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On the adjacent eccentric distance sum of graphs

Abstract. In this paper we show bounds for the adjacent eccentric distance sum of graphs in terms of Wiener index, maximum degree and minimum degree. We extend some earlier results of Hua and Yu [*Bounds for the Adjacent Eccentric Distance Sum*, International Mathematical Forum, Vol. 7 (2002) no. 26, 1289–1294].

The adjacent eccentric distance sum index of the graph G is defined as

$$\xi^{sv}(G) = \sum_{v \in V(G)} \frac{\varepsilon(v) D(v)}{\deg(v)},$$

where $\varepsilon(v)$ is the eccentricity of the vertex v, $\deg(v)$ is the degree of the vertex v and $D(v) = \sum_{u \in V(G)} d(u,v)$ is the sum of all distances from the vertex v.

1. Introduction. In this paper we will consider simple connected graphs. Let us start with a few definitions and notations. Let $G = (V(G), E(G))$ be a simple connected graph. For two vertex disjoint graphs G and F by $G \cup F$ we denote the vertex disjoint union of G and F and by $G + F$ we denote the join of the graphs. Moreover, by $2G$, we denote the graph $G \cup G$. If H is a subgraph of G, then by $G - H$ we denote the graph obtained from G by deleting all edges of H. By $\overline{G}$ we denote the complement of the graph G.

For a vertex $v \in V(G)$, by $\deg(v)$ we denote the degree of v in G. By the symbol $\delta(G)$ (resp. $\Delta(G)$) we denote the minimum degree (resp. maximum degree) over all vertices of G. A graph G is r-regular if all vertices of G have degree r. A graph G is $(\Delta(G), r)$-regular if all vertices of G have degree

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in the set $\{r, \Delta(G)\}$ with integer r, $r \neq \Delta(G)$. For vertices $u, v \in V(G)$ we define a distance $d(u,v)$ as the length of the shortest path between u and v. What is more, $D(v)$ denotes the sum of all distances from the vertex v. The eccentricity $\varepsilon(v)$ of a vertex v is the maximum from the distances between v and all other vertices. The minimum eccentricity over all vertices is denoted by $\mathrm{rad}(G)$ and called the radius of the graph G, while the maximum eccentricity is denoted by $\mathrm{diam}(G)$ and called the diameter of the graph G. Let K_n be a complete graph and P_n a path on n vertices.

Let S_i be the set of vertices of the eccentricity i in the graph G and let $n_i = |S_i|$, where $1 \leq i \leq \mathrm{diam}(G)$. Let

$$\delta_{\epsilon>2}(G) = \begin{cases} \min\{\deg(y) | y \in V(G) \backslash (S_1 \cup S_2)\}, & S_i \neq \emptyset \text{ for } i > 2 \\ 1, & S_i = \emptyset \text{ for } i > 2 \end{cases}$$

$$\Delta_{\epsilon>2}(G) = \begin{cases} \max\{\deg(y) | y \in V(G) \backslash (S_1 \cup S_2)\}, & S_i \neq \emptyset \text{ for } i > 2 \\ \Delta(G), & S_i = \emptyset \text{ for } i > 2 \end{cases}$$

and

$$\Delta_{\epsilon=2}(G) = \begin{cases} \max\{\deg(y) | y \in S_2\}, & S_2 \neq \emptyset \\ \Delta(G), & S_2 = \emptyset. \end{cases}$$

For other notation and terminology not defined here, the reader is referred to [1].

The Wiener index – the oldest topological index and probably the most used one is defined as a sum of the distances between all pairs of vertices in a graph G:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{v \in V(G)} D(v).$$

The adjacent eccentric distance sum index (shortly AEDS) has been introduced some time ago as follows

$$\xi^{sv}(G) = \sum_{v \in V(G)} \frac{\varepsilon(v) D(v)}{\deg(v)}.$$

The index is studied in [7] (see also references) for some molecular graphs and in [4] some relations to Wiener index are presented. Some mathematical properties of other molecular topological indices and their application for predicting biological and physical properties have been investigated in [2] –[8]. In this paper we give additional properties of the adjacent eccentric distance sum index for simple connected graphs.

2. Bounds for adjacent eccentric distance sum index. Hongbo Hua and Guihai Yu [4] presented and proved a few theorems. Motivated by this we were trying to find a more general bounds for the adjacent eccentric distance sum index, but let us now focus on the theorems.

