On the Affine Connections that Give Rise to a Given Curvature (*).

DOMINIC G. B. EDELEN (Bethlehem, Pennsylvania)

Summary. – *A problem of both theoretical and practical importance is that of characterizing the collection of all affine connections that gives rise to a given curvature structure on a subset of a differentiable manifold of finite dimension. This problem is solved in closed form in Section three. We also show that the cardinality of the collection of all distinct connections that give the same curvature is that of the continuum, and that the connections of any two curvature structures can be brought into a 1-to-1 correspondence.*

1. – C^∞-modules of antiexact differential forms.

Let U be an open region of an n-dimensional differentiable manifold that is starlike with respect to one of its points $P_0 \subset U$. By this, we mean that U can be covered with a coordinate patch with specific coordinate functions (x^α) such that P_0 has coordinates (x_0^α) and, for any point $P \in U$ with coordinates (x^α), the set of points with coordinates $(x_0^\alpha + \lambda(x^\alpha - x_0^\alpha))$ belong to U for all $\lambda \in [0, 1]$. All geometric object fields on U are assumed to be evaluated in terms of the specific coordinate functions (x^α) relative to which U is starlike.

The graded associative algebra of exterior forms on U is denoted by $\Lambda(U)$: $\Lambda^0(U)$ denotes the class of C^∞ functions on U and $\Lambda^k(U)$ denotes the C^∞-module of differential forms of degree k on U. The exterior product and the exterior derivative are denoted by $\wedge$ and d, respectively, while $\lrcorner$ is used to denote the operation of inner multiplication as defined in reference [1].

We define a vector field $X(x^\alpha)$ on U in the preferred coordinate system (x^α) by

$$X(x^\alpha) = (x^\beta - x_0^\beta)\, \partial/\partial x^\beta \,, \tag{1.1}$$

and note that

$$X\big(x_0^\alpha + \lambda(x^\alpha - x_0^\alpha)\big) = \lambda X(x^\alpha) \tag{1.2}$$

for all $\lambda \in [0, 1]$. Define the homotopy operator H on $\Lambda^k(U)$, $k = 0, \ldots, n$, in the

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standard way by

$$(H\omega)(x^\alpha) = \int_0^1 X(x^\alpha) \lrcorner\, \omega\big(x_0^\alpha + \lambda(x^\alpha - x_0^\alpha)\big)\lambda^{k-1} d\lambda\,. \tag{1.3}$$

Although some of the properties of H given in the following Lemma are trivial consequences of the others in the list, they are given explicitly in order to simplify proofs later on.

LEMMA 1.1. – *The operator H ha the following properties:*

H_1: *H maps $\Lambda^k(U)$ into $\Lambda^{k-1}(U)$, $k \geqslant 1$, and commutates with addition and multiplication by constants,*

H_2: *$dH + Hd =$ identity for $k \geqslant 1$, $(Hdf)(x^\alpha) - f(x_0^\alpha) = f(x^\alpha)$ for $k = 0$,*

H_3: $(HH\omega)(x^\alpha) = 0$, $(H\omega)(x_0) = 0$,

H_4: $HdH = H$, $dHd = d$,

H_5: $HdHd = Hd$, $dHdH = dH$, $(dH)(Hd) = 0$, $(Hd)(dH) = 0$,

H_6: $X(x^\alpha) \lrcorner\, (H\omega)(x^\alpha) = 0$, $HX \lrcorner = 0$.

PROOF. – Property H_1 follows directly from the definition of H, while H_2 is simply the standard result that is used in establishing the Poincaré lemma. Since $X(x_0^\alpha) = 0$ from (1.1), (1.3) yields $(H\omega)(x_0) = 0$. When use is made of (1.2), (1.3) gives

$$HH\omega = \int_0^1\int_0^1 X(x^\alpha) \lrcorner \{\mu X(x^\alpha) \lrcorner\, \omega(x_0^\alpha + \lambda\mu(x^\alpha - x_0^\alpha)\}\lambda^{k-1}\mu^{k-2} d\lambda\, d\mu = 0$$

and hence H_3 is established. Properties H_4 follow directly by allowing H and d to act on $dH + Hd =$ identity and using H_3; *i.e.* $H = H(\text{identity}) = H(dH + Hd) = HdH + HHd = HdH$. Allowing H and d to act on properties H_4 established properties H_5. Properties H_6 are established in exactly the same way as that used to establish $(HH\omega) = 0$; *i.e.*, (1.2) can be used to obtain a factor of the form $X(x^\alpha) \lrcorner \{X(x^\alpha) \lrcorner (\;)\}$ under the integral sign, which, of course, vanishes.

