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On the Algebra of Test Functions for Field Operators

Decomposition of Linear Functionals into Positive Ones

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Abstract. It is shown that every continuous linear functional on the field algebra can be defined by a vector in the Hilbert space of some representation of the algebra. The functionals which can be written as a difference of positive ones are characterized. By an example it is shown that a positive functional on a subalgebra does not always have an extension to a positive functional on the whole algebra.

1. Introduction

The formulation of the reconstruction theorem of Wightman [1] in terms of positive functionals on the tensor algebra over a space of test functions [2] provides a natural framework for a study of the nonlinear restrictions on the Wightman distributions. This is the reason why the properties of this algebra are of interest and have been the subject of several investigations [2–7]. This is also the motivation for the present paper, although we shall here ignore the linear conditions of field theory and be concerned with the positive linear functionals in general. As a space of test functions we take Schwartz space $\mathscr{S}$ and we denote the algebra by $\underline{\mathscr{S}}$. It is shown that there exist so many positive functionals that the corresponding Hilbert norms define a topology on $\underline{\mathscr{S}}$ which is identical to the usual one. This topology, however, is not well adapted to the order structure on $\underline{\mathscr{S}}$ in the sense that continuous linear functionals need not be of the form $(T_1 - T_2) + i(T_3 - T_4)$ with positive functionals T_i. This is connected with the fact that the multiplication on the algebra is not continuous in both variables jointly. Let τ denote the usual topology on $\underline{\mathscr{S}}$ and $\mathscr{N}$ the strongest convex topology such that the multiplication is a jointly continuous bilinear map $\underline{\mathscr{S}}[\tau] \times \underline{\mathscr{S}}[\tau] \to \underline{\mathscr{S}}[\mathscr{N}]$. It is shown that functionals of the above form are exactly the $\mathscr{N}$-continuous functionals.

Finally we consider the problem of extending a positive functional from a subalgebra of $\underline{\mathscr{S}}$ to a positive functional on the whole algebra. In some cases this is shown to be possible, but an example is also given where no extension is a linear combination of positive functionals.

2. Notations and Basic Properties of the Field Algebra

The basic properties of $\underline{\mathscr{S}}$ and its connection with Wightman field theory are discussed in [2, 3, 5, 6]. We shall therefore only review the notation briefly and state some additional properties which are mainly a consequence of the fact that $\mathscr{S}$ is a nuclear space.

The field algebra is the locally convex direct sum $\underline{\mathscr{S}} = \bigoplus_{n=0}^{\infty} \mathscr{S}_n$ where $\mathscr{S}_0 = \mathbb{C}$ and $\mathscr{S}_n = \mathscr{S}(\mathbb{R}^{d \cdot n})$ is Schwartz space of C^∞-functions $\mathbb{R}^{d \cdot n} \to \mathbb{C}$ of rapid decrease. We denote the direct sum topology on $\underline{\mathscr{S}}$ by τ, for its properties see [3, 5]. The dual space is the product $\underline{\mathscr{S}}' = \prod_{n=0}^{\infty} \mathscr{S}_n'$, usually equipped with the strong topology. The elements of $\underline{\mathscr{S}}$ are thus sequences

$$\underline{f} = (f_0, f_1, \ldots, f_n, 0, \ldots)$$

where all but a finite number of $f_\nu \in \mathscr{S}_\nu$ are equal to zero, whereas $\underline{\mathscr{S}}'$ consists of arbitrary sequences

$$T = (T_0, T_1, \ldots)$$

with $T_\nu \in \mathscr{S}_\nu'$. As an algebra $\underline{\mathscr{S}}$ is the completion of the tensor algebra over $\mathscr{S}_1$, the multiplication is defined by

$$(\underline{f} \times \underline{g})_n(x_1, \ldots, x_n) = \sum_{\mu + \nu = n} f_\mu(x_1, \ldots, x_\mu)\, g_\nu(x_{\mu+1}, \ldots, x_n),$$

with the unit element $1 = (1, 0, 0, \ldots)$. An involution $*$ is defined by

$$(\underline{f}^*)_n(x_1, \ldots, x_n) = \overline{f_n(x_n, \ldots, x_1)}.$$

The multiplication is continuous in each variable separately, it is jointly continuous as a bilinear map $\bigoplus_{n=0}^{N} \mathscr{S}_n \times \bigoplus_{n=0}^{N} \mathscr{S}_n \to \bigoplus_{n=0}^{2N} \mathscr{S}_n$ for $N < \infty$. The involution is an anti-isomorphism, there is a basis of continuous norms p with $p(\underline{f}^*) = p(\underline{f})$. The Hermitean part of $\underline{\mathscr{S}}$

$$\underline{\mathscr{S}}_h = \{\underline{f} \in \underline{\mathscr{S}} \mid \underline{f}^* = \underline{f}\}$$

is a real vector space with the cone of positive elements

$$\underline{\mathscr{S}}^+ = \left\{\underline{f} \in \underline{\mathscr{S}} \mid \underline{f} = \sum_{i=1}^{\infty} \underline{f}^{i*} \times \underline{f}^i, \underline{f}^i \in \underline{\mathscr{S}}\right\}.$$

$\underline{\mathscr{S}}^+$ defines an order relation on $\underline{\mathscr{S}}: \underline{f} \leqq \underline{g}$ iff $\underline{g} - \underline{f} \in \underline{\mathscr{S}}^+$. A linear functional which is positive on $\underline{\mathscr{S}}^+$ is automatically continuous [6]. It has to be noted, however, that positivity on $\underline{\mathscr{S}}^+$ is a stronger condition than positivity on squares $\underline{f}^* \times \underline{f}$ if no continuity is assumed, because $\underline{\mathscr{S}}^+$ contains infinite sums of squares.

The dual space of $\underline{\mathscr{S}}_h$ is identical to the space of Hermitean or real functionals on $\underline{\mathscr{S}}$:

$$\underline{\mathscr{S}}_h' = \{T \in \underline{\mathscr{S}}' \mid T = T^*\}, \quad \text{where} \quad T^*(\underline{f}) := \overline{T(\underline{f}^*)}.$$

Every element can be decomposed into its real and imaginary parts, so

$$\underline{\mathscr{S}} = \underline{\mathscr{S}}_h + i\underline{\mathscr{S}}_h, \qquad \underline{\mathscr{S}}' = \underline{\mathscr{S}}_h' + i\underline{\mathscr{S}}_h'.$$

Positive functionals are Hermitean because

$$\underline{\mathscr{S}}_h = \underline{\mathscr{S}}^+ - \underline{\mathscr{S}}^+. \tag{1}$$

They form the dual cone

$$\underline{\mathscr{S}}^{+\prime} = \{T \in \underline{\mathscr{S}}' \mid T(\underline{f}) \geqq 0 \text{ for all } \underline{f} \in \underline{\mathscr{S}}^+\} \subset \underline{\mathscr{S}}_h'.$$

That $\underline{\mathscr{S}}^{+\prime}$ is in fact a cone and not merely a wedge, i.e.

$$\underline{\mathscr{S}}^{+\prime} \cap -\underline{\mathscr{S}}^{+\prime} = \{0\}$$

follows from (1) by duality. According to [7], $\underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime}$ is dense in $\underline{\mathscr{S}}_h'$. This follows also from the stronger result that $\underline{\mathscr{S}}^+$ is a closed cone [4] or from Theorem 5 below.

In the proof of Theorem 1 we shall make use of the fact that the spaces $\mathscr{S}_n$ are nuclear. The topology of a nuclear space can always be defined by Hilbert seminorms [9, 10], i.e. seminorms which derive from a scalar product. For $\mathscr{S}_n$ we can even find Hilbert norms [11]. Let h^μ, h^ν be Hilbert norms on $\mathscr{S}_\mu, \mathscr{S}_\nu$ respectively, and denote by $\langle f, g\rangle^\mu, \langle f, g\rangle^\nu$ the corresponding scalar products. On the tensor product $\mathscr{S}_\mu \otimes \mathscr{S}_\nu$ one can define many different norms $h^\mu \otimes h^\nu$ with the property

$$h^\mu \otimes h^\nu(f \otimes g) = h^\mu(f)\, h^\nu(g).$$

The biggest one is the trace norm

$$h^\mu \otimes_\pi h^\nu(u) = \inf\left\{\sum_i h^\mu(f^i)\, h^\nu(g^i) \,\middle|\, \sum_i f^i \otimes g^i = u\right\},$$

because every norm with the above property satisfies

$$h^\mu \otimes h^\nu\left(\sum_i f^i \otimes g^i\right) \leqq \sum_i h^\mu \otimes h^\nu(f^i \otimes g^i) = \sum_i h^\mu(f^i)\, h^\nu(g^i).$$

In particular, $h^\mu \otimes_\pi h^\nu$ is greater than the Hilbert-Schmidt norm

$$h^\mu \otimes_\sigma h^\nu\left(\sum_i f^i \otimes g^i\right) = \left(\sum_{i,j} \langle f^i, f^j\rangle^\mu \langle g^i, g^j\rangle^\nu\right)^{1/2}$$

which on the other hand dominates the usual operator norm[1]

$$h^\mu \otimes_\varepsilon h^\nu(u) = \sup\{|T \otimes S(u)| \mid T \in \mathscr{S}_\mu', S \in \mathscr{S}_\nu', |T| \leqq h^\mu, |S| \leqq h^\nu\}.$$

[1] One can think of the tensor product of two Hilbert spaces $\mathscr{H}_1 \otimes \mathscr{H}_2$ as the set of linear operators $\mathscr{H}_1 \to \mathscr{H}_2$ with finite dimensional range. Hence the names for these three products. The products $\otimes_\pi$ and $\otimes_\varepsilon$ can of course be defined for arbitrary seminorms [9, 10].