Theorem 2.1 (Hua and Yu [4]). *Let G be a connected graph on n vertices. Then*

$$\xi^{sv}(G) \geq n_1 + \frac{2n(n-n_1)}{n-2}$$

with equality holding if and only if $G \simeq K_n - \frac{n-n_1}{2}K_2$, $n-n_1$ is even, where $K_n - kK_2$ is a graph obtained from K_n by deleting k independent edges for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

The next theorem presents us the inequality holding for the adjacent eccentric distance sum index and total eccentricity.

Theorem 2.2 (Hua and Yu [4]). *Let G be a connected graph on $n \geq 3$ vertices. Then*

$$\xi^{sv}(G) \geq \zeta(G)$$

with equality holding if and only if $G \simeq K_n$.

The next theorem we want to present is simply connected with the Wiener index.

Theorem 2.3 (Hua and Yu [4]). *Let G be a connected graph on $n \geq 3$ vertices with the minimum degree δ. Then*

$$\xi^{sv}(G) \leq \frac{2(n-\delta)}{\delta} W(G)$$

with equality holding if and only if $G \simeq K_n$, or $G \simeq K_n - \frac{n}{2}K_2$ for even n.

Let us now consider the first extended result of the Theorem 2.1.

Theorem 2.4. *Let G be a connected graph on n vertices. Then*

$$\xi^{sv}(G) \geq n_1 - 2n_2 + \frac{4n_2(n-1)}{n-2} + 3(n-n_1-n_2)\left(2 + \frac{6}{n-3} - \frac{n-2}{\delta_{\epsilon>2}(G)}\right).$$

Moreover,

$$\xi^{sv}(G) \geq n_1 - 2n_2 + \frac{4n_2(n-1)}{\Delta_{\epsilon=2}(G)} - 3(n-n_1-n_2)\left(1 - \frac{2n-1}{\Delta_{\epsilon>2}(G)}\right).$$

Proof. Let $S_1 = \{v_1, v_2, \ldots, v_{n_1}\}$ be the set of vertices with eccentricity equal to 1 and $S_2 = \{u_1, u_2, \ldots, u_{n_2}\}$ the set of vertices with eccentricity equal to 2. Let for $y \in V(G)$, $N_i(y)$ be the set of vertices at the distance i from the vertex y, where $1 \leq i \leq \varepsilon(y)$.

By the definition we have:

$$
\begin{aligned}
\xi^{sv}(G) &= \sum_{i=1}^{n_1} \frac{\varepsilon(v_i)D(v_i)}{\deg(v_i)} + \sum_{i=1}^{n_2} \frac{\varepsilon(u_i)D(u_i)}{\deg(u_i)} + \sum_{y\in V(G)\setminus(S_1\cup S_2)} \frac{\varepsilon(y)D(y)}{\deg(y)} \\
&\geq n_1 + 2\sum_{i=1}^{n_2} \frac{D(u_i)}{\deg(u_i)} + 3 \sum_{y\in V(G)\setminus(S_1\cup S_2)} \frac{D(y)}{\deg(y)} \\
&= n_1 + 2\sum_{i=1}^{n_2} \frac{\deg(u_i) + 2(n-\deg(u_i)-1)}{\deg(u_i)} \\
&\quad + 3 \sum_{y\in V(G)\setminus(S_1\cup S_2)} \frac{1}{\deg(y)} \sum_{i=1}^{\varepsilon(y)} i\cdot |N_i(y)| \\
&\geq n_1 + 2\sum_{i=1}^{n_2} \frac{-\deg(u_i) + 2n - 2}{\deg(u_i)} \\
&\quad + 3 \sum_{y\in V(G)\setminus(S_1\cup S_2)} \frac{\deg(y) + 2|N_2(y)| + 3(n-1-\deg(y)-|N_2(y)|)}{\deg(y)} \\
&= n_1 - 2n_2 + 4\sum_{i=1}^{n_2} \frac{n-1}{\deg(u_i)} \qquad (2.1) \\
&\quad + 3 \sum_{y\in V(G)\setminus(S_1\cup S_2)} \frac{-|N_2(y)| + 3(n-1) - 2\deg(y)}{\deg(y)} \\
&\geq n_1 - 2n_2 + \frac{4(n-1)n_2}{n-2} - 6(n-n_1-n_2) \\
&\quad + \frac{9(n-1)(n-n_1-n_2)}{n-3} - 3 \sum_{y\in V(G)\setminus(S_1\cup S_2)} \frac{|N_2(y)|}{\deg(y)} \\
&\geq n_1 - 2n_2 + \frac{4n_2(n-1)}{n-2} + 6(n-n_1-n_2) + \frac{18}{n-3}(n-n_1-n_2) \\
&\quad - 3(n-2) \sum_{y\in V(G)\setminus(S_1\cup S_2)} \frac{1}{\deg(y)} \\
&\geq n_1 - 2n_2 + \frac{4n_2(n-1)}{n-2} \\
&\quad + 3(n-n_1-n_2)\left(2 + \frac{6}{n-3} - \frac{n-2}{\delta_{\epsilon>2}(G)}\right).
\end{aligned}
$$