Property H_2 allows us to make the following definitions.

DEFINITION. $(dH\omega)(x^\alpha) = \omega_e(x^\alpha)$ is the *exact part* of $\omega \in \Lambda^k(U)$.

DEFINITION. $(Hd\omega)(x^\alpha) = \omega_a(x^\alpha)$ is the *antiexact part* of $\omega \in \Lambda^k(U)$ for $k \geqslant 1$. Since no elements of $\Lambda^0(U)$ is exact, we make the following agreement.

AGREEMENT. – An element of $\Lambda^0(U)$ is its own antiexact part.

Let the set $\mathcal{A}^k(U)$, $k = 0, \ldots, n$ be defined by

$$\mathcal{A}^k(U) = \begin{cases} \Lambda^0(U)\,, & k = 0 \\ \{\omega \in \Lambda^k(U) | X \lrcorner\, \omega = 0\,, \quad \omega(x_0^\alpha) = 0\}\,, & k \geqslant 1\,. \end{cases} \tag{1.4}$$

DEFINITION. – The elements of $\mathcal{A}^k(U)$ are referred to as *antiexact* differential forms of degree k.

The reason for this definition is that $dH + Hd =$ identity for forms of positive degree while every $dH\omega$ is exact and every $Hd\omega$ belongs to $\mathcal{A}^k(U)$ for $\omega \in \Lambda^k(U)$; that is, every differential form of positive degree is the sum of an exact form and an antiexact form. The importance of antiexact forms is a direct consequence of the results established in the following Lemma.

LEMMA 1.2. – *Antiexact forms possess the following properties*:

A_1: $\mathcal{A}^k(U) \subset \ker(H)$,

A_2: $\omega \in \mathcal{A}^k(U)$, $\gamma \in \mathcal{A}^l(U)$ *implies* $\omega \wedge \gamma \in \mathcal{A}^{k+l}(U)$,

A_3: $\mathcal{A}^k(U)$ *is a* C^∞*-module over* $\mathcal{A}^0$,

A_4: H *is the inverse of* d *on* $\mathcal{A}^k(U)$ *for* $\gamma \geqslant 1$.

PROOF. – If $\omega \in \mathcal{A}^k(U)$, then $X(x^\alpha) \lrcorner\, \omega(x^\alpha) = 0$ and hence

$$H\omega = \int_0^1 X(x^\alpha) \lrcorner\, \omega(x_0^\alpha + \lambda(x^\alpha - x_0^\alpha))\, \lambda^{k-1}\, d\lambda =$$

$$= \int_0^1 \frac{1}{\lambda} X(x_0^\alpha + \lambda(x_0^\alpha - x^\alpha)) \lrcorner\, \omega(x_0^\alpha + \lambda(x^\alpha - x_0^\alpha))\, \lambda^{k-1}\, d\lambda = 0\,.$$

Thus, A_1 is established. Clearly, A_2 holds for elements of $\mathcal{A}^0(U) = \Lambda^0(U)$ and hence it sufficies to establish A_2 for $k \geqslant 1$. Under the hypotheses of A_2, we know that $\omega \wedge \gamma \varepsilon \Lambda^{k+l}(U)$ and $(\omega \wedge \gamma)(x_0^\alpha) = 0$ since the first factor ω vanishes at (x_0^α) because $k \geqslant 1$. Further, $X \lrcorner\, (\omega \wedge \gamma) = (X \lrcorner\, \omega) \wedge \gamma + (-1)^k \omega \wedge (X \lrcorner\, \gamma) = 0$ since $X \lrcorner\, \omega = 0$, $X \lrcorner\, \gamma = 0$ by hypothesis. Thus, $X \lrcorner\, (\omega \wedge \gamma) = 0$, $(\omega \wedge \gamma)(x_0^\alpha) = 0$ and hence $\omega \wedge \gamma \in \mathcal{A}^{k+l}(U)$; A_2 is established. A_3 then follows directly from A_2 since the set $\mathcal{A}^k(U)$ is closed under addition and also under (exterior) multiplication by all elements of $\mathcal{A}^0(U) = \Lambda^0(U)$. Property H_2 gives, for $\omega \in \Lambda^k(U)$, $k \geqslant 1$, $\omega = dH\omega + Hd\omega$. However, A_1 shows that $H\omega = 0$ for $\omega \in A^k(U)$, so that we have $\omega = Hd\omega$, and A_4 is established.