Now the $\mathscr{S}_\mu$, $\mathscr{S}_\nu$ are nuclear spaces, and this implies that it makes no difference which of these products one uses [9, 10]: For fixed h^μ, h^ν they are not equal:

$$h^\mu \otimes_\varepsilon h^\nu \leqq h^\mu \otimes_\sigma h^\nu \leqq h^\mu \otimes_\pi h^\nu ,$$

but there are always other continuous norms k^μ, k^ν such that

$$h^\mu \otimes_\pi h^\nu \leqq k^\mu \otimes_\varepsilon k^\nu \leqq k^\mu \otimes_\sigma k^\nu .$$

The families $\{h^\mu_\alpha \otimes_\varepsilon h^\nu_\beta\}$, $\{h^\mu_\alpha \otimes_\sigma h^\nu_\beta\}$, and $\{h^\mu_\alpha \otimes_\pi h^\nu_\beta\}$ define therefore all the same topology on $\mathscr{S}_\mu \otimes \mathscr{S}_\nu$ if $\{h^\mu_\alpha\}$, $\{h^\nu_\beta\}$ is a basis of continuous Hilbert norms for the topology of $\mathscr{S}_\mu$, $\mathscr{S}_\nu$ respectively. The completion $\mathscr{S}_\mu \hat{\otimes} \mathscr{S}_\nu$ is isomorphic to $\mathscr{S}_{\mu+\nu}$ [10]. In particular, one can define the topology of $\mathscr{S}_n$ by a system of Hilbert norms of the form

$$h^1 \otimes_\sigma \cdots \otimes_\sigma h^1$$

with Hilbert norms h^1 on $\mathscr{S}_1$.

The topology of $\underline{\mathscr{S}}$ is defined as the locally convex direct sum of the $\mathscr{S}_n$ topologies, i.e. it is given by the norms

$$p(\underline{f}) = \sum_{n=0}^{\infty} h^n(f_n)$$

where the h^n run through a basis of continuous Hilbert norms on $\mathscr{S}_n$, multiplied with arbitrary coefficients. It is, however, not difficult to see that we get an equivalent family of norms if we take the Hilbert direct sum

$$h(\underline{f}) = \left(\sum_{n=0}^{\infty} h^n(f_n)^2 \right)^{1/2} .$$

Although this is probably well known we give a proof:

Lemma 1. *There exist constants* c_ν^2 *such that* $\left(\sum_{\mu=0}^{\infty} x_\mu \right)^2 \leqq \sum_{\nu=0}^{\infty} c_\nu^2 x_\nu^2$ *for all finite sequences of real numbers* $\{x_\nu\}$.

Proof. Let $a_{\mu\nu} = -1$ for $\mu \neq \nu$, $a_{\mu\mu} = c_\mu^2 - 1$. We have to choose c_ν^2 such that $\sum_{\mu,\nu} x_\mu a_{\mu\nu} x_\nu \geqq 0$. For this it is sufficient that the determinant of every submatrix $A_n = (a_{\mu\nu})_{\mu,\nu \leqq n}$ is greater than zero. We define c_ν^2 by induction: Choose $c_0^2 > 1$ and assume then that $\det A_{n-1} > 0$. We have

$$\det A_n = a_{nn} \cdot \det A_{n-1} + R_{n-1}$$

where R_{n-1} does not contain a_{nn}. So $\det A_n > 0$ if

$$c_n^2 - 1 = a_{nn} > -R_{n-1} \cdot (\det A_{n-1})^{-1} .$$

We conclude this section with a few remarks on representations of $\underline{\mathscr{S}}$. By representation we shall in the following mean weakly continuous $*$-representation, i.e. a mapping of $\underline{\mathscr{S}}$ on linear operators $A(\underline{f})$ with a common dense domain of definition D in some Hilbert space $\mathscr{H}$ such that

(i) $A(\underline{f})\, D \subset D \quad$ for all $\quad \underline{f} \in \underline{\mathscr{S}}$.

(ii) $A(\alpha \underline{f} + \beta \underline{g}) = \alpha A(\underline{f}) + \beta A(\underline{g})$,
$A(\underline{f} \times \underline{g}) = A(\underline{f}) \cdot A(\underline{g})$,
$A(\underline{f}^*) = A(\underline{f})^* \quad$ on $\quad D$.

(iii) $\langle \varphi, A(\underline{f})\, \psi \rangle$ is continuous in $\underline{f}$ for all $\varphi, \psi \in D$.

If $\mathscr{H}, D, A$ is a representation, then $T(\underline{f}) = \langle \varphi, A(\underline{f})\, \varphi \rangle$ is a positive functional on $\underline{\mathscr{S}}$ for all $\varphi \in D$, and conversely, a positive functional T defines by the GNS-construction a cyclic representation with cyclic vector Ω, domain of definition $D_0 = \{A(\underline{f})\, \Omega | \underline{f} \in \underline{\mathscr{S}}\}$ and $\|A(\underline{f})\, \Omega\| = T(\underline{f}^* \times \underline{f})^{1/2}$.

Although the multiplication on $\underline{\mathscr{S}}$ is not jointly continuous, it is jointly continuous as a map $\mathscr{S}_n \times \mathscr{S}_n \to \mathscr{S}_{2n}$ and this is sufficient for

Lemma 2. *A weakly continuous $*$-representation of $\underline{\mathscr{S}}$ is strongly continuous, i.e. $\underline{f} \mapsto \|A(\underline{f})\, \varphi\|$ is continuous for all $\varphi \in D$. As a consequence, $h(\underline{f}) = T(\underline{f}^* \times \underline{f})^{1/2}$ is a continuous seminorm on $\underline{\mathscr{S}}$ for all $T \in \underline{\mathscr{S}}^{+\prime}$.*

Proof. $\|A(\underline{f})\varphi\| = \left\| \sum_n A(f_n)\varphi \right\| \leqq \sum_n \|A(f_n)\varphi\| = \sum_n \langle \varphi, A(f_n^* \times f_n)\varphi \rangle^{1/2}$.

By weak continuity there are continuous norms p_{2n} on $\mathscr{S}_{2n}$ such that $\langle \varphi, A(f_n^* \times f_n)\, \varphi \rangle \leqq p_{2n}(f_n^* \times f_n)$. Since $\mathscr{S}_{2n} = \mathscr{S}_n \hat{\otimes}_\pi \mathscr{S}_n$, we can choose $p_{2n} = q_n \otimes_\pi q_n$ with continuous norms q_n on $\mathscr{S}_n$ and we can take q_n such that $q_n(f_n^*) = q_n(f_n)$. Hence $\|A(\underline{f})\, \varphi\| \leqq \sum_{n=0}^{\infty} q_n(f_n)$ so $\|A(\underline{f})\, \varphi\|$ is continuous. The last statement follows immediately because a positive T defines a representation with $\|A(\underline{f})\, \varphi\| = h(\underline{f})$.

3. A Class of Positive Functionals

If T is a positive functional, then $h(\underline{f}) = T(\underline{f}^* \times \underline{f})^{1/2}$ is a continuous Hilbert seminorm on $\underline{\mathscr{S}}$ by Lemma 2. The collection of all such seminorms defines a topology on $\underline{\mathscr{S}}$ which is in any case weaker than τ. It is in fact identical to τ as we are now going to show. The method is a generalization of a construction used in [13] to show that every sequence of real numbers is a difference of two sequences of positive type (cf. also Lemma 2 in [7]).