The last two inequalities hold by $|N_2(y)| \leq n-2-\deg(y)$ and by the definition of $\delta_{\epsilon>2}(G)$. Thus we get the result.

Moreover, we can apply $\Delta_{\epsilon=2}(G)$ and $\Delta_{\epsilon>2}(G)$ in the lines 7–8 of the inequality (2.1) to count the following relation:

$$\xi^{sv}(G) \geq n_1 - 2n_2 + \frac{4n_2(n-1)}{\Delta_{\epsilon=2}(G)} - 3(n-n_1-n_2)\left(1 - \frac{2n-1}{\Delta_{\epsilon>2}(G)}\right).$$

The proof is done. □

We will now try to find a graph for which the equality holds.

Notice that if $n_1 \neq 0$, then $n-n_1-n_2 = 0$ and we have the result of Theorem 2.1 by the first inequality of Theorem 2.4, and the second inequality of Theorem 2.4 leads to the following second extension of Theorem 2.1.

Proposition 2.5. *Let G be a connected graph on n vertices with $n_1 \neq 0$. Let $\Delta(G)$ be the maximum vertex degree in G and $\delta(G)$ be the minimum vertex degree in G. Then*

$$\xi^{sv}(G) \geq 3n_1 - 2n + \frac{4(n-1)(n-n_1)}{\Delta_{\epsilon=2}(G)}.$$

The equality holds for all $\Delta(G)$-regular graphs G with the diameter 2 and for all $(\delta(G), n-1)$-regular graphs G, where $\delta(G) < n-1$. In particular the equality is satisfied for $G = K_{n_1} + C_{n-n_1}$ with $n_1 \geq 1$.

Moreover, we get the following new result.

Proposition 2.6. *Let $n_1 = 0$ and let*

$$c(G) = \min\{n - 1 - \deg(y) - |N_2(y)| : \; y \in V(G)\backslash(S_1 \cup S_2)\}.$$

Then

$$\xi^{sv}(G) \geq \frac{4n_2(n-1)}{n-2} - 2n_2 - 3(n-n_2)\left(1 - \frac{2n-1}{\Delta_{\epsilon>2}(G)}\right).$$

Moreover,

$$\xi^{sv}(G) \geq \frac{4n_2(n-1)}{\Delta_{\epsilon=2}(G)} - 2n_2 - 3(n-n_2)\left(1 - \frac{2(n-1)+c(G)}{\Delta_{\epsilon>2}(G)}\right).$$

The equality holds for an infinite family of graphs with $\mathrm{diam}(G) = 3$.

Proof. The first inequality holds immediately by Theorem 2.4. The second inequality holds by applying the definition of $c(G)$ in the lines 7–8 of the inequality (2.1). The equality holds for $G = K_{2t} - B_{t-1,t-1} = \overline{B_{t-1,t-1}}$, where $t \geq 2$ and $B_{t-1,t-1}$ is the tree (double star) of order $2t$ with exactly two adjacent vertices of degree t (see Figure 1). In this case $\mathrm{rad}(G) = 2$, $\mathrm{diam}(G) = 3$, $c(G) = 1$ and $|S_3| = 2$. Similarly the graph obtained from $K_{|S_3|t}$ by joining new vertices y_i for $1 \leq i \leq |S_3|$ with t–sets of vertices of $K_{|S_3|t}$ pairwise disjoint satisfies the equality. □

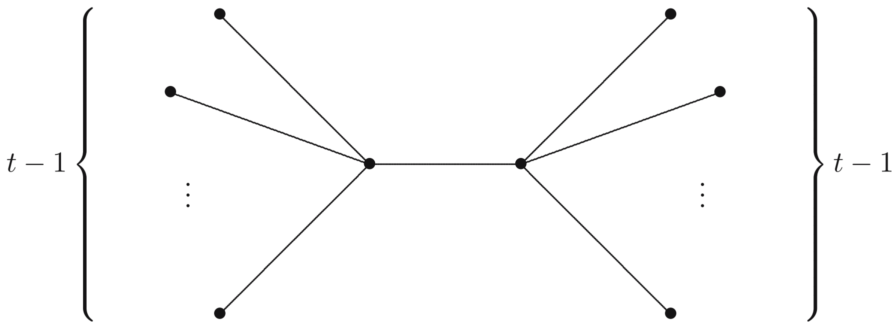

FIGURE 1. The graph $B_{t-1,t-1}$ with $t > 1$.