REMARK. – For the case $k = 0$, property H_2 gives $f(x^\alpha) = f(x_0^\alpha) + (Hdf)(x^\alpha)$. Accordingly, the operator H can be used to invert the operator d on $\mathcal{A}^k(U)$ for all values of k.

The following results are straightforward consequences of the properties H_1 through H_6 and A_1 through A_4, so we simply state them without proof.

LEMMA 1.3. – *If* $\omega \in \Lambda^k(U)$, $k \geq 1$ *satisfies* $X \lrcorner\, \omega = 0$, $X \lrcorner\, d\omega = 0$, *then* $\omega = 0$.

LEMMA 1.4. – *Any* $\omega \in \Lambda^k(U)$, $k \geq 1$, *has the unique representation* $\omega = d\mu_1 + \mu_2$ *under the conditions* $\mu_1 \in \mathcal{A}^{k-1}(U)$, $\mu_2 \in \mathcal{A}^k(U)$. *If these conditions are satisfied, then* $\mu_2 = H(d\omega)$ *is unique, while* $\mu_1 = H(\omega)$ *is unique for* $k > 1$ *and* $\mu_1 = H(\omega) + \text{constant}$ *for* $k = 1$.

2. – Affine connections and curvatures.

Let $\mathbf{\Gamma}$ denote the n-by-n matrix of affine connection 1-forms $((\Gamma^i_j)) = ((\gamma^i_{jk}\, dx^k))$ and let $\mathbf{\Theta}$ denote the n-by-n matrix of curvature 2-forms of the connection $\mathbf{\Gamma}$. Thus, $\mathbf{\Gamma}$ and $\mathbf{\Theta}$ are related by the second half of Cartan's structure equations

$$d\mathbf{\Gamma} = \mathbf{\Gamma} \wedge \mathbf{\Gamma} + \mathbf{\Theta}\,, \tag{2.1}$$

where the matrix exterior product $\mathbf{\Gamma}\wedge\mathbf{\Gamma}$ is defined by

$$\mathbf{\Gamma}\wedge\mathbf{\Gamma} = ((\Gamma^j_k \wedge \Gamma^k_i))\,.$$

The following results were first obtained by representing each differential form by $dH\omega + Hd\omega$ and using the fact that H is the inverse of d on $\mathcal{A}^k(U)$ in order to integrate the resulting equations that obtain from (2.1). The following method of proof is significantly simpler than such a procedure. We make the following agreement in order to simplify the notation.

AGREEMENT 2.1. – If each of the elements of an n-by-n matrix $\boldsymbol{\psi}$ of differential forms of fixed degree belongs to a collection $K(U)$ on the set U, then we shall simply write $\boldsymbol{\psi} \in K(U)$. Thus, the assumption that each of the entries of $\mathbf{\Gamma}$ is an element of $\Lambda^1(U)$ is simply written $\mathbf{\Gamma} \in \Lambda^1(U)$.

LEMMA 2.1. – *Let* $\boldsymbol{A} \in \mathcal{A}^0(U)$ *be nonsingular and let* $\boldsymbol{\theta} \in \mathcal{A}^1(U)$. *The connection*

$$\mathbf{\Gamma} = (d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\theta})\boldsymbol{A}^{-1} \tag{2.2}$$

has the curvature

$$\mathbf{\Theta} = \boldsymbol{A}(d\boldsymbol{\theta} - \boldsymbol{\theta}\wedge\boldsymbol{\theta})\boldsymbol{A}^{-1}\,. \tag{2.3}$$

PROOF. – The result is clear with $\boldsymbol{A} = \boldsymbol{E} =$ identity matrix. For $\boldsymbol{A} \neq \boldsymbol{E}$, (2.3) is easily established by direct substitution of (2.2) into (2.1).