Theorem 1. *For every continuous seminorm p on $\underline{\mathscr{S}}$ there is a positive functional T such that*

$$p(\underline{f})^2 \leqq T(\underline{f}^* \times \underline{f}) \quad \textit{for all} \quad \underline{f} \in \underline{\mathscr{S}}\,.$$

Proof. The Hilbert norms

$$h_k(f) = \Big(\sum_{|\alpha|, \varkappa \leqq k} \int (1+|x|^2)^{\varkappa} |D^{\alpha} f(x)|^2 \, dx\Big)^{1/2}$$

form a basis of norms for the topology of $\mathscr{S}_1$ [11]. We can also write this as

$$h_k(f) = \langle f, M_k f \rangle_0^{1/2}$$

where $\langle f, g \rangle_0$ is the usual scalar product in L_2 and

$$M_k = \sum_{|\alpha|, \varkappa \leqq k} (-1)^{|\alpha|} D^{\alpha} (1+|x|^2)^{\varkappa} D^{\alpha}$$

is a continuous operator $\mathscr{S}_1 \to \mathscr{S}_1$, positive with respect to the L_2 scalar product. We denote $\langle f, M_k g \rangle_0$ by $\langle f, g \rangle_k$. On $\mathscr{S}_n$ we have the basis of Hilbert norms

$$h_k^n = h_k \otimes_\sigma \cdots \otimes_\sigma h_k$$

with the corresponding scalar products

$$\langle f, g \rangle_k^n = \langle f, M_k^n g \rangle_0^n$$

where $M_k^n = M_k \otimes \cdots \otimes M_k$.

After these preliminary remarks we begin the construction of T by defining $T_{2\nu+1} = 0$ for all $\nu = 0, 1, \ldots$. We are then going to define by induction Hilbert norms $q_n = c_n \cdot h_{k_n}^n$ and distributions $T_{2n} \in \mathscr{S}_{2n}'$, where $c_n > 0$ and $h_{k_n}^n$ is of the above form, such that the following three conditions are satisfied:

(i) $q(\underline{f}) \geqq p(\underline{f})$ for all $\underline{f} = (f_0, \ldots, f_n)$, $q(\underline{f}) = \Big(\sum_{\nu=0}^{n} q_\nu(f_\nu)^2\Big)^{1/2}$.

(ii) There is an $\varepsilon_n > 0$ such that $T(\underline{f}^* \times \underline{f}) \geqq (1+\varepsilon_n) \cdot q(\underline{f})^2$.

(iii) There exist continuous seminorms $q_{\mu+\nu,\nu}$ on $\mathscr{S}_\nu$ for $0 \leqq \mu < \nu \leqq 2n$, $\mu + \nu \leqq 2n$, such that

$$|T_{\mu+\nu}(f_\mu^* \times g_\nu)| \leqq q_\mu(f_\mu) \cdot q_{\mu+\nu,\nu}(g_\nu) \quad \text{for all} \quad f_\mu \in \mathscr{S}_\mu, g_\nu \in \mathscr{S}_\nu.$$

We begin at $n = 0$: p_0 is a continuous seminorm on $\mathbb{C}$ so there is a $c_0 > 0$ with $p_0(a) \leqq c_0 |a| =: q_0(a)$. Define $T_0 = 2c_0^2$ and $\varepsilon_0 = 1$. The conditions above are all satisfied. Assume now that they are valid with $n-1$ in place of n. Choose $q_n = c_n \cdot h_{k_n}^n$ such that

$$q_n \geqq \max\{h_0^n, p_n, q_{n,n}, q_{n+1,n}, \ldots, q_{2(n-1),n}\}. \tag{2}$$

This is possible because every finite set of continuous norms is dominated by some norm of the type h_k^n. By Lemma 1 we may suppose that the given norm p is the Hilbert direct sum of the norms p_n so (i) is satisfied. Now define a linear functional on $\mathscr{S}_n \otimes \mathscr{S}_n$:

$$T_{2n}\Big(\sum_i f_n^i \times g_n^i\Big) = \lambda_{2n} \cdot c_n^2 \sum_i \langle f_n^{i*}, g_n^i \rangle_{k_n}^n$$

with a constant λ_{2n} to be fixed shortly. We have

$$|T_{2n}(f)| \leqq \text{const}\, h^n_{k_n} \otimes_\pi h^n_{k_n}(f)\,,$$

for $f \in \mathscr{S}_n \otimes \mathscr{S}_n$, so T_{2n} defines a continuous functional on $\mathscr{S}_{2n} = \mathscr{S}_n \hat{\otimes} \mathscr{S}_n$. To check (ii) let $\hat{\underline{f}} = (f_0, \ldots, f_{n-1})$, $\underline{f} = (f_0, \ldots, f_n) = (\hat{\underline{f}}, f_n)$. Then

$$T(\underline{f}^* \times \underline{f}) = T(\hat{\underline{f}}^* \times \hat{\underline{f}}) + 2\mathrm{Re}\, T(\hat{\underline{f}}^* \times f_n) + T(f_n^* \times f_n)\,.$$

The mixed term can be estimated by (iii) and (2):

$$|T(\hat{\underline{f}}^* \times f_n)| \leqq \sum_{\mu=0}^{n-1} |T_{\mu+n}(f_\mu^* \times f_n)| \leqq \sum_{\mu=0}^{n-1} q_\mu(f_\mu) \cdot q_{\mu+n,n}(f_n)$$

$$\leqq \left(\sum_{\mu=0}^{n-1} q_\mu(f_\mu)\right) \cdot q_n(f_n) \leqq n^{1/2} \cdot q(\hat{\underline{f}}) \cdot q_n(f_n)\,.$$

By (ii), $T(\hat{\underline{f}}^* \times \hat{\underline{f}}) \geqq (1 + \varepsilon_{n-1}) \cdot q(\hat{\underline{f}})^2$ with $\varepsilon_{n-1} > 0$. Let $\varepsilon_n = 1/2\varepsilon_{n-1}$. Since $T(f_n^* \times f_n) = \lambda_{2n} q_n(f_n)^2$ and $q(\underline{f})^2 = q(\hat{\underline{f}})^2 + q_n(f_n)^2$ by definition, we have

$$T(\underline{f}^* \times \underline{f}) - (1 + \varepsilon_n) \cdot q(\underline{f})^2 \geqq (1 + 2\varepsilon_n) \cdot q(\hat{\underline{f}})^2 - 2n^{1/2} \cdot q(\hat{\underline{f}}) \cdot q_n(f_n)$$

$$+ \lambda_{2n} \cdot q_n(f_n)^2 - (1 + \varepsilon_n) \cdot q(\hat{\underline{f}})^2 - (1 + \varepsilon_n) \cdot q_n(f_n)^2$$

$$= \varepsilon_n q(\hat{\underline{f}})^2 - 2n^{1/2} q(\hat{\underline{f}})\, q_n(f_n) + (\lambda_{2n} - 1 - \varepsilon_n) \cdot q_n(f_n)^2 \geqq 0$$

$$\text{if} \quad \lambda_{2n} \geqq n \cdot \varepsilon_n^{-1} + 1 + \varepsilon_n\,.$$

It remains to verify (iii). For $\mu + \nu \leqq 2(n-1)$ it is valid by assumption and for $\mu + \nu = 2n - 1$ because $T_{2n-1} = 0$. One has therefore to find continuous seminorms $q_{2n,\nu}$ for $\nu = n+1, \ldots, 2n$ such that

$$|T_{2n}(f_\mu^* \times g_\nu)| \leqq q_\mu(f_\mu) \cdot q_{2n,\nu}(g_\nu)$$

for $\mu + \nu = 2n$, ν as above. By definition

$$T_{2n}(f_n \times g_n) = \text{const}\,\langle f_n^*, M^n_{k_n} g_n \rangle^n_0$$

and thus by Cauchy-Schwarz

$$|T_{2n}(f_n \times g_n)| \leqq h^n_0(f_n) \cdot r^n(g_n)$$

where $r^n(g) = \text{const}\, h^n_0(M^n_{k_n} g)$ is a continuous seminorm on $\mathscr{S}_n$. This implies

$$|T_{2n}| \leqq h^n_0 \otimes_\pi r^n\,.$$

Now let $\mu = 2n - \nu \leqq n - 1$. We have

$$h^n_0 = h^\mu_0 \otimes_\sigma h^{n-\mu}_0 \leqq h^\mu_0 \otimes_\pi h^{n-\mu}_0$$

and thus

$$|T_{2n}| \leqq (h_0^\mu \otimes_\pi h_0^{n-\mu}) \otimes_\pi r^n = h_0^\mu \otimes_\pi (h_0^{n-\mu} \otimes_\pi r^n) \leqq q_\mu \otimes_\pi q_{2n,\nu}$$

and $q_{2n,\nu} := h_0^{n-\mu} \otimes_\pi r^n$ is a continuous seminorm on $\mathscr{S}_\nu$.

As an immediate corollary we have

Theorem 2. *For every continuous linear functional $S \in \underline{\mathscr{S}}'$ there is a cyclic representation $\mathscr{H}, A, \Omega$ of $\underline{\mathscr{S}}$ and a vector $\varphi \in \mathscr{H}$ such that*

$$S(\underline{f}) = \langle \varphi, A(\underline{f})\, \Omega \rangle .$$

Proof. By the previous theorem there is a $T \in \underline{\mathscr{S}}^{+\prime}$ with

$$|S(\underline{f})| \leqq T(\underline{f}^* \times \underline{f})^{1/2} .$$

The proposition is thus a consequence of the Riesz lemma: T defines a cyclic representation $\mathscr{H}, A, \Omega$ of $\underline{\mathscr{S}}$ with cyclic vector Ω and $\|A(\underline{f})\, \Omega\| = T(\underline{f}^* \times \underline{f})^{1/2}$. On the dense set $D_0 = \{A(\underline{f})\, \Omega \,|\, \underline{f} \in \underline{\mathscr{S}}\}$, S defines a continuous linear functional

$$s(A(\underline{f})\, \Omega) := S(\underline{f})$$

and is therefore given by a vector $\varphi \in \mathscr{H}$.

