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ON THE ALGEBRAIC FORMULATION OF DYNAMICAL MODELS

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ABSTRACT

An algebraic method for the solution of some dynamical problems for a non-relativistic system with known symmetry properties is given, in the case of the so-called broken symmetry. Examples of soluble models are given. One of these, based on the non-compact group Sp(6, R), gives a mass formula for baryons depending on three parameters which reproduces the known masses within electromagnetic mass differences. The model implies the relation

$$\frac{Y^* - \Sigma}{\Sigma - \Lambda} \approx \frac{5}{2}$$

in good agreement with experiment. The mass formula in the model can be considered as an eigenvalue solution of a differential equation in an appropriate Riemannian manifold.

Relation with the proposed models and charge algebras is discussed.

ON THE ALGEBRAIC FORMULATION OF DYNAMICAL MODELS

I. INTRODUCTION

In a previous work[1] it was shown how one can give an algebraic formulation of the dynamical problem of some classical systems whose symmetry properties may be represented by simple Lie groups.

The interest of this approach lies in its generality which, we may hope, will help provide some insight into the dynamical properties of systems with no classical analogue for which the symmetry properties are the experimentally best known features.

These symmetry properties are conventionally expressed by the existence of a Lie algebra which, in a Lagrangian formulation of the theory, commute both with the interaction and mass operator.

In the present approach the Lie algebra considered does not commute with the mass operator and it is precisely the commutation relations of the latter with the generators of the Lie algebra which determine the dynamical properties of the system, and in particular its mass spectrum.

It is known that in general the existence of discrete mass spectra put severe restrictions on the possibility of mixing internal with space time symmetries. We thus restrict our consideration, for the moment, to non-relativistic systems at rest.

The outline of the method is the following: let S_o be the symmetry algebra of a system with the property of commuting with the mass operator M_o which in turn is a non-commuting member of a larger algebra S leaving S_o as a subalgebra; the algebra S, called dynamical algebra[2)*] or spectrum-generating algebra[3] (SGA) is characterized by having representations which are a direct sum of the discrete representations of S_o belonging to the different eigenvalues of M_o.

Once S_o, S and M_o are known, the Casimir operators of S,

* The concept of dynamical algebra as introduced by Barut differs from ours in that it implies S_o resulting from a contraction of S.

for the value corresponding to the desired representations, allow the determination of M_0 as a function of the Casimir operators of S_0 .

By this method, the dynamical problem of maximal degenerate systems[1] (for which M_0 depends from only one quantum number) can easily be solved.

We shall now try to apply the method to the case of so-called broken symmetry. That is, the mass operator M_0 is substituted by a new operator M which will be invariant only with respect to a subgroup s_0 of S_0 . This will obviously induce a lowering of the degeneracy of the mass eigenvalues but we will see that if the breaking is obtained with the criterion of introducing in the mass operator only commuting operators belonging to S_0 or to its enveloping algebra, the Casimir operator of the original group S can still give an exact eigenvalue equation for M .

We shall see further that when the mass breaking is attained according to the above rule, in the field theory obtained by operating a second quantization on the eigenstates of the mass operator, the "charges" build up the algebra S_0 even after the symmetry is broken.

We shall give some examples of application, and a model suitable for reproducing some aspects of baryon physics.

II. THE METHOD

Let S_0 be a symmetry algebra for the mass operator M_0 which in turn is a generator of a larger algebra S containing S_0 as a subalgebra.

Let the system be a maximal degenerate one. The Casimir operator C of S will be of the form

$$(1) \qquad C = C_0 + M_0^2 + A$$

where C_0 is the Casimir operator of S_0 and A depends on C and C_0 only. C determines the representations of S and we shall be interested in those which contain the desired ones of S_0 once and only once. Then

Eq. (1) determines M_o as a function of C_o and each eigenvalue of M_o defines a multiplet of eigenstates belonging to it.

It is known that in general we can choose a number of commuting operators belonging to the algebra S_o or to its enveloping algebra that remove partially or completely the degeneracy of M_o .

Let us suppose we are able to identify the subgroup s_o of S_o which remains after the symmetry is broken; this identification is in general possible by direct inspection of the experimental data suggesting which are the good quantum numbers.

This identification determines the set of operators which removes the degeneracy of M_o . If we are now able to decompose C_o in (1) in terms of c_o, the Casimir operator of s_o, and operators O_i commuting among themselves with M_o and with s_o and define M as the sum (or more generally a function) of these operators and M_o , the new relation obtained from (1) will be of the type:

$$(2) \qquad M = M\ (C_o\ ,\ c_o\ ,\ O_i)$$

and will determine the mass splitting inside the multiplets of M_o . In general we shall also request that the operators to be added to M_o be traceless and irreducible with respect to the unbroken symmetry S_o .