LEMMA 2.2. – *Let* $\mathbf{\Gamma} \in \Lambda^1(U)$ *be given. There exists a nonsingular* $\boldsymbol{A} \in \Lambda^0(U)$ *and a* $\boldsymbol{\theta} \in \mathcal{A}^1(U)$ *such that* (2.2) *holds.*

PROOF. – Since (2.2) gives

$$\mathbf{\Gamma} \boldsymbol{A} = d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\theta} \tag{2.2a}$$

it is sufficient to establish the existence of a nonsingular matrix $\boldsymbol{A} \in \mathcal{A}^0(U)$ and a $\boldsymbol{\theta} \in \mathcal{A}^1(U)$ such that (2.2*a*) holds. Since $\boldsymbol{\theta} \in \mathcal{A}^1(U)$, $\boldsymbol{A}\boldsymbol{\theta} \in \mathcal{A}^1(U)$ from the module property of $\mathcal{A}^k(U)$, and hence $\boldsymbol{A}\boldsymbol{\theta} \in \ker(H)$ by A_1. Applying H to both sides of (2.2*a*) gives $H(\mathbf{\Gamma}\boldsymbol{A}) = H(d\boldsymbol{A})$. However, $\boldsymbol{A} \in \mathcal{A}^0(U) = \Lambda^0(U)$ and hence $(Hd\boldsymbol{A})(x^\alpha) = \boldsymbol{A}(x^\alpha) - \boldsymbol{A}(x_0^\alpha)$. We thus have the following linear integral equation for the determination of $\boldsymbol{A}$:

$$\boldsymbol{A}(x^\alpha) = \boldsymbol{A}(x_0^\alpha) + (H\mathbf{\Gamma}\boldsymbol{A})(x^\alpha)\,. \tag{2.4}$$

Since all quantities are assumed to be C^∞ geometric object fields, standard existence theorems show that (2.4) possesses a nonsingular solution on some open set N that contains the point (x_0^α) provided $a_0 = \det(\boldsymbol{A}(x_0^\alpha)) \neq 0$. If we set $a(x^\alpha) = \det(\boldsymbol{A}(x^\alpha))$ then, at all points of N, (2.4) yields $da = a\,\mathrm{tr}\,(d\boldsymbol{A}\,\boldsymbol{A}^{-1}) = a\,\mathrm{tr}\,(dH(\mathbf{\Gamma}\boldsymbol{A})\boldsymbol{A}^{-1})$, where tr denotes the trace. However, $dH\boldsymbol{\omega} = \boldsymbol{\omega} - Hd\boldsymbol{\omega}$ by H_2, so that we have $da = a\,\mathrm{tr}\,(\mathbf{\Gamma} - Hd(\mathbf{\Gamma}\boldsymbol{A})\boldsymbol{A}^{-1})$ and hence $\ln|a| - \ln|a_0| = \mathrm{tr}\,H(\mathbf{\Gamma} - Hd(\mathbf{\Gamma}\boldsymbol{A})\boldsymbol{A}^{-1}) = \mathrm{tr}\,H(\mathbf{\Gamma})$ because $Hd(\mathbf{\Gamma}\boldsymbol{A})\boldsymbol{A}^{-1} \in \ker(H)$. Accordingly, $a(x^\alpha) = a_0 \exp[\mathrm{tr}\,H(\mathbf{\Gamma})]$ and we conclude that $\boldsymbol{A}(x^\alpha)$ is nonsingular throughout U if $\boldsymbol{A}$ is nonsingular at the point (x_0^α). Since $d\boldsymbol{A}$ is the exact part of $\mathbf{\Gamma}\boldsymbol{A}$, the unique decomposition given by Lemma 1.4 and (2.3) show that $\boldsymbol{A}\boldsymbol{\theta} = Hd(\mathbf{\Gamma}\boldsymbol{A})$ is the antiexact part of $\mathbf{\Gamma}\boldsymbol{A}$. Thus,

$$\boldsymbol{\theta} = \boldsymbol{A}^{-1}Hd(\mathbf{\Gamma}\boldsymbol{A}) \tag{2.5}$$

is antiexact from the module property of $\mathcal{A}^k(U)$, and the result is established.