4. Decomposition of Linear Functionals

Positive functionals on $\underline{\mathscr{S}}$ are continuous in the direct sum topology τ as mentioned in Section 2. They have in fact a stronger continuity property which follows from Lemma 2 and is not shared by all functionals in $\underline{\mathscr{S}}'$:

Lemma 3. *For every $T \in \underline{\mathscr{S}}^{+\prime}$ there is a continuous norm p on $\underline{\mathscr{S}}$ such that for all $\underline{f}, \underline{g} \in \underline{\mathscr{S}}$*

$$|T(\underline{f}^* \times \underline{g})| \leqq p(\underline{f}) \cdot p(\underline{g}) , \tag{3}$$

i.e. T defines a jointly continuous sesquilinear form on $\underline{\mathscr{S}}$

$$(\underline{f}, \underline{g}) \mapsto T(\underline{f}^* \times \underline{g}) .$$

Proof. The statement is just Lemma 2 combined with the Cauchy-Schwarz inequality.

If T_1 and T_2 are positive functionals, then Lemma 3 is also true for $T_1 - T_2$ because $|T_1 - T_2| \leqq |T_1| + |T_2|$. The following example shows that (3) is not valid for all $T \in \underline{\mathscr{S}}_h'$. For simplicity of notation we take $\mathscr{S} = \mathscr{S}(\mathbb{R}^1)$.

Example. Define $T=(T_0, T_1, \dots)$ as follows: $T_0=1$, $T_n=\delta^{(n)}\otimes\cdots\otimes\delta^{(n)}$. (Derivative of the δ-function in each variable.) Let $\underline{f}=(0, f_1)$, $\underline{g}=(0,0,\dots,0,g_N)$. We have

$$|T(\underline{f}^*\times\underline{g})|=|T_{N+1}(f_1^*\times g_N)|=|f_1^{(N+1)}(0)\cdot g_N^{(N+1,\dots,N+1)}(0,\dots,0)|.$$

If the last factor is $\neq 0$, then (3) would imply

$$|f_1^{(N+1)}(0)|\leqq p^1(f_1)\cdot c_N$$

with a continuous seminorm p^1 on $\mathscr{S}_1$ and

$$c_N=p^N(g_N)\cdot g_N^{(N+1,\dots,N+1)}(0,\dots,0)^{-1}.$$

This is not possible for all N because a single continuous seminorm on $\mathscr{S}$ can only dominate derivatives up to some finite order.

Thus, although $\underline{\mathscr{S}}^{+\prime}-\underline{\mathscr{S}}^{+\prime}$ is dense in $\underline{\mathscr{S}}_h'$ we have

Lemma 4. $\underline{\mathscr{S}}^{+1}-\underline{\mathscr{S}}^{+\prime}\neq\underline{\mathscr{S}}_h'$, and

Lemma 5. *The product $\underline{f}\times\underline{g}$ is not jointly continuous.*

Proof. Otherwise $(\underline{f},\underline{g})\mapsto T(\underline{f}^*\times\underline{g})$ would be jointly continuous for all $T\in\underline{\mathscr{S}}$ as a composition of continuous mappings.

Notation. By $\mathscr{N}$ we denote the strongest locally convex topology on $\underline{\mathscr{S}}$ such that the multiplication on $\underline{\mathscr{S}}$ is a jointly continuous bilinear mapping

$$m:\underline{\mathscr{S}}[\tau]\times\underline{\mathscr{S}}[\tau]\to\underline{\mathscr{S}}[\mathscr{N}].$$

Since m is surjective ($\underline{\mathscr{S}}$ has a unit element) this topology exists. An absolutely convex set $U\subset\underline{\mathscr{S}}$ is an $\mathscr{N}$-neighbourhood of zero if and only if $m^{-1}(U)$ is a neighbourhood of zero in $\underline{\mathscr{S}}[\tau]\times\underline{\mathscr{S}}[\tau]$. The bilinear mapping m is by definition continuous and defines therefore a continuous linear mapping of the tensor product

$$M:\underline{\mathscr{S}}[\tau]\otimes_\pi\underline{\mathscr{S}}[\tau]\to\underline{\mathscr{S}}[\mathscr{N}]$$

and $\mathscr{N}$ is also the strongest convex topology such that M is continuous. $\underline{\mathscr{S}}[\mathscr{N}]$ is therefore isomorphic to the quotient space

$$\underline{\mathscr{S}}[\mathscr{N}]\cong\underline{\mathscr{S}}[\tau]\otimes_\pi\underline{\mathscr{S}}[\tau]/\operatorname{Ker}M. \tag{4}$$

$\mathscr{N}$ can be defined explicitly by the norms

$$\hat{p}_\alpha(\underline{f})=\inf\left\{\sum_i p_\alpha(\underline{g}^i)\cdot p_\alpha(\underline{h}^i)\,\middle|\,\sum_i \underline{g}^i\times\underline{h}^i=\underline{f}\right\} \tag{5}$$

where $\{p_\alpha\}$ is a basis of norms for τ. It makes no difference whether the infimum is taken over the decompositions of $\underline{f}$ into a finite or an infinite number of products because

$$\sum_{i=1}^\infty \underline{g}^i\times\underline{h}^i=\sum_{i=1}^N \underline{g}^i\times\underline{h}^i+\left(\sum_{i=N+1}^\infty \underline{g}^i\times\underline{h}^i\right)\times 1$$

and the last term becomes arbitrarily small for $N \to \infty$ if the sum converges. $\mathcal{N}$ is weaker than τ because

$$\hat{p}(\underline{f}) \leqq p(1) \cdot p(\underline{f}). \tag{6}$$

If p is a seminorm with $p(\underline{f}^*) = p(\underline{f})$ then $|T(\underline{f})| \leqq \hat{p}(\underline{f})$ for all $\underline{f} \in \underline{\mathcal{S}}$ if and only if $|T(\underline{f}^* \times \underline{g})| \leqq p(\underline{f}) \cdot p(\underline{g})$ for all $\underline{f}, \underline{g} \in \underline{\mathcal{S}}$. Positive functionals and their linear combinations are therefore $\mathcal{N}$-continuous. The converse follows from Theorem 1 as we are now going to show. In fact a little more can be proven: An $\mathcal{N}$-equicontinuous set of Hermitean functionals can be expressed as the set of differences of positive functionals in some $\mathcal{N}$-equicontinuous set. (A set C is equicontinuous if there is a continuous seminorm $\hat{p}$ with $|T| \leqq \hat{p}$ for all $T \in C$.) Such a connection between order and topology has a name [12]:

Definition. A cone K in a topological vector space is normal if there is a basis of neighbourhoods of zero U with $(U+K) \cap (U-K) = U$.

Theorem 3. (Schaefer). *Let E be a real topological vector space with a cone K and dual cone $K' \subset E'$. K is normal if and only if every equicontinuous set $C \subset E'$ is of the form $C_1 - C_1$ with an equicontinuous set $C_1 \subset K'$.*

Proof. See [12], Proposition 1.22, p. 73.

We now have the following corollary of Theorem 1:

Theorem 4. *$\underline{\mathcal{S}}^+$ is a normal cone in $\underline{\mathcal{S}}_h[\mathcal{N}]$.*

Proof. Let $C \subset \underline{\mathcal{S}}_h[\mathcal{N}]$ be equicontinuous. There is then a τ-norm p such that $|S| \leqq \hat{p}$ for all $S \in C$. By Theorem 1 there is a $T \in \underline{\mathcal{S}}^{+\prime}$ with $T(\underline{f}^* \times \underline{f}) \geqq p(\underline{f})^2$ and by Lemma 2 a q such that $|T| \leqq \hat{q}$. We write $S = (S+T) - T$. $(S+T)$ is positive because $(S+T)(\underline{f}^* \times \underline{f}) \geqq T(\underline{f}^* \times \underline{f}) - |S(\underline{f}^* \times \underline{f})| \geqq p(\underline{f})^2 - p(\underline{f})^2 = 0$, and both T and $(S+T)$ are continuous with respect to the norm $\hat{p} + \hat{q}$.