The corresponding eigenstates will be given by those linear combinations of the eigenstates of M_o (belonging to the same eigenvalue) which are simultaneous eigenstates of the operators added to M_o to obtain M .

By this procedure no arbitrary parameter is inserted in (2). But it may happen that some of the commuting operators which are necessary to lower the degeneracy of M_o do not appear from the decomposition of C_o ; in such a case one can try to find another group S from which to start or simply add these operators in both members of (2). Obviously if this second heuristic method is adopted for each new addition a new arbitrary parameter will be added in the mass formula.

III. APPLICATIONS

In order to gain insight into the method, we shall start from a model having some kind of similarity to elementary particles and in which the mathematical algorithm is simple enough not to hide the physical content.

In order to simplify the algebraic calculations we shall use as far as possible the raising-lowering operators technique.

a) The six-dimensional oscillator model.

It is known that some of the properties of elementary particles [3)] can be represented by a model of two uncoupled oscillators with creation and annihilation operators:

(3)
$$a_1=\frac{1}{2\sqrt{2}}\left[p_1-q_2-i(q_1+p_2)\right], \qquad a_4=\frac{1}{2\sqrt{2}}\left[p_4-q_5-i(q_4+p_5)\right],$$
$$a_2=\frac{1}{2\sqrt{2}}\left[p_1+q_2-i(q_1-p_2)\right], \qquad a_5=\frac{1}{2\sqrt{2}}\left[p_4+q_5-i(q_4-p_5)\right],$$
$$a_3=\frac{1}{2}\left[p_3-iq_3\right], \qquad a_6=\frac{1}{2}\left[p_6-iq_6\right],$$

$$a_1^\dagger=\frac{1}{2\sqrt{2}}\left[p_1-q_2+i(q_1+p_2)\right],\ etc.,\ with\ \left[a_i, a_j^\dagger\right]=\delta_{ij}$$

In case of isotropy the Hamiltonian

(4)
$$M_0=\frac{\omega}{2}\sum_1^6{}_i\,\left[a_i^\dagger, a_i\right]_+$$

commutes with the algebra U(6) of generators,

(5)
$$\tilde{E}^i_j=\frac{1}{2}\left[a_i^\dagger, a_j\right]_+ ,$$
$$\left[\tilde{E}^i_j, \tilde{E}^k_e\right]=\delta_j^k\,\tilde{E}^i_e-\delta^i_e\,\tilde{E}^k_j$$

and every eigenstate of the system belonging to SU(6) representations of the type (N, 0000), given by

(6)
$$\frac{1}{\sqrt{\alpha_1!\dots\alpha_6!}}(a_1^\dagger)^{\alpha_1}\dots(a_6^\dagger)^{\alpha_6}\,|0\rangle \quad , \quad \alpha_1+\dots+\alpha_6=N$$

with degeneracy $\frac{(N+5)!}{N!\ 5!}$

is contained once and only once in two infinite-dimensional representations of Sp(6, R) of generators:

(7)
$$E^i_j = \frac{1}{2}[a^\dagger_i, a_j]_+ - \frac{1}{6}\delta^i_j \, E$$
$$E = \Sigma_k \frac{1}{2}[a^\dagger_k, a_k]_+$$
$$E^{ij}_0 = a^\dagger_i a^\dagger_j \quad , \qquad E^0_{ij} = a_i a_j$$

whose commutation relations are easily obtained using (3).

The second order Casimir operator Q_6 of Sp(6, R)

(8)
$$Q_6 = C_6 - \frac{5}{6}\frac{M_0^2}{\omega^2} - 12$$

has the same eigenvalue -39/2 for both representations and from this value the dependence of M_0 from C_6 , the second order Casimir operator of SU(6), is immediately obtained:

$$M_0^2 = \frac{6}{5}\omega^2\left(C_6 + \frac{15}{2}\right)$$

Now let us suppose that the symmetry is broken by letting

$$a_i = a_i \quad , \qquad b_i = a_{i+3} \quad , \qquad i = 1, \ldots, 3$$

and the two three-dimensional oscillators correspond to different frequencies ω_1 and ω_2 . It is clear that the commutation relations of the a's and b's will remain unchanged and will be so for the operators (3) which are defined in terms of the a's and b's . Only the Hamiltonian will be changed:

(9)
$$M_0 \to M = \frac{\omega_1}{2}\sum_1^3{}_i [a^\dagger_i, a_i]_+ + \frac{\omega_2}{2}\sum_1^3{}_i [b^\dagger_i, b_i]_+$$

and will be symmetric only with respect to the algebra $SU(3)_a \times SU(3)_b$.

