-3} \left[\mu^3 \right. \\ &\quad - 2(m+n-1)\mu^2 + \left(m^2 + n^2 + 2mn - 4\right)\mu \\ &\quad \left. - 2(m+n-2)^2\right]. \end{aligned} \quad (29)$$

On equating to zero and extracting eigenvalues from the equation above, the theorem follows.

Note. When $m = n$, the Laplacian polynomial is

$$\begin{aligned} \left|\mu I - C\left(K_n \circ_e K_n\right)\right| &= (\mu - n)^{2n-3} \\ &\quad \cdot \left[\mu^3 - 2(2n-1)\mu^2 + \left(4n^2 - 4\right)\mu - 8(n-1)^2\right] \\ &= (\mu - n)^{2n-3} \left[\mu - 2\right] \left[\mu - 2(n-1)^2\right]. \end{aligned} \quad (30)$$

The Laplacian eigenvalues are n, $2n - 3$ times, $2(n - 1)$ twice, 2, and 0 once.

Since $\operatorname{adg}(K_m \circ_e K_n) = (n^2 - n - 1)/(2n - 2)$ the Laplacian energy is

$$\begin{aligned} E_L\left(K_n \circ_e K_n\right) &= \left|0 - \frac{n^2 - n - 1}{2n-2}\right| + \left|2 - \frac{n^2 - n - 1}{2n-2}\right| \\ &\quad + \left|n - \frac{n^2 - n - 1}{2n-2}\right|(2n-3) \\ &\quad + \left|2(n-1) - \frac{n^2 - n - 1}{2n-2}\right| \times (2) \\ &= \frac{2n^3 + n^2 - 5n + 6}{2n-2} \quad \text{for } n = 2, 3, 4, \\ &= \frac{2n^3 + 3n^2 - 15n + 9}{2n-2} \quad \text{for } n \geq 5. \end{aligned} \quad (31)$$

□

2.3. Signless Laplacian Energy. Now we consider the case of signless Laplacian matrix of the coalescence of complete graphs and deduce the corresponding energy. Before we do so, consider the following results.

Lemma 7 (see [17]). *If G is any graph with p vertices and q edges, then characteristic polynomial of line graph $L(G)$ in terms of Q (signless Laplacian) polynomial is given by*

$$P\left[L(G) : \lambda\right] = (\lambda + 2)^{q-p} Q\left[G : \lambda + 2\right]. \quad (32)$$

Lemma 8 (see [18]). *If G is any graph with p vertices and q edges, then characteristic polynomial of subdivision graph $S(G)$ in terms of Q (signless Laplacian) polynomial is given by*

$$P\left[S(G) : \lambda\right] = \lambda^{q-p} Q\left[G : \lambda^2\right]. \quad (33)$$

Theorem 9. *The signless Laplacian energy of the vertex coalescence of complete graphs K_m and K_n is given by*

$$\begin{aligned} &E_{SL}\left(K_m \circ_v K_n\right) \\ &= \frac{|mn - 3m - n + 5|(m-2) + |mn - 3n - m + 5|(n-2) + \left|2mn - 3.5m - 3.5n + 6 + \sqrt{a}\right| + \left|2mn - 3.5m + 3.5n + 6 - \sqrt{a}\right| + |2mn - m - n + 7|}{m+n-2}, \end{aligned} \quad (34)$$

where $a = (m - n)^2 + m + n - 1.75$.

Proof. From the degree matrix and adjacency matrix of the vertex coalescence $(K_m \circ_v K_n)$, we have the signless Laplacian matrix:

$$Q\left(K_m \circ_v K_n\right) = \begin{bmatrix} m+n-2 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & m-1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & m-1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & m-1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & n-1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & n-1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & n-1 \end{bmatrix}. \quad (35)$$

The signless Laplacian polynomial is then

$$\left|\nu I - Q\left(K_m \circ_v K_n\right)\right| = \begin{vmatrix} \nu-(m+n-2) & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & \nu-(m-1) & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \nu-(m-1) & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \nu-(m-1) & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 & \nu-(n-1) & -1 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & \nu-(n-1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & \nu-(n-1) \end{vmatrix}. \quad (36)$$

Performing $C_1 + (1/(\nu - 2m + 3)) \sum_{i=2}^{n} C_i$,

$$\left|\nu I - Q\left(K_m \circ_v K_n\right)\right| = \begin{vmatrix} \nu-(m+n-2)-\dfrac{m-1}{\nu-(m-1)} & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & \nu-(m-1) & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \nu-(m-1) & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \nu-(m-1) & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 & \nu-(n-1) & -1 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & \nu-(n-1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & \nu-(n-1) \end{vmatrix}. \quad (37)$$