The functionals in $\underline{\mathcal{S}}^{+\prime} - \underline{\mathcal{S}}^{+\prime}$ can now be characterized:

Theorem 5. *The following are equivalent for a Hermitean linear functional S on $\underline{\mathcal{S}}$:*

(i) *$S \in \underline{\mathcal{S}}^{+\prime} - \underline{\mathcal{S}}^{+\prime}$.*

(ii) *S is $\mathcal{N}$-continuous.*

(iii) *$(\underline{f}, \underline{g}) \mapsto S(\underline{f}^* \times \underline{g})$ is a jointly continuous sesquilinear form.*

(iv) *There is a cyclic representation $\mathcal{H}, A, \Omega$ of $\underline{\mathcal{S}}$ and a bounded Hermitean operator B on $\mathcal{H}$, commuting weakly with the representation*[2] *such that*

$$S(\underline{f}) = \langle B\Omega, A(\underline{f})\,\Omega\rangle.$$

[2] I.e. $\langle \varphi, BA(\underline{g})\,\psi\rangle = \langle A(\underline{g}^*)\,\varphi, B\psi\rangle$ for all $\varphi, \psi \in D_0 = \{A(\underline{f})\Omega \mid \underline{f} \in \underline{\mathcal{S}}\}$.

Proof. (i) implies (iii) by Lemma 3 and (i) follows from (ii) by Theorem 4. Equivalence of (ii) and (iii) follows immediately from the definition of $\mathcal{N}$ and the fact that $*$ is an antiisomorphism. (iv) implies (iii): $|S(\underline{f}^* \times \underline{g})| = |\langle B\Omega, A(\underline{f}^* \times \underline{g})\Omega\rangle| = |\langle A(\underline{f})\Omega, BA(\underline{g})\Omega\rangle| \leqq \|A(\underline{f})\Omega\| \cdot \|B\| \cdot \|A(\underline{g})\Omega\| \leqq \|B\|^{1/2} p(\underline{f}) \cdot \|B\|^{1/2} p(\underline{g})$ by Lemma 2. Finally, (iii) implies (iv): Let $|S(\underline{f}^* \times \underline{g})| \leqq p(\underline{f}) \cdot p(\underline{g})$. By Theorem 1 there is a $T \in \underline{\mathscr{S}}^{+\prime}$ with $p(\underline{f})^2 \leqq T(\underline{f}^* \times \underline{f})$ and S defines a bounded sequilinear form on the Hilbert space of the corresponding representation. It is therefore given by a bounded operator by the Riesz lemma, which is Hermitean since S is, and commutes weakly with the representation because $S(\underline{f}^* \times (\underline{g} \times \underline{h})) = S((\underline{g}^* \times \underline{f})^* \times \underline{h})$.

The topology $\mathcal{N}$ has the advantage of being better adapted to the order structure and the positive functionals than τ. We collect its basic properties:

Theorem 6. (i) *$\underline{\mathscr{S}}[\mathcal{N}]$ is a Hausdorff locally convex space with dual space $\underline{\mathscr{S}}[\mathcal{N}]' = (\underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime}) + i(\underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime})$. There is a basis of continuous norms n such that $0 \leqq \underline{f} \leqq \underline{g}$ implies $n(\underline{f}) \leqq n(\underline{g})$.*

(ii) *$\mathcal{N}$ is strictly weaker than τ, but both topologies are identical when restricted to $\bigoplus_{\nu=0}^{N} \mathscr{S}_\nu$ for $N < \infty$.*

(iii) *Let Q_N be the projection $Q_N(\underline{f}) = (f_0, \ldots, f_N, 0, \ldots)$. If $\mathcal{M} \subset \underline{\mathscr{S}}$ has the property $Q_N \mathcal{M} \subset \mathcal{M}$ for all $N < \infty$, then the closure of $\mathcal{M}$ is the same in both topologies.*

(iv) *The bounded sets are the same in $\underline{\mathscr{S}}[\mathcal{N}]$ and $\underline{\mathscr{S}}[\tau]$.*

(v) *$\underline{\mathscr{S}}[\mathcal{N}]$ is nuclear and complete, but neither bornological nor barrelled.*

For the verification of these properties it is convenient to have a more explicit description of the $\mathcal{N}$-continuous norms:

Lemma 6. *Let $p(\underline{f}) = \sum_n p^n(f_n)$ be a τ-norm and $\hat{p}$ the corresponding $\mathcal{N}$-norm* (5). *Then*

$$\hat{p}(\underline{f}) = \sum_n \hat{p}^n(f_n)$$

where

$$\hat{p}^n(f_n) = \inf\left\{ \sum_{\mu+\nu=n} p^\mu \otimes_\pi p^\nu(f_{\mu\nu}) \Big| \sum_{\mu+\nu=n} f_{\mu\nu} = f_n \right\}.$$

Furthermore,

$$\hat{p}_{\{c_n\}}(\underline{f}) = \sum_n c_n \cdot \hat{p}^n(f_n)$$

is $\mathcal{N}$-continuous for all sequences $c_n \geqq 0$.

Proof. Consider the latter statement first. Let $\{d_n\}$ be a sequence such that $c_n \leqq \min_{\mu+\nu=n} \{d_\mu \cdot d_\nu\}$. (Construct it by induction: Take $d_0 = \max\{c_0^{1/2}, 1\}$

and suppose then one has constructed $d_0, \ldots, d_{n-1}$ with $d_\nu > 0$ and $c_{\mu+\nu} \leqq d_\mu \cdot d_\nu$ for all $\mu, \nu \leqq n-1$. Define

$$d_n = \max\{1, c_n \cdot d_0^{-1}, \ldots, c_{2n-1} \cdot d_{n-1}^{-1}, c_{2n}^{1/2}\}.)$$

Now

$$\hat{p}_{\{c_n\}}(\underline{g} \times \underline{h}) = \sum_n c_n \cdot \hat{p}^n\Big(\sum_{\mu+\nu=n} g_\mu \times h_\nu\Big) \leqq \sum_{\mu,\nu} d_\mu \cdot d_\nu \cdot p^\mu(g_\mu) \cdot p^\nu(h_\nu)$$
$$= p_{\{d_n\}}(\underline{g}) \cdot p_{\{d_n\}}(\underline{h}),$$

so $\hat{p}_{\{c_n\}}$ is $\mathcal{N}$-continuous. If $d_n = c_n \equiv 1$ and $\underline{f} = \sum_i \underline{g}^i \times \underline{h}^i$, we also have by taking the infimum over decompositions of $\underline{f}$ that $\sum_n \hat{p}^n(f_n) \leqq \hat{p}(\underline{f})$. Conversely, since $\mathcal{S}_\mu \hat{\otimes}_\pi \mathcal{S}_\nu = \mathcal{S}_{\mu+\nu}$ we have by the definition of $\hat{p}$ (5) that $\hat{p}(f_{\mu\nu}) \leqq p^\mu \otimes_\pi p^\nu(f_{\mu\nu})$ for all $f_{\mu\nu} \in \mathcal{S}_{\mu+\nu}$, so if $f_n = \sum_{\mu+\nu=n} f_{\mu\nu}$, then

$$\hat{p}(\underline{f}) \leqq \sum_n \hat{p}(f_n) \leqq \sum_n \sum_{\mu+\nu=n} \hat{p}(f_{\mu\nu}) \leqq \sum_n \sum_{\mu+\nu=n} p^\mu \otimes_\pi p^\nu(f_{\mu\nu})$$

and thus $\hat{p}(\underline{f}) \leqq \sum_n \hat{p}^n(f_n)$.

Proof of Theorem 6. (i) By Lemma 6 there exist $\mathcal{N}$-continuous norms so $\underline{\mathcal{S}}[\mathcal{N}]$ is Hausdorff. The dual space is generated by the positive functionals according to Theorem 5 which says just this for the real and imaginary part of $\underline{\mathcal{S}}[\mathcal{N}]'$. The last part of the statement follows from Theorem 4 and Proposition 1.5, p. 63, in [12]. The norms n have the form $n(\underline{f}) = \sup\{|T(\underline{f})| \,|\, T \in \underline{\mathcal{S}}^{+\prime}, |T| \leqq \hat{q}\}$ where $\hat{q}$ is some $\mathcal{N}$-norm.

(ii) $\mathcal{N}$ is strictly weaker than τ by (6) and Lemma 4. If $p^n = p^1 \otimes_\pi \cdots \otimes_\pi p^1$, then $p^\mu \otimes_\pi p^\nu = p^{\mu+\nu}$ for all μ and ν, so $\hat{p}^n = p^n$ in Lemma 6. Thus $\sum_n p^n(f_n)$ is $\mathcal{N}$-continuous, but these norms form a basis for the topology of $\bigoplus_{n=0}^N \mathcal{S}_n$.

(iii) Since τ is finer than $\mathcal{N}$ the τ-closure $\bar{\mathcal{M}}^\tau$ is in any case contained in the $\mathcal{N}$-closure $\bar{\mathcal{M}}^{\mathcal{N}}$. Let $\underline{f} = (f_0, \ldots, f_N) \notin \bar{\mathcal{M}}^\tau$. There is then a τ-neighbourhood of 0, U, such that $(\underline{f} + U) \cap \mathcal{M} = \emptyset$. We can choose U such that $Q_N U \subset U$ [e.g. the unit ball of a norm $q(\underline{f}) = \Sigma q^n(f_n)$]. By (ii) there is an $\mathcal{N}$-neighbourhood of 0, V, with $Q_N V \subset Q_N U \subset U$. Therefore, $Q_N((\underline{f} + V) \cap \mathcal{M}) \subset (Q_N \underline{f} + Q_N V) \cap Q_N \mathcal{M} \subset (\underline{f} + U) \cap \mathcal{M} = \emptyset$ because $Q_N \underline{f} = \underline{f}$ and $Q_N \mathcal{M} \subset \mathcal{M}$ by assumption. So $(\underline{f} + V) \cap \mathcal{M} = \emptyset$ and $\underline{f} \notin \bar{\mathcal{M}}^{\mathcal{N}}$.