Now we present the first extension of Theorem 2.3.

Theorem 2.7. *Let G be a connected graph on $n \geq 3$ vertices with minimum degree $\delta = \delta(G)$. Let M_1 be the set of vertices with the minimum degree. Then*

$$\xi^{sv}(G) \leq 2\frac{n-\delta}{\delta}W(G) - \frac{n}{(\delta+1)\delta} \sum_{v \in V(G)\setminus M_1} D(v).$$

Equivalently

$$\xi^{sv}(G) \leq \frac{2(n-\delta-1)}{\delta+1}W(G) + \frac{n}{\delta(\delta+1)} \sum_{v \in M_1} D(v).$$

Proof.

$$\begin{aligned}
\xi^{sv}(G) &= \sum_{v \in V(G)} \frac{\varepsilon(v)D(v)}{\deg(v)} \\
&\leq \sum_{v \in V(G)} \frac{(n-\deg(v))D(v)}{\deg(v)} \\
&\leq \sum_{v \in M_1} \frac{(n-\delta)D(v)}{\delta} + \sum_{v \in V(G)\setminus M_1} \frac{(n-\delta-1)D(v)}{\delta+1} \\
&= \frac{(n-\delta)(\delta+1)}{\delta(\delta+1)} \sum_{v \in M_1} D(v) + \frac{(n-\delta-1)\delta}{(\delta+1)\delta} \sum_{v \in V(G)\setminus M_1} D(v)
\end{aligned}$$

$$= \frac{n\delta + n - \delta^2 - \delta}{\delta(\delta+1)} \sum_{v \in M_1} D(v) + \frac{n\delta - \delta^2 - \delta + n}{\delta(\delta+1)} \sum_{v \in V(G)\setminus M_1} D(v)$$
$$- \frac{n}{(\delta+1)\delta} \sum_{v \in V(G)\setminus M_1} D(v)$$
$$= 2\frac{n-\delta}{\delta} W(G) - \frac{n}{(\delta+1)\delta} \sum_{v \in V(G)\setminus M_1} D(v)$$
$$= \frac{2(n-\delta-1)}{\delta+1} W(G) + \frac{n}{\delta(\delta+1)} \sum_{v \in M_1} D(v).$$

□

Moreover, we get the following result.

Proposition 2.8. *The equality in Theorem 2.7 holds for an infinite family of graphs.*

Proof. Notice that $G = 2K_1 + K_{n-2}$ has $\delta(G) = n-2$. Thus $\xi^{sv}(G) = \frac{4}{n-2}W(G) - n$. So we get the upper bound. □

Now we present the next extension of Theorem 2.3.

Theorem 2.9. *Let G be a connected graph on $n \geq 3$ vertices with minimum degree $\delta = \delta(G)$. Let δ_2, δ_3 be the second (third) minimum degree, respectively. Let M_1 be the set of vertices with degree equal to the minimum degree and let M_2 be the set of vertices with degree equal to the second minimum degree. Then*

$$\xi^{sv}(G) \leq 2\frac{n-\delta}{\delta} W(G) + \frac{n(\delta-\delta_2)}{\delta\delta_2} \sum_{v \in M_2} D(v)$$
$$+ \frac{n(\delta-\delta_3)}{\delta\delta_3} \sum_{v \in V(G)\setminus(M_1 \cup M_2)} D(v).$$

Equivalently

$$\xi^{sv}(G) \leq \frac{2(n-\delta_2)}{\delta_2} W(G) + \frac{n(\delta_2-\delta_3)}{\delta_2\delta_3} \sum_{v \in V(G)\setminus(M_1 \cup M_2)} D(v)$$
$$- \frac{n(\delta-\delta_2)}{\delta\delta_2} \sum_{v \in M_1} D(v).$$

Proof.