The eigenstates of the M will still be of the type

(10)
$$\frac{1}{\sqrt{\alpha_1! \ldots \alpha_6!}}(a^\dagger_1)^{\alpha_1} \ldots (b^\dagger_3)^{\alpha_6} |0\rangle \quad , \qquad \begin{matrix} \alpha_1 + \alpha_2 + \alpha_3 = N_a \\ \alpha_4 + \alpha_5 + \alpha_6 = N_b \end{matrix}$$

with degeneracy lowered to
$$\frac{1}{4}(N_a+1)(N_a+2)(N_b+1)(N_b+2)$$

That is, we shall have a splitting of the previous levels due to the breaking of the symmetry. But the spectrum-generating algebra remains the same, and we can still obtain from its Casimir operator a mass formula which is exact. We can write in fact:

$$M = \frac{\omega_1+\omega_2}{2\omega} M_0 + \frac{\omega_1-\omega_2}{2\omega} M_1 = c_0 M_0 + c_1 M_1 \quad , \tag{11}$$

with

$$\frac{1}{\omega} M_1 = \frac{1}{2} \sum_{1}^{3} {}_i \left\{ [a_i^\dagger, a_i]_+ - [b_i^\dagger, b_i]_+ \right\}$$

Cartan operator of SU(6) invariant of $SU_a(3) \times SU_b(3)$. If we now reduce the Casimir operator of $Sp(6, R)$ with respect to $SU_a(3) \times SU_b(3)$ we obtain:

$$Q_6 = C_3^a + C_3^b - \frac{1}{3\omega^2}(M_1^2 + M_0^2) - \frac{33}{2} \tag{12}$$

where C_3^a and C_3^b are the Casimir operators of the two SU(3) subgroups. If we now note that $M_1 = \omega\left(\sqrt{3/2\ C_3^a + 9/4} - \sqrt{3/2\ C_3^b + 9/4}\right)$ and we substitute M_0 obtained from (11) in (12), we obtain:

$$\begin{aligned} M &= \omega_1 \sqrt{3/2\, C_3^a + 9/4} + \omega_2 \sqrt{3/2\, C_3^b + 9/4} = \\ &= \omega_1 (j_a + 3/2) + \omega_2 (j_b + 3/2) \end{aligned} \tag{13}$$

with j_a, $j_b = 0, 1, 2, \ldots$ defined by $C_3^a = 2/3\ j_a (j_a + 3)$, a formula which is exact, but contains two arbitrary parameters.

In general, a similar result is obtained if we substitute for M_0 any function of M_0 and M_1 instead of just the linear one. If we wish to obtain the form of M for the broken symmetry with no arbitrary parameters but depending only on the chosen non-compact group $Sp(6,\mathbb{R})$, we shall simply substitute in (5):

$$M_0^2 \longrightarrow M^2 = [M_0^2 + M_1^2] = \frac{\omega^2}{2} \left\{ \left[\sum_1^3 {}_i [a_i^\dagger, a_i]_+\right]^2 + \left[\sum_1^3 {}_i [b_i^\dagger, b_i]_+\right]^2 \right\} \tag{14}$$

Then we obtain from Eq. (12):

(15) $$M^2 = 3/2\,\omega^2[C_3^a + C_3^b] + 9/2\,\omega^2 = \omega^2[(j_a + 3/2)^2 + (j_b + 3/2)^2]$$

exact for all states labelled by M_0 and M_1 eigenvalues.

b) The Elliott model and its generalization for hadrons

We shall now show that the same algebra may also account for the baryon mass spectra. To this end we shall first formulate with our method the Elliott model.[4] As is known, the Elliott model amounts to considering the nucleons bound in the nucleus in an elastic potential well to which a quadrupole type interaction is added.

Let $a_i^{\alpha\dagger}$, a_i^{α} be the creation-annihilation operators for the system where the index $\alpha = 1, \ldots, A$ refers to the nucleon in the nucleus and i refers to the spatial degree of freedom (for simplicity we shall not consider spin and isotopic spin here).