(iv) Since the sequence $\{c_n\}$ in Lemma 6 can grow arbitrarily fast, an $\mathcal{N}$-bounded set must lie in $\bigoplus_{n=0}^N \mathcal{S}_n$ for some N and so be τ-bounded by (ii).

(v) $\underline{\mathcal{S}}[\mathcal{N}]$ is nuclear because it is isomorphic to the quotient of the nuclear space $\underline{\mathcal{S}}[\tau] \otimes_\pi \underline{\mathcal{S}}[\tau]$ and the closed subspace $\ker M$ (4). (Ker M

is closed since $\mathcal{N}$ is Hausdorff.) A τ-continuous linear functional is bounded on the τ-bounded sets and thus on the $\mathcal{N}$-bounded sets by (iv). Since τ-continuity does not imply $\mathcal{N}$-continuity $\underline{\mathcal{S}}[\mathcal{N}]$ is not bornological. Let $p = \bigoplus_n p^n$ be a τ-norm. The unit ball $\bar{U}_p = \{\underline{f} \mid p(\underline{f}) \leqq 1\}$ is $\mathcal{N}$-closed by (iii) and thus a barrel in $\underline{\mathcal{S}}[\mathcal{N}]$. It is not an $\mathcal{N}$-neighbourhood of 0 if p is not $\mathcal{N}$-continuous. Completeness of $\underline{\mathcal{S}}[\mathcal{N}]$ can be proven in a similar way as for a direct sum of complete spaces (cf. [8], p. 215) using Lemma 6.

At the end of this section two remarks:

1. The ordered vector space $\underline{\mathcal{S}}^{+\prime} - \underline{\mathcal{S}}^{+\prime}$ is not a vector lattice, i.e. for a $S \in \underline{\mathcal{S}}^{+\prime} - \underline{\mathcal{S}}^{+\prime}$ there is in general no smallest $T \geqq 0$ with $S + T \geqq 0$. This can be seen as follows:

Let $S = (0, S_1, 0, \ldots)$ with a real $S_1 \in \mathcal{S}_1'$. By Theorems 6 (ii) and 5, $S \in \underline{\mathcal{S}}^{+\prime} - \underline{\mathcal{S}}^{+\prime}$ so there is a $T \geqq 0$ such that $T + S \geqq 0$. Let λ be a real number > 0, $\alpha_\lambda T = (T_0, \lambda T_1, \lambda^2 T_2, \ldots)$. This is also positive because $\alpha_\lambda T(\underline{f}^* \times \underline{f}) = T(\alpha_\lambda \underline{f}^* \times \alpha_\lambda \underline{f}) \geqq 0$, and so $\lambda^{-1} \alpha_\lambda T \geqq 0$ and $\lambda^{-1} \alpha_\lambda T + S = \lambda^{-1} \alpha_\lambda (T + S) \geqq 0$. If $0 \leqq F \leqq \lambda^{-1} \alpha_\lambda T$ for all $\lambda > 0$, then $0 \leqq F_0 \leqq \lambda^{-1} T_0$ which implies $F_0 = 0$ and therefore $F = 0$ by Cauchy-Schwarz, so $\sup\{S, 0\}$ does not exist.

2. The order topology on $\underline{\mathcal{S}}$ is by definition the finest convex topology with the property that every order interval $[\underline{f}, \underline{g}] = \{\underline{h} \mid \underline{f} \leqq \underline{h} \leqq \underline{g}\}$ is bounded (cf. [12], p. 118). This topology is here identical to the Mackey topology τ. (Proposition 1.29, p. 77 and Proposition 1.16, p. 123 in [12].)

5. On the Extension of Positive Functionals

In this section we consider the problem of extending a positive functional from a subspace to the whole of $\underline{\mathcal{S}}$. For applications in field theory two types of subspaces are particularly of interest, subalgebras and the spaces $\bigoplus_{n=0}^{N} \mathcal{S}_n$. We shall here only consider the former case and even restrict ourselfs to special subalgebras. As for the latter case we only mention that the extension problem does not always have a solution: Consider for instance $T = (0, \ldots, 0, T_{2N})$ on $\bigoplus_{n=0}^{2N} \mathcal{S}_n$. This is positive on $\underline{\mathcal{S}}^+ \cap \left(\bigoplus_{n=0}^{2N} \mathcal{S}_n \right)$ if only T_{2N} is positive on $\mathcal{S}_{2N} \cap \underline{\mathcal{S}}^+$. For functionals in $\underline{\mathcal{S}}^{+\prime}$, however, $T_0 = 0$ implies $T = 0$.

If $\mathcal{B}$ is a $*$-subalgebra of $\underline{\mathcal{S}}$ we define

$$\mathcal{B}^+ = \left\{ \underline{f} = \sum_i \underline{f}^{i*} \times \underline{f}^i \,\middle|\, \underline{f}^i \in \mathcal{B} \right\}.$$

It is not true in general that $\mathscr{B}^+ = \mathscr{B} \cap \underline{\mathscr{S}}^+$. For instance, if

$$\mathscr{B} = \{\underline{f} \in \mathscr{S} \mid f_\nu = 0 \text{ for } 1 \leqq \nu \leqq 2N-1\},$$

then $(0, \ldots, 0, f_N)^* \times (0, \ldots, 0, f_N)$ is in $\mathscr{B} \cap \underline{\mathscr{S}}^+$ but not in $\mathscr{B}^+$. We shall, however, be concerned with subalgebras which are tensor algebras over closed, $*$-invariant subspaces $\mathscr{A}_1 \subset \mathscr{S}_1$, i.e. algebras of the type

$$\underline{\mathscr{A}} = \bigoplus_{n=0}^{\infty} \mathscr{A}_n$$

where $\mathscr{A}_0 = \mathbb{C}$ and $\mathscr{A}_n = \mathscr{A}_1 \hat{\otimes} \cdots \hat{\otimes} \mathscr{A}_1$. For this case we have

Lemma 7. *If $\sum_i \underline{f}^{i*} \times \underline{f}^i \in \underline{\mathscr{A}}$ with $\underline{f}^i \in \underline{\mathscr{S}}$, then $\underline{f}^i \in \underline{\mathscr{A}}$ for all i. In particular*

(i) $\underline{\mathscr{A}}^+ = \underline{\mathscr{A}} \cap \underline{\mathscr{S}}^+$

(ii) *$\underline{f}, \underline{g} \in \underline{\mathscr{S}}^+$, $\underline{f} + \underline{g} \in \underline{\mathscr{A}}$ implies $\underline{f}, \underline{g} \in \underline{\mathscr{A}}^+$.*

Proof. Since $\mathscr{A}_1$ is closed we have $\mathscr{A}_1 = (\mathscr{A}_1^\perp)^\perp$, where $\mathscr{A}_1^\perp \subset \mathscr{S}_1'$ is the annihilator of $\mathscr{A}_1$. More generally

$$\mathscr{A}_n = \{f_n \in \mathscr{S}_n \mid r^\mu \otimes t \otimes s^\nu(f_n) = 0 \text{ for all } t \in \mathscr{A}_1^\perp, r^\mu \in \mathscr{S}_\mu', s^\nu \in \mathscr{S}_\nu', \mu + \nu + 1 = n\}.$$

Suppose $\underline{f} = \sum_i \underline{f}^{i*} \times \underline{f}^i \in \underline{\mathscr{A}}$. For some N, $\underline{f} \in \bigoplus_{n=0}^{2N} \mathscr{S}_n$. Consider the $2N$-component $f_{2N}^i = \sum_i f_N^{i*} \times f_N^i \in \mathscr{A}_{2N}$. For all $t \in \mathscr{A}_1^\perp$, $r^\mu \in \mathscr{S}_\mu'$, $s^\nu \in \mathscr{S}_\nu'$ with $\mu + \nu + 1 = N$ we have $(r^\mu \otimes t \otimes s^\nu)^* \otimes (r^\mu \otimes t \otimes s^\nu) \in \mathscr{A}_{2N}^\perp$ and thus

$$0 = (r^\mu \otimes t \otimes s^\nu)^* \otimes (r^\mu \otimes t \otimes s^\nu)\, f_{2N} = \sum_{i=1}^{\infty} |(r^\mu \otimes t \otimes s^\nu)\, f_N^i|^2,$$

so $(r^\mu \otimes t \otimes s^\nu)\, f_N^i = 0$ for all i and thus $f_N^i \in \mathscr{A}_N$ for all i. Suppose now it has been proven for $m \geqq n+1$ that $f_m^i \in \mathscr{A}_m$ and therefore also $f_m^{i*} \in \mathscr{A}_m$ for all i. We have

$$f_{2n} = \sum_{i=1}^{\infty} \sum_{\lambda+\kappa=2n} f_\lambda^{i*} \times f_\kappa^i = \sum_{i=1}^{\infty} f_n^{i*} \times f_n^i + \sum_{i=1}^{\infty} \sum_{\lambda+\kappa=2n, \lambda \neq \kappa} f_\lambda^{i*} \times f_\kappa^i.$$

The functionals $(r^\mu \otimes t \otimes s^\nu)^* \otimes (r^\mu \otimes t \otimes s^\nu)$ with $t \in \mathscr{A}_1$ and $\mu + \nu + 1 = n$ annihilate f_{2n} and also all $f_\lambda^{i*} \times f_\kappa^i$ with λ or $\kappa \geqq n+1$ by assumption. Hence

$$\sum_{i=1}^{\infty} |(r^\mu \otimes t \otimes s^\nu)\, f_n^i|^2 = 0$$

and therefore $f_n^i \in \mathscr{A}_n$. We have thus proven that $\underline{f}^i \in \underline{\mathscr{A}}$ for all i. (i) and (ii) follow immediately.