Now that we have established that every matrix $\mathbf{\Gamma}$ of affine connection has the form (2.2) for $\boldsymbol{\theta} \in \mathcal{A}^1(U)$, it follows from Lemma 2.1 that every matrix $\mathbf{\Theta}$ of curvature forms has the structure given by (2.3). This establishes the following corollary on noting that $H(\boldsymbol{\mu}) \in \mathcal{A}^1(U)$ for every $\boldsymbol{\mu} \in \Lambda^2(U)$.

COROLLARY 2.1. – *Every matrix* $\mathbf{\Theta}$ *of curvature forms on* U *can be written as*

$$\mathbf{\Theta} = \boldsymbol{A}\big(dH(\boldsymbol{\mu}) - H(\boldsymbol{\mu})\wedge H(\boldsymbol{\mu})\big)\boldsymbol{A}^{-1} \tag{2.6}$$

for some $\boldsymbol{A} \in \mathcal{A}^0(U) = \Lambda^0(U)$ *and some* $\boldsymbol{\mu} \in \Lambda^2(U)$.

Multiplication of (2.6) by $\boldsymbol{A}^{-1}$ on the left and $\boldsymbol{A}$ on the right yields $\boldsymbol{A}^{-1}\mathbf{\Theta}\boldsymbol{A} = dH(\boldsymbol{\mu}) - H(\boldsymbol{\mu})\wedge H(\boldsymbol{\mu})$. Thus, since $H(\boldsymbol{\mu})\wedge H(\boldsymbol{\mu}) \in \mathcal{A}^2(U)$ by A_2, allowing H to act on both sides of this equation give $H(\boldsymbol{A}^{-1}\mathbf{\Theta}\boldsymbol{A}) = HdH(\boldsymbol{\mu}) = H(\boldsymbol{\mu})$. Thus, if we define $\boldsymbol{\theta}_A \in \mathcal{A}^1(U)$ for given $\mathbf{\Theta}$ by $\boldsymbol{\theta}_A = H(\boldsymbol{A}^{-1}\mathbf{\Theta}\boldsymbol{A})$, we arrive at the following result.

LEMMA 2.3. – *Every matrix* $\mathbf{\Theta}$ *of curvature forms on* U *can be written as*

$$\mathbf{\Theta} = \boldsymbol{A}(d\boldsymbol{\theta}_A - \boldsymbol{\theta}_A \wedge \boldsymbol{\theta}_A)\boldsymbol{A}^{-1} \tag{2.7}$$

where

$$\boldsymbol{\theta}_A = H(\boldsymbol{A}^{-1}\mathbf{\Theta}\boldsymbol{A}) \tag{2.8}$$

and a matrix of connection forms that gives rise to $\mathbf{\Theta}$ *is given by*

$$\mathbf{\Gamma}_A = (d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\theta}_A)\boldsymbol{A}^{-1}. \tag{2.9}$$

3. – The connective support of a given matrix of curvature forms.

Let $C(\mathbf{\Theta}; U)$ denote the collection of all matrices of connection forms that give rise to the same matrix $\mathbf{\Theta}$ of curvature forms on the starlike region U.

DEFINITION 3.1. – The collection $C(\mathbf{\Theta}; U)$ is the *connective support* of $\mathbf{\Theta}$.

The results obtained in the previous two sections provide the information whereby a characterization of $C(\mathbf{\Theta}; U)$ can be obtained.

THEOREM 3.1. – *Every member of the connective support of a given matrix* $\mathbf{\Theta}$ *of curvature forms is given by*

$$\mathbf{\Gamma}_B = (d\boldsymbol{B} + \boldsymbol{B}\boldsymbol{\theta}_B)\boldsymbol{B}^{-1}, \qquad \boldsymbol{\theta}_B = H(\boldsymbol{B}^{-1}\boldsymbol{\theta}\boldsymbol{B}) \tag{3.1}$$

for some nonsingular $\boldsymbol{B} \in \mathcal{A}^0(U)$.