Then, the generators

$$E_{j\beta}^{i\alpha} = \frac{1}{2}[a_i^{\alpha\dagger}, a_j^{\beta}]_+ \qquad \begin{matrix} i, j = 1, \ldots 3 \\ \alpha, \beta = 1, \ldots A \end{matrix}$$

build up the algebra of U(3A) and together with

$$E_0^{i\alpha, j\beta} = a_i^{\alpha\dagger} a_j^{\beta\dagger}$$

$$E_{i\alpha, j\beta}^{0} = a_i^{\alpha} a_j^{\beta}$$

the spectrum-generating algebra Sp(3A, R).

If we define

(16) $$M_0 = \frac{\omega}{2} \sum_{1}^{3}{}_i \sum_{1}^{A}{}_\alpha [a_i^{\alpha\dagger}, a_i^{\alpha}]_+$$

we easily obtain

$$Q_{3A} = C_{3A} + \frac{1-3A}{3A} \frac{M_0^2}{\omega^2} - \frac{3A}{2}(3A+1)$$

from which we get

(17) $$\frac{M_0}{\omega} = j_{3A} + \frac{3A}{2}$$

We know also that M_0 commutes with the algebra SU(3) with generators

(18) $$E^i_j = \frac{1}{2} \sum_{1\alpha}^{A} [a^{\alpha\dagger}_i, a^{\alpha}_j]_+ - \frac{1}{3} \delta^i_j \frac{M_0}{\omega}$$

This means that the M_0 eigenstates build up irreducible representations of this algebra also.

If we re-arrange these generators in such a way as to obtain among them the generators of the SO(3) algebra of orbital angular momenta

(19) $$L_+ = -\sum_{1\alpha}^{A} (a^{\alpha\dagger}_3 a^{\alpha}_2 + a^{\alpha\dagger}_1 a^{\alpha}_3) \qquad L_3 = \sum_{1\alpha}^{A} (a^{\alpha\dagger}_1 a^{\alpha}_1 - a^{\alpha\dagger}_2 a^{\alpha}_2)$$
$$L_- = \sum_{1\alpha}^{A} (a^{\alpha\dagger}_3 a^{\alpha}_1 + a^{\alpha\dagger}_2 a^{\alpha}_3)$$

the other five generators are

$$q_{+2} = -\sqrt{6} \sum_\alpha a^{\alpha\dagger}_1 a^{\alpha}_2 \quad , \quad q_{-2} = -\sqrt{6} \sum_\alpha a^{\alpha\dagger}_2 a^{\alpha}_1$$

(19') $$q_{+1} = -\sqrt{3} \sum_\alpha (a^{\alpha\dagger}_3 a^{\alpha}_2 - a^{\alpha\dagger}_1 a^{\alpha}_3) \quad , \quad q_{-1} = -\sqrt{3} \sum_\alpha (a^{\alpha\dagger}_3 a^{\alpha}_1 - a^{\alpha\dagger}_2 a^{\alpha}_3)$$

$$q_0 = \sum_\alpha (2 a^{\alpha\dagger}_3 a^{\alpha}_3 - a^{\alpha\dagger}_1 a^{\alpha}_1 - a^{\alpha\dagger}_2 a^{\alpha}_2)$$

The Casimir operator of SU(3) now appears:

(20) $$C_3 = \frac{1}{2} L^2 - q^2$$

where $q^2 = \frac{1}{6} \sum_m (-)^m q_m q_{-m}$, and both commute with M_0.
The simultaneous eigenstates of M_0, L^2, q^2, C_3 will now be built up by particular linear combination of the symmetric SU(3A) eigenstates.

Following our procedure, we can now substitute for M_0, M given by:

$$M = M_0 + \gamma q^2$$

with γ arbitrary parameter, which amounts to adding to the elastic potential a collective quadrupole interaction. From (17) and (20) we obtain the Elliott energy spectrum:

(21) $$M = \omega\left(j_{3A} + \frac{3A}{2}\right) + \frac{\gamma}{2} L^2 - \gamma C_3$$

where $L^2 = \ell(\ell+1)$ and $C_3(\mu,\nu)$ is determined.

Let us now take a similar model to describe baryons. We shall take the same algebra of the preceding paragraph to describe a hadron system; we have only to interpret the indices of the a and a^+ as relating to the internal degree of freedom of quarks (spin and unitary spin).