For C^*-algebras the extension of a positive functional from a subalgebra with unit to the whole algebra causes no problems [14]. The proof depends on the fact that the unit element is an interior point of the cone of positive elements. On the other hand it is not difficult to see that $\mathscr{S}^+$ has no interior points at all [6]. Moreover, Lemma 6(ii) means that $\mathscr{A}^+$ is an extremal surface of $\mathscr{S}^+$ and this makes some other convenient criteria useless (e.g. [12], p. 83). The necessary and sufficient conditions e.g. in [12], [15] or [6] are not very handy in the present case. We are going to see that this question has no equally simple answer as for C^*-algebras (which is not so surprising because the structures are quite different). A related question which will also be treated is the following: Is the finest convex topology $\mathscr{N}_{\mathscr{A}}$ on $\mathscr{A}$, with the property that $\mathscr{A}[\tau]\times\mathscr{A}[\tau]\to\mathscr{A}[\mathscr{N}_{\mathscr{A}}]$ is jointly continuous, identical to the restriction $\mathscr{N}|\mathscr{A}$?

Theorem 7. *If $\mathscr{A}_1$ has a topological complement in $\mathscr{S}_1$ then every positive functional on $\mathscr{A}$ has an extension in $\mathscr{S}^{+\prime}$ and $\mathscr{N}_{\mathscr{A}}=\mathscr{N}|\mathscr{A}$.*

Proof. Let π_1 be a continuous projection of $\mathscr{S}_1$ on $\mathscr{A}_1$. If $\pi_1(f^*)\neq\pi_1(f)^*$, we can find a real projection by taking $\operatorname{Re}\pi_1=1/2(\pi_1+\pi_1^*)$ which is a projection on $\mathscr{A}_1$ because $\mathscr{A}_1=\mathscr{A}_1^*$. So we may assume that $\pi_1(f^*)=\pi_1(f)^*$. Define $\pi_0=id|\mathbb{C}$ and $\pi_n=\pi_1\otimes\cdots\otimes\pi_1$. $\pi=\bigoplus_{n=0}^{\infty}\pi_n$ is then a continuous $*$-algebra homomorphism $\mathscr{S}\to\mathscr{A}$ with $\pi|\mathscr{A}=id$. If T is positive on $\mathscr{A}$, then $T\circ\pi$ is a positive extension to $\mathscr{S}$.

Let U be an $\mathscr{N}_{\mathscr{A}}$-neighbourhood of 0 in $\mathscr{A}$. By definition, there is a τ-neighbourhood $V\subset\mathscr{A}$ such that $m(V\times V)\subset U$ and since π is continuous, there is a τ-neighbourhood $W\subset\mathscr{S}$ with $\pi W\subset V$. Thus, $m(W\times W)\cap\mathscr{A}=\pi(m(W\times W))=m(\pi W\times\pi W)\subset m(V\times V)\subset U$, and $m(W\times W)$ is an $\mathscr{N}$-neighbourhood of 0 so $\mathscr{N}|\mathscr{A}$ is finer than $\mathscr{N}_{\mathscr{A}}$. The inverse is trivial.

Theorem 8. *Let $\mathscr{A}_1\subset\mathscr{S}_1$ be a closed $*$-invariant subspace and T a positive linear functional on $\mathscr{A}$. If there exists a continuous norm p^1 on $\mathscr{S}_1$ and constants c_n such that $|T_n|\leqq c_n\cdot p^1\otimes_\pi\cdots\otimes_\pi p^1$, then T has an extension to a positive functional on $\mathscr{S}$.*

Proof. Let h^1 be a continuous Hilbert norm on $\mathscr{S}_1$ with $p^1\leqq h^1$ and $h^1(f^*)=h^1(f)$. This norm defines a topology on $\mathscr{A}_1$, let $\bar{\mathscr{A}}_1$ be the completion and $\bar{\mathscr{A}}=\bigoplus_{n=0}^{\infty}\bar{\mathscr{A}}_1\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\bar{\mathscr{A}}_1$ the corresponding tensor algebra with the topology of a locally convex direct sum. $\mathscr{A}$ is dense in $\bar{\mathscr{A}}$ and we can extend $*$ to a continuous involution on $\bar{\mathscr{A}}$. Furthermore, the multiplication is jointly continuous on $\bar{\mathscr{A}}$: By definition the topology on $\bar{\mathscr{A}}$ is given by the norms $h_{\{c_n\}}(f)=\Sigma c_n h^n(f_n)$ where $h_n=h^1\otimes_\pi\cdots\otimes_\pi h^1$ and $c_n>0$. As in the proof of Lemma 6 we can find d_n such that $c_n\leqq\min_{\mu+\nu=n}\{d_\mu\cdot d_\nu\}$, and show that $h_{\{c_n\}}(f\times g)\leqq h_{\{d_n\}}(f)\cdot h_{\{d_n\}}(g)$. The positive cone $\bar{\mathscr{A}}^+ = \{\Sigma \underline{f}^{i*} \times \underline{f}^i | \underline{f}^i \in \bar{\mathscr{A}}\}$ is therefore contained in the closure of $\underline{\mathscr{A}}^+$ in $\bar{\mathscr{A}}$. Every linear functional T on $\underline{\mathscr{A}}$ with $|T_n| \leqq c_n \cdot h^n$ defines a continuous functional on $\bar{\mathscr{A}}$ which is positive on $\bar{\mathscr{A}}^+$ if it is on $\underline{\mathscr{A}}^+$. Let π_1 be the projection $\mathscr{S}_1 \to \mathscr{A}_1$ with respect to the Hilbert norm h^1. As in Theorem 7, the corresponding $*$-algebra homomorphism $\pi : \underline{\mathscr{S}} \to \bar{\mathscr{A}}$ defines the extension $T \circ \pi$.

The following counterexample shows that Theorem 7 is not true for arbitrary subspaces $\mathscr{A}_1 \subset \mathscr{S}_1$. For simplicity of notation we consider functions of one variable, i.e. $\mathscr{S}_1 = \mathscr{S}(\mathbb{R}^1)$. For $\mathscr{A}_1$ we take the subspace

$$\mathscr{S}^0 = \{f \in \mathscr{S}_1 | d^n f/dx^n(0) = 0 \text{ for all } n\}.$$

$\mathscr{S}^0$ has no topological complement in $\mathscr{S}(\mathbb{R}^1)$.

Theorem 9. *There are positive functionals on $\underline{\mathscr{S}}^0$ which have no extension in $\underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime}$. In particular, $\mathscr{N}_{\underline{\mathscr{S}}^0} \neq \mathscr{N} | \underline{\mathscr{S}}^0$.*

Remark. The restriction of $\underline{\mathscr{S}}^{+\prime}$ to $\underline{\mathscr{S}}^0$ is strongly dense in $(\underline{\mathscr{S}}^0)^{+\prime}$. This follows simply from $\underline{\mathscr{S}}^{0+} = \underline{\mathscr{S}}^0 \cap \underline{\mathscr{S}}^+$ by duality and the fact that strong and weak closure of a convex set in $\underline{\mathscr{S}}^{0\prime}$ is the same thing. (This because $\underline{\mathscr{S}}^0$ is reflexive and the strong topology on $\underline{\mathscr{S}}^{0\prime}$ therefore compatible with the duality between $\underline{\mathscr{S}}^0$ and $\underline{\mathscr{S}}^{0\prime}$.) But if there exists one positive functional R on $\underline{\mathscr{S}}^0$ which has no extension in $\underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime}$, then such functionals are also dense: If T has an extension in $\underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime}$ then $T + \varepsilon R$ cannot have an extension for any $\varepsilon > 0$.