PROOF. – If $\mathbf{\Gamma}_B \in C(\mathbf{\Theta}; U)$ then $\mathbf{\Gamma}_B$ satisfies

$$d\mathbf{\Gamma}_B = \mathbf{\Gamma}_B \wedge \mathbf{\Gamma}_B + \mathbf{\Theta} \tag{3.2}$$

and, by exterior differentiation of (3.2), we obtain the identity

$$d\mathbf{\Theta} = \mathbf{\Gamma}_B \wedge \mathbf{\Theta} - \mathbf{\Theta} \wedge \mathbf{\Gamma}_B. \tag{3.3}$$

We now use Lemma 2.2 to infer the existence of a $\boldsymbol{B} \in \mathcal{A}^0(U)$ and a $\boldsymbol{\mu} \in \mathcal{A}^1(U)$ such that

$$\mathbf{\Gamma}_B = (d\boldsymbol{B} + \boldsymbol{B}\boldsymbol{\mu})\boldsymbol{B}^{-1}, \tag{3.4}$$

in which case, Lemma 2.1 gives the curvature

$$\mathbf{\Theta}_B = \boldsymbol{B}(d\boldsymbol{\mu} - \boldsymbol{\mu} \wedge \boldsymbol{\mu})\boldsymbol{B}^{-1}. \tag{3.5}$$

The theorem will thus be established upon showing that $\boldsymbol{\Theta}_B = \boldsymbol{\Theta}$. Define the quantity $\boldsymbol{\theta}_B \in \mathcal{A}^1(U)$ by

$$\boldsymbol{\theta}_B = H(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}). \tag{3.6}$$

Exterior differentiation of (3.6) and use of $dH + Hd =$ identity yields

$$d\boldsymbol{\theta}_B = \boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B} - Hd(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}). \tag{3.7}$$

However, $Hd(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}) = H(\boldsymbol{B}^{-1}d\boldsymbol{\Theta}\boldsymbol{B} - \boldsymbol{B}^{-1}d\boldsymbol{B}\wedge\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B} + \boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}\wedge\boldsymbol{B}^{-1}d\boldsymbol{B})$, so that the use of the identity (3.3) together with (3.4) shows that

$$Hd(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}) = H(\boldsymbol{\mu}\wedge\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B} - \boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}\wedge\boldsymbol{\mu}). \tag{3.8}$$

Since $\boldsymbol{\mu} \in \mathcal{A}^1(U)$, it follows from H_2 that $H(\boldsymbol{\mu}\wedge\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}) = H\big(\boldsymbol{\mu}\wedge dH(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}) + \boldsymbol{\mu}\wedge Hd(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B})\big) = H\big(\boldsymbol{\mu}\wedge dH(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B})\big) = H(\boldsymbol{\mu}\wedge d\boldsymbol{\theta}_B)$, where the last equality is obtained by use of (3.6). Accordingly, (3.8) yields $Hd(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}) = H(\boldsymbol{\mu}\wedge d\boldsymbol{\theta}_B - d\boldsymbol{\theta}_B\wedge\boldsymbol{\mu})$, and (3.7) yields the identity

$$\boldsymbol{\Theta} = \boldsymbol{B}\big(d\boldsymbol{\theta}_B + H(\boldsymbol{\mu}\wedge d\boldsymbol{\theta}_B - d\boldsymbol{\theta}_B\wedge\boldsymbol{\mu})\big)\boldsymbol{B}^{-1}. \tag{3.9}$$

Thus, (3.5) and (3.9) give $\boldsymbol{\Theta} = \boldsymbol{\Theta}_B$ if and only if

$$d\boldsymbol{\theta}_B + H(\boldsymbol{\mu}\wedge d\boldsymbol{\theta}_B - d\boldsymbol{\theta}_B\wedge\boldsymbol{\mu}) = d\boldsymbol{\mu} - \boldsymbol{\mu}\wedge\boldsymbol{\mu} \tag{3.10}$$

holds. Allowing H to act on both sides of this equality, and noting that $\boldsymbol{\mu}\wedge\boldsymbol{\mu} \in \mathcal{A}^2(U)$ by A_2, it follows from H_3, A_1 and A_4 that

$$\boldsymbol{\theta}_B = Hd\boldsymbol{\theta}_B = Hd\boldsymbol{\mu} = \boldsymbol{\mu}.$$