Then, as usual, we shall assign the baryons (built up by $3 + 2n$ quarks) to the symmetric representations of SU(6). Taking once again Sp(6, R) as the spectrum-generating algebra we obtain for the unbroken symmetry*:

(22) $$M_0 = \omega(j_6 + 3)$$

We shall now break the symmetry with respect to SU(3) x SU(2) when the first algebra refers to isotopic spin and the second to ordinary spin. The Casimir operator C_6 of SU(6) will correspondingly break:

(23) $$C_6 = 5(C_2 - C_3)$$

valid for all symmetric representations of SU(6). Again C_3, C_2 and C_6 commute with M_0. As in the Elliott model we shall now substitute M_0 by M given by

(24) $$M = M_0 + \alpha C_2$$

(which amounts to adding a spin-dependent interaction) and we get

(25) $$M = \omega(j_6 + 3) + \alpha\left[1/5\, C_6 + C_3\right]$$

with α arbitrary constant.

The eigenstates of M will be those linear combinations of the symmetric eigenstates of SU(6) which are simultaneous eigenstates of C_2 and C_3. These quadratic operators in the frame of the SU(6) algebra are

* Here, obviously, the indices have no relation to spatial degree of freedom and correspondingly there is no place for interpretation of spatial potential.

neither traceless nor irreducible. In order to break the symmetry further we shall decompose all tensors in (25) in traceless and irreducible parts; we thus obtain

$$C_3 = \frac{1}{2} C_3 + J(J+1) - \frac{5}{14} C_6 + \frac{1}{2} C_3 - J(J+1) - \frac{1}{10} C_6 + \frac{16}{35} C_6$$

Substituting this expression and noting that $\frac{1}{2} C_3 - J(J+1) - \frac{1}{10} C_6$ is identically zero for the symmetric representation of SU(6) we obtain

$$M = \omega(j_6+3) + \alpha\left[\tfrac{1}{2} C_3 + J(J+1) + \tfrac{3}{10} C_6\right] \tag{26}$$

Now, breaking the SU(3) symmetry in SU(2) x U(1), where SU(2) refers to isotopic spin, we obtain:

$$\tfrac{1}{2} C_3 = I(I+1) - \tfrac{1}{4} Y^2 + \tfrac{1}{3} C_3 + T_3^3 \tag{27}$$

Inserting this into (26) and adding the traceless operator T_3^3, commuting with C_3 and I^2, to M we get

$$\overline{M} = \omega(j_6+3) + \alpha\left[J(J+1) + \tfrac{1}{3} C_3 + \tfrac{3}{10} C_6 + I(I+1) - \tfrac{1}{4} Y^2\right] \tag{28}$$

In breaking the symmetry for SU(3) to SU(2) x U(1) we could also have changed the frequencies of the six-dimensional oscillator, putting

$$\omega_1 = \omega_2 = \omega_4 = \omega_5 = \omega_a$$

$$\omega_3 = \omega_6 = \omega_b$$

This would have changed M_o to M_o' given by:

$$M_o' = \frac{2\omega_a + \omega_b}{3\omega} M_o + (\omega_a - \omega_b) Y$$

and again M_o' commutes with all the preceding operators inserted in M. We finally obtain

$$\mathcal{M} = \omega'(j_6+3) + \alpha\left[J(J+1) + \tfrac{1}{3} C_3 + \tfrac{3}{10} C_6 + I(I+1) - \tfrac{1}{4} Y^2\right] + \beta Y \tag{29}$$

For $\omega' = .1241$, $\alpha = .039$, $\beta = -.1955$, the (29) fits the experimental baryon masses within the e.m. mass differences, i.e., within .5%. The mean mass of the next SU(6) representation $\{252\}$ would be 2.167 BeV.

Considering that for the symmetric representations $C_3 = 2J(J+1) + \frac{1}{5} C_6$, we see that (29) can also be written in the form:

$$\mathcal{M} = \mathcal{M}_o + \alpha\left[\tfrac{5}{3} J(J+1) + I(I+1) - \tfrac{1}{4} Y^2\right] + \beta Y \tag{30}$$

with $m_o = 1{,}066$ BeV, for the $\{56\}$ and $m_o = 1.648$ BeV for the $\{252\}$ representations respectively. From (30) one obtains the relation

$$2\,\frac{Y_1^* - \Sigma}{\Sigma - \Lambda} = 5 \tag{31}$$

as compared with the experimental value 5.01.

It must be pointed out that (29) is exact in the frame of the model; that is, (29) represents the exact eigenvalues of m for the eigenstates of the SU(6) symmetric multiplets which are eigenstates of I , J , Y and C_3 .

One can also try to trace back from (29) a differential operator which gives the eigenvalue equation in an appropriate symmetric space.

Following RĄCZKA [6] it is sufficient to consider