Proof of Theorem 9. The notation becomes simpler if we use the fact that $\mathscr{S}(\mathbb{R}^1)$ is isomorphic to the space $\mathscr{C}_c^\infty[-1, 1]$ of C^∞-function on $\mathbb{R}$ with compact support in the closed interval $[-1, 1]$ (cf. [10], p. 529). This isomorphism is given by the transformation of variables

$$x = (1-t)^{-1} - (1+t)^{-1} \in [-1, 1] \quad \text{for} \quad t \in [-\infty, \infty],$$

and the topology on $\mathscr{C}_c^\infty$ by the Hilbert norms

$$h_k(f) = \left(\sum_{\kappa=0}^{k} \int_{-1}^{1} |f^{(\kappa)}(x)|^2 \, dx \right)^{1/2}$$

with the scalar product

$$\langle f, g \rangle_k = \langle f, M_k g \rangle_0, \qquad M_k = 1 - d^2/dx^2 + \cdots + (-1)^k \, d^{2k}/dx^{2k}.$$

$\mathscr{S}^0$ is isomorphic to $\{f \in \mathscr{C}_c^\infty[-1, 1] | f^{(n)}(0) = 0 \text{ for all } n\}$. By abuse of notation, we shall in the following write $f \in \mathscr{S}$ and mean the corresponding function in $\mathscr{C}_c^\infty[-1, 1]$.

Let θ be the step function: $\theta(t)=0$ for $t<0$ and 1 otherwise. Functions in $\mathcal{S}^0$ have a zero of infinite order at 0, so cutting with θ defines a continuous $*$-algebra homomorphism $\alpha_\theta: \underline{\mathcal{S}}^0 \to \underline{\mathcal{S}}^0$:

$$\alpha_\theta f_n(t_1, \ldots, t_n) = \theta(t_1) \ldots \theta(t_n) f_n(t_1, \ldots, t_n).$$

If T is a positive functional on $\underline{\mathcal{S}}$, then $R = T \circ \alpha_\theta$ is positive on $\underline{\mathcal{S}}^0$. In the proof of Theorem 1 we considered functionals of the form $T_0 = 1$, $T_{2n-1} = 0$, $T_{2n}(f \otimes g) = c_{2n} \cdot \langle f^*, g \rangle_{k_n}^n$, and it was shown that $T = (T_0, T_1, \ldots)$ is a positive functional on $\underline{\mathcal{S}}$ if the sequences $\{c_{2n}\}$ and $\{k_n\}$ grow sufficiently fast for $n \to \infty$. We claim that $R = T \circ \alpha_\theta$ has no extension in $\underline{\mathcal{S}}^{+\prime} - \underline{\mathcal{S}}^{+\prime}$ if the sequence $\{k_n\}$ is not bounded. As a first step towards a proof of this we consider the following functionals on $\mathcal{S}^0 \hat{\otimes} \mathcal{S}^0$:

$$s_k(f \otimes g) = \langle \theta \bar{f}, \theta g \rangle_k = \sum_{\kappa=0}^{k} \int_{-1}^{1} f^{(\kappa)}(x)\, \theta(x)\, g^{(\kappa)}(x)\, dx.$$

s_k is real and symmetric in f and g. Any two extensions of s_k to a linear functional on $\mathcal{S} \hat{\otimes} \mathcal{S}$ differ at most by terms with support in $\{0\} \times \mathbb{R}^1 \cup \mathbb{R}^1 \times \{0\}$, so the symmetric, continuous extensions have the form

$$s_k(f \otimes g) = \sum_{\kappa=0}^{k} \int_{-1}^{1} f^{(\kappa)}(x) \theta(x) g^{(\kappa)}(x) dx + \sum_{\lambda=0}^{L_k} (f^{(\lambda)}(0) t_\lambda^k(g) + t_\lambda^k(f) g^{(\lambda)}(0)) \tag{7}$$

with $t_\lambda^k \in \mathcal{S}'$.

Lemma 7. *Let $\{k_n\}$ be a sequence of natural numbers and s_{k_n} of the form (7). If there exist continuous seminorms q and q_n such that*

$$|s_{k_n}(f \otimes g)| \leqq q(f) \cdot q_{k_n}(g) \tag{8}$$

for all $f, g \in \mathcal{S}$ and all k_n, then $\{k_n\}$ is bounded.

Proof of the Lemma. There is a Hilbert norm

$$h_M(f) = \left(\sum_{\kappa=0}^{M} \int |f^{(\kappa)}(x)|^2 \, dx \right)^{1/2}$$

with $q \leqq h_M$. Suppose $\{k_n\}$ is not bounded. Then for some n, $k_n = N \geqq M+1$ and

$$|s_N(f \otimes g)| \leqq q(f) \cdot q_N(g) \leqq h_{N-1}(f) \cdot q_N(g). \tag{9}$$

Partial integration in (7) yields

$$s_N(f \otimes g) = \sum_{\nu=0}^{N-1} \int f^{(\nu)}(x)\, \theta(x)\, g^{(\nu)}(x)\, dx - \int f^{(N-1)}(x)\, \theta(x)\, g^{(N+1)}(x)\, dx$$
$$- f^{(N-1)}(0)\, g^{(N)}(0) + \sum_{\lambda=0}^{L} (f^{(\lambda)}(0)\, t_\lambda(g) + t_\lambda(f)\, g^{(\lambda)}(0)).$$

For the first two terms an estimate of the form (9) is valid (eventually with another q_N). We have therefore also

$$\left|\sum_{\lambda=0}^{L} (f^{(\lambda)}(0)\, t_\lambda(g) + t_\lambda(f)\, g^{(\lambda)}(0)) - f^{(N-1)}(0)\, g^{(N)}(0)\right| \leqq h_{N-1}(f)\, \tilde{q}_N(g) \quad (10)$$

with some continuous seminorm $\tilde{q}_N$. Let $g \in \mathscr{S}^0$ and $f^{(\lambda)}(0) = 0$ with the exception of $\lambda = N-1$ and $\lambda = N$ respectively. In these cases (10) reduces to

$$|f^{(N-1)}(0)\, t_{N-1}(g)| \leqq h_{N-1}(f)\, q_N(g)$$

and

$$|f^{(N)}(0)\, t_N(g)| \leqq h_{N-1}(f)\, q_N(g)\,.$$

Now $f^{(N-1)}(0)$ and $f^{(N)}(0)$ are not dominated by the norm h_{N-1} because derivatives of order $\leqq N$ can be arbitrarily large at one point while the integral $h_{N-1}(f) = \left(\sum_{\kappa=0}^{N-1} \int |f^{(\kappa)}(x)|^2\, dx\right)^{1/2}$ remains bounded. Therefore $t_{N-1}(g) = t_N(g) = 0$, so t_{N-1} and t_N must have support in $\{0\}$. Let $g^{(\lambda)}(0) = f^{(\lambda)}(0) = 0$ with the exception of $\lambda = N-1$ and $\lambda = N$. Then (10) takes the form

$$\left|\sum_{m,n} a_{mn} f^{(m)}(0)\, g^{(n)}(0) - f^{(N-1)}(0)\, g^{(N)}(0)\right| \leqq h_{N-1}(f)\, \tilde{q}_N(g)$$

with constants a_{mn} which are symmetric in m and n because the sum over λ in (10) is symmetric in f and g. This inequality can only be true if the last term on the left side is compensated by $a_{N-1,N} f^{(N-1)}(0)\, g^{(N)}(0)$, i.e. $a_{N-1,N} = 1$. But in that case $a_{N,N-1} = 1$ and $f^{(N)}(0)\, g^{(N-1)}(0)$ can even less be estimated by $h_{N-1}(f)\, \tilde{q}_N(g)$ so (9) cannot be true and we have a contradiction to the assumption that $\{k_n\}$ is unbounded.

We now bring the proof of our theorem to an end. Consider any Hermitean extension of the functional R. For $f_2, \ldots, f_n$ fixed the functional

$$f \otimes g \mapsto R_{2n}(f \otimes f_2^* \otimes \cdots \otimes f_n^* \otimes f_n \otimes \cdots \otimes f_2 \otimes g)$$

is an extension of const$\langle \theta \bar{f}, \theta g \rangle_{k_n}$ and is therefore of the form (7). If $R \in \underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime}$, then R is $\mathscr{N}$-continuous and there is a continuous norm q_1 (not depending on n) and continuous norms q_{2n-1} such that

$$|R_{2n}(f \otimes g_{2n-1})| \leqq q_1(f) \cdot q_{2n-1}(g_{2n-1})\,.$$

But this would imply (8) which is not possible if $\{k_n\}$ is unbounded. So there is no extension in $\underline{\mathscr{S}}^{+\prime} - \underline{\mathscr{S}}^{+\prime}$ and R is not $\mathscr{N}$-continuous. It is in any case $\mathscr{N}_{\underline{\mathscr{S}}^0}$-continuous, so $\mathscr{N}_{\underline{\mathscr{S}}^0}$ is strictly finer than $\mathscr{N} | \underline{\mathscr{S}}^0$.

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