$$\begin{aligned}
\xi^{sv}(G) &= \sum_{v\in V(G)} \frac{\varepsilon(v)D(v)}{\deg(v)} \le \sum_{v\in V(G)} \frac{(n-\deg(v))D(v)}{\deg(v)}\\
&\le \sum_{v\in M_1} \frac{(n-\delta)D(v)}{\delta} + \sum_{v\in M_2} \frac{(n-\delta_2)D(v)}{\delta_2}\\
&\quad + \sum_{v\in V(G)\setminus(M_1\cup M_2)} \frac{(n-\delta_3)D(v)}{\delta_3}\\
&= \frac{n-\delta}{\delta}\sum_{v\in M_1} D(v) + \frac{n-\delta_2}{\delta_2}\sum_{v\in M_2} D(v)\\
&\quad + \frac{n-\delta_3}{\delta_3}\sum_{v\in V(G)\setminus(M_1\cup M_2)} D(v)\\
&= \frac{(n-\delta)\delta_2\delta_3}{\delta\delta_2\delta_3}\sum_{v\in M_1} D(v) + \frac{(n-\delta_2)\delta\delta_3}{\delta\delta_2\delta_3}\sum_{v\in M_2} D(v)\\
&\quad + \frac{(n-\delta_3)\delta\delta_2}{\delta\delta_2\delta_3}\sum_{v\in V(G)\setminus(M_1\cup M_2)} D(v)\\
&\le \frac{n-\delta}{\delta}2W(G) + \frac{n\delta_3(\delta-\delta_2)}{\delta\delta_2\delta_3}\sum_{v\in M_2} D(v)\\
&\quad + \frac{n\delta_2(\delta-\delta_3)}{\delta\delta_2\delta_3}\sum_{v\in V(G)\setminus(M_1\cup M_2)} D(v)\\
&= 2\frac{n-\delta}{\delta}W(G) + \frac{n(\delta-\delta_2)}{\delta\delta_2}\sum_{v\in M_2} D(v)\\
&\quad + \frac{n(\delta-\delta_3)}{\delta\delta_3}\sum_{v\in V(G)\setminus(M_1\cup M_2)} D(v)\\
&= 2\frac{n-\delta}{\delta}W(G) + \frac{n(\delta-\delta_2)}{\delta\delta_2}\sum_{v\in M_2} D(v)\\
&\quad + \frac{n(\delta-\delta_3)}{\delta\delta_3}\left(2W(G) - \sum_{v\in M_1\cup M_2} D(v)\right)\\
&= W(G)\left(\frac{2(n-\delta)\delta_3}{\delta\delta_3} + \frac{2n(\delta-\delta_3)}{\delta\delta_3}\right) + \frac{n(\delta-\delta_2)}{\delta\delta_2}\sum_{v\in M_2} D(v)\\
&\quad - \frac{n(\delta-\delta_3)}{\delta\delta_3}\sum_{v\in M_1\cup M_2} D(v)\\
&= \frac{2(n-\delta_3)}{\delta_3}W(G) + \frac{n(\delta_3-\delta_2)}{\delta_2\delta_3}\sum_{v\in M_2} D(v) - \frac{n(\delta-\delta_3)}{\delta\delta_3}\sum_{v\in M_1} D(v).
\end{aligned} \tag{2.2}$$

Moreover, by the lines 10–11 of the formula (2.2) we get the equivalent relation:

$$\xi^{sv}(G) \leq \frac{2(n-\delta_2)}{\delta_2}W(G) + \frac{n(\delta_2-\delta_3)}{\delta_2\delta_3}\sum_{v\in V(G)\setminus(M_1\cup M_2)} D(v)$$
$$- \frac{n(\delta-\delta_2)}{\delta\delta_2}\sum_{v\in M_1} D(v).$$

□

Proposition 2.10. *The equality in Theorem 2.9 holds for an infinite family of graphs.*

Proof. Notice that we get the upper bound for all graphs isomorphic to $K_{n_1} + 2K_1$. □

By Theorem 2.9 we have the following result.

Proposition 2.11. *If* $V(G)\setminus(M_1 \cup M_2) = \emptyset$ *then*

$$\xi^{sv}(G) \leq \frac{2(n-\delta_2)}{\delta_2}W(G) - \frac{n(\delta-\delta_2)}{\delta\delta_2}\sum_{v\in M_1} D(v)$$

or equivalently

$$\xi^{sv}(G) \leq 2\frac{n-\delta}{\delta}W(G) + \frac{n(\delta-\delta_2)}{\delta\delta_2}\sum_{v\in M_2} D(v).$$

In future study we will characterize extremal graphs with respect to the adjacent eccentric distance sum index among all n-vertex graphs from some families of connected graphs.

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