Again performing $C_1 + (1/(\nu - 2m + 3))\sum_{j=m+1}^{m+n-1} C_j$ then directly expanding along first column, we obtain

$$\begin{aligned}\left|\nu I - Q\left(K_m \circ_v K_n\right)\right| &= [\nu - (m+n-3)] \\ &\cdot [\nu - (m-2)]^{m-2}[\nu - (n-2)]^{n-2} \\ &\cdot \left[\nu^2 - (2m+2n-5)\nu + 4mn - 6m - 6n + 8\right].\end{aligned} \tag{38}$$

The signless Laplacian eigenvalues are $(m-2)(m-2)$ times, $(n-2)(n-2)$ times, $(m+n-2.5) \pm \sqrt{(m-n)^2 + m + n - 1.75}$, and $m+n-3$.

Now the $\mathrm{adg}(K_m \circ_e K_n) = (m(m-1) + n(n-1) - 1)/(m+n-2)$; hence, the theorem follows. □

Corollary 10. *From Lemma 7, the energy (adjacency energy) of line graph of $K_m \circ_v K_n$ is given by*

$$\begin{aligned}E\left[L\left(K_m \circ_v K_n\right)\right] &= \left(m^2 + n^2 - 3m - 3n + 2\right) \\ &+ (m-2)|m-4| + (n-2)|n-4| \\ &+ (2m+2n-9) \quad m, n \geq 3.\end{aligned} \tag{39}$$

In particular for $m, n \geq 4$,

$$\begin{aligned}E\left[L\left(K_m \circ_v K_n\right)\right] &= 2(m-2)^2 + 2(n-2)^2 - 3 \\ &+ 2\sqrt{(m-n)^2 + m + n - 1.75}.\end{aligned} \tag{40}$$

Corollary 11. *From Lemma 8, the energy (adjacency) of subdivision graph of $K_m \circ_v K_n$ where $m, n \geq 2$, is given by*

$$\begin{aligned}E\left[S\left(K_m \circ_v K_n\right)\right] &= 2\sqrt{m+n-3} + 2\sqrt{m-2}(m-2) \\ &+ 2\sqrt{n-2}(n-2) \\ &+ \sqrt{2}\sqrt{(2m+2n-5)} \\ &\pm 2\sqrt{m^2 + n^2 - 11m - 11n - 8}.\end{aligned} \tag{41}$$

Theorem 12. *The signless Laplacian energy of the edge coalescence of complete graphs K_m and K_n is given by*

$$\begin{aligned}E_{SL}\left(K_m \circ_e K_n\right) &= \left|\frac{2mn - 5m - 5n + 10}{m+n-2}\right|(m+n-2) \\ &+ \left|\frac{mn - n^2 - 3m - n + 6}{m+n-2}\right|(m-3) \\ &+ \left|\frac{mn - m^2 - 3n - m + 6}{m+n-2}\right|(n-3) \\ &+ \left|\alpha - \frac{m^2 + n^2 - m - n - 2}{m+n-2}\right| \\ &+ \left|\beta - \frac{m^2 + n^2 - m - n - 2}{m+n-2}\right| \\ &+ \left|\delta - \frac{m^2 + n^2 - m - n - 2}{m+n-2}\right|,\end{aligned} \tag{42}$$

where α, β, and δ are roots of the cubic equation:

$$\begin{aligned}&x^3 - (3m + 3n - 10)x^2 \\ &+ 2\left(m^2 + n^2 + 4mn - 11m - 11n + 20\right)x \\ &- 4(m+n-4)(m-2)(n-2) = 0.\end{aligned} \tag{43}$$

Proof. From the degree and adjacency matrix, the signless Laplacian matrix of the edge coalescence is

$$Q\left(K_m \circ_e K_n\right) = \begin{bmatrix} m+n-3 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & m+n-3 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & m-1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & m-1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & n-1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 & 1 & n-1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & n-1 \end{bmatrix}. \tag{44}$$

The signless Laplacian polynomial is then given by

$$\left|\nu I - Q\left(K_m \circ_e K_n\right)\right| = \begin{vmatrix} \nu-(m+n-3) & -1 & \cdots & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & \nu-(m+n-3) & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \nu-(m-1) & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \nu-(m-1) & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & 0 & \nu-(n-1) & 1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & 0 & -1 & \nu-(n-1) & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & \cdots & 0 & -1 & -1 & \cdots & \nu-(n-1) \end{vmatrix}. \tag{45}$$