In this case (3.10) is identically satisfied because

$$H(\boldsymbol{\mu}\wedge d\boldsymbol{\theta}_B - d\boldsymbol{\theta}_B\wedge\boldsymbol{\mu}) = H(\boldsymbol{\mu}\wedge d\boldsymbol{\mu} - d\boldsymbol{\mu}\wedge\boldsymbol{\mu}) = -Hd(\boldsymbol{\mu}\wedge\boldsymbol{\mu}) = -\boldsymbol{\mu}\wedge\boldsymbol{\mu},$$

and the theorem is established.

Clearly, Theorem 3.1 establishes a 1-to-1 correspondence between the collection of all nonsingular $\boldsymbol{B} \in \mathcal{A}^0(U)$ and the elements of $C(\boldsymbol{\Theta}; U)$. The cardinality of $C(\boldsymbol{\Theta}; U)$ is thus easily established once the distinct elements of $C(\boldsymbol{\Theta}; U)$ have been determined. To this end, we need the following lemma.

LEMMA 3.1. – *If $\boldsymbol{\Gamma}_A$ and $\boldsymbol{\Gamma}_B$ are two elements of $C(\boldsymbol{\Theta}; U)$ that correspond to the two nonsingular elements $\boldsymbol{A}$ and $\boldsymbol{B}$ of $\mathcal{A}^0(U)$, then $\boldsymbol{\Gamma}_A = \boldsymbol{\Gamma}_B$ if and only if $d(\boldsymbol{A}\boldsymbol{B}^{-1}) = \boldsymbol{0}$.*

PROOF. – By hypothesis,

$$\boldsymbol{\Gamma}_A = (d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\theta}_A)\boldsymbol{A}^{-1}, \quad \boldsymbol{\theta}_A = H(\boldsymbol{A}^{-1}\boldsymbol{\Theta}\boldsymbol{A}), \tag{3.11}$$

$$\boldsymbol{\Gamma}_B = (d\boldsymbol{B} + \boldsymbol{B}\boldsymbol{\theta}_B)\boldsymbol{B}^{-1}, \quad \boldsymbol{\theta}_B = H(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}), \tag{3.12}$$

and hence there exists a nonsingular $\boldsymbol{C} \in \mathcal{A}^0(U)$ such that $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}$. When this is substituted into (3.11) and (3.12) is used to eliminate the resulting $d\boldsymbol{B}$, we obtain

$$\boldsymbol{\Gamma}_A = \boldsymbol{\Gamma}_B + \boldsymbol{B}(d\boldsymbol{C} + \boldsymbol{C}\boldsymbol{\theta}_A - \boldsymbol{\theta}_B\boldsymbol{C})\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}. \tag{3.13}$$

Thus, $\boldsymbol{\Gamma}_A = \boldsymbol{\Gamma}_B$ if and only if

$$d\boldsymbol{C} = \boldsymbol{\theta}_B\boldsymbol{C} - \boldsymbol{C}\boldsymbol{\theta}_A. \tag{3.14}$$

However, since $\boldsymbol{C} \in \mathcal{A}^0(U)$, $\boldsymbol{C} = \boldsymbol{C}(x_0^\alpha) + Hd\boldsymbol{C}$ by H_2, and hence (3.14) and the module property of $\mathcal{A}^1(U)$ gives $\boldsymbol{C} = \boldsymbol{C}(x_0^\alpha)$. We thus obtain

$$d\boldsymbol{C} = d(\boldsymbol{A}\boldsymbol{B}^{-1}) = \boldsymbol{0}. \tag{3.15}$$