Performing $C_1, C_2 + (1/(\nu - 2m+4)) \sum_{i=3}^{m} C_i$,

$$\left|\nu I - Q\left(K_m \circ_e K_n\right)\right| = \begin{vmatrix} \nu-(m+n-3)-\dfrac{m-2}{\nu-2m+4} & -1-\dfrac{m-2}{\nu-2m+4} & \cdots & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1-\dfrac{m-2}{\nu-2m+4} & \nu-(m+n-3)-\dfrac{m-2}{\nu-2m+4} & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & \nu-(m-1) & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \cdots & \nu-(m-1) & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & 0 & \nu-(n-1) & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & 0 & -1 & \nu-(n-1) & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & \cdots & 0 & -1 & -1 & \cdots & \nu-(n-1) \end{vmatrix}. \tag{46}$$

Again performing $C_1, C_2 + (1/(\nu-2)) \sum_{j=3}^{m+n-2} C_j$ and expanding directly yield

$$\left|\nu I - Q\left(K_m \circ_e K_n\right)\right| = Q\left(K_{m-2} : \nu\right) Q\left(K_{n-2} : \nu\right) \cdot \begin{vmatrix} \nu-(m+n-3)-\dfrac{m-2}{\nu-(2m+4)}-\dfrac{n-2}{\nu-(2n+4)} & -1-\dfrac{m-2}{\nu-(2m+4)}-\dfrac{n-2}{\nu-(2n+4)} \\ -1-\dfrac{m-2}{\nu-(2m+4)}-\dfrac{n-2}{\nu-(2n+4)} & \nu-(m+n-3)-\dfrac{m-2}{\nu-(2m+4)}-\dfrac{n-2}{\nu-(2n+4)} \end{vmatrix}. \tag{47}$$

On performing elementary operations, we finally arrive at

$$\begin{aligned} &\left|\nu I - Q\left(K_m \circ_e K_n\right)\right| \\ &\quad = \left[\nu-(m+n-4)\right]\left[\nu-(m-2)\right]^{m-3}\left[\nu-(n-2)\right]^{n-3}\left[\nu^3\right. \\ &\quad - (3m+3n-10)\,\nu^2 \\ &\quad + \left(2m^2+n^2+4mn-11m-11n+20\right)\nu \\ &\quad \left. - 4(m+n-4)(m-2)(n-2)\right]. \end{aligned} \tag{48}$$

On equating to zero and extracting eigenvalues from the equation above, the theorem follows. □

Corollary 13. *From Lemma 7, the energy (adjacency) of line graph of $K_m \circ_e K_n$ is given by*

$$E\left[L\left(K_m \circ_e K_n\right)\right] = \left|m^2 + n^2 - 3m - 3n + 2\right| + |m + n - 6| + |m - 4|(m - 3) + |n - 4|(n - 3) + \left|\lambda_1\right| + \left|\lambda_2\right| + \left|\lambda_3\right|, \tag{49}$$

where, λ_1, λ_2, and λ_3 are the roots of the cubic equation:

$$\lambda^3 - (3m + 3n - 16)\lambda^2 + \left(2m^2 + 2n^2 + 8mn - 34m - 34n + 92\right)\lambda + 4\left(m^2 + n^2 + 4mn - 14m - 14n + 32\right) - (m + n - 4)(mn - 2m - 2n + 4) = 0. \tag{50}$$

Corollary 14. *From Lemma 8, the energy (adjacency) of subdivision graph of $K_m \circ_e K_n$ where $m, n \geq 2$ is given by*

$$E\left[S\left(K_m \circ_e K_n\right)\right] = 2\sqrt{m + n - 4} + 2\sqrt{m - 2}(m - 3) + 2\sqrt{n - 2}(n - 3) + \left|\sqrt{\lambda_1}\right| + \left|\sqrt{\lambda_2}\right| + \left|\sqrt{\lambda_3}\right|, \tag{51}$$

where λ_1, λ_2, and λ_3 are the roots of the equation

$$\lambda^3 - (3m + 3n - 10)\lambda^2 + 2\left(m^2 + n^2 + 4mn - 11m - 11n + 20\right)\lambda - 4(m + n - 4)(m - 2)(n - 2) = 0. \tag{52}$$

Competing Interests

The authors declare that they have no competing interests.

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