However, with $\boldsymbol{C} = \boldsymbol{C}_0 = \boldsymbol{C}(x_0^\alpha)$, (3.14) becomes

$$\boldsymbol{0} = \boldsymbol{\theta}_B\boldsymbol{C}_0 - \boldsymbol{C}_0\boldsymbol{\theta}_A = \boldsymbol{\theta}_B\boldsymbol{C}_0 - \boldsymbol{C}_0 H(\boldsymbol{A}^{-1}\boldsymbol{\Theta}\boldsymbol{A}) = \boldsymbol{\theta}_B\boldsymbol{C}_0 - \boldsymbol{C}_0 H(\boldsymbol{C}_0^{-1}\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B}\boldsymbol{C}) =$$
$$= \boldsymbol{\theta}_B\boldsymbol{C}_0 - H(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B})\boldsymbol{C}_0 = \boldsymbol{\theta}_B\boldsymbol{C}_0 - \boldsymbol{\theta}_B\boldsymbol{C}_0$$

since H commutes with multiplication by constants by H_1.

Clearly, $d(\boldsymbol{A}\boldsymbol{B}^{-1}) = \boldsymbol{0}$ defines an equivalence relation on the collection of all nonsingular matrices belonging to $\mathcal{A}^0(U)$, and Lemma 3.1 thus yields the following result.

THEOREM 3.2. – *The distinct elements of the connective support of a given matrix* $\boldsymbol{\Theta}$ *of curvature forms can be placed in a* 1*-to-*1 *correspondence with the equivalence classes of all nonsingular n-by-n matrices of elements of* $\mathcal{A}^0(U)$ *under the equivalence relation* $d(\boldsymbol{A}\boldsymbol{B}^{-1}) = \boldsymbol{0}$. *The cardinality of the distinct elements of* $C(\boldsymbol{\Theta}; U)$ *is thus that of the continuum.*

COROLLARY 3.1. – *The elements of* $C(\boldsymbol{\Theta}_1; U)$ *can be placed in a* 1*-to-*1 *correspondence with the elements of* $C(\boldsymbol{\Theta}_2; U)$ *for any two matrices* $\boldsymbol{\Theta}_1$ *and* $\boldsymbol{\Theta}_2$ *of curvature forms on* U.

PROOF. – For each nonsingular $\boldsymbol{B} \in \mathcal{A}^0(U)$, we have

$$\boldsymbol{\Gamma}_{1B} = (d\boldsymbol{B} + \boldsymbol{B}\boldsymbol{\theta}_{1B})\boldsymbol{B}^{-1}, \quad \boldsymbol{\theta}_{1B} = H(\boldsymbol{B}^{-1}\boldsymbol{\Theta}_1\boldsymbol{B}),$$
$$\boldsymbol{\Gamma}_{2B} = (d\boldsymbol{B} + \boldsymbol{B}\boldsymbol{\theta}_{2B})\boldsymbol{B}^{-1}, \quad \boldsymbol{\theta}_{2B} = H(\boldsymbol{B}^{-1}\boldsymbol{\Theta}_2\boldsymbol{B}),$$

from (3.1) and the result follows.

The proof of Theorem 3.1 established the identity

$$\boldsymbol{\Theta} = \boldsymbol{B}(d\boldsymbol{\theta}_B - \boldsymbol{\theta}_B \wedge \boldsymbol{\theta}_B)\boldsymbol{B}^{-1}, \qquad \boldsymbol{\theta}_B = H(\boldsymbol{B}^{-1}\boldsymbol{\Theta}\boldsymbol{B})$$

for any curvature form $\boldsymbol{\Theta}$ and any nonsingular $\boldsymbol{B} \in \mathcal{A}^0(U)$. Thus, if we take $\boldsymbol{B} = \boldsymbol{E}$, we obtain the following result.

THEOREM 3.3. – *Every matrix* $\boldsymbol{\Theta}$ *of curvature forms is uniquely characterized by a corresponding* $\boldsymbol{\theta} = H(\boldsymbol{\Theta}) \in \mathcal{A}^1(U)$ *through*

$$\boldsymbol{\Theta} = d\boldsymbol{\theta} - \boldsymbol{\theta} \wedge \boldsymbol{\theta} \tag{3.16}$$

and hence the collection of all matrices of curvature forms on U *can be placed in a* 1-*to*-1 *correspondence with all distinct* n-*by*-n *matrices of elements of* $\mathcal{A}^1(U)$. *The cardinality of the collection of all matrices of curvature forms on* U *is thus that of the continuum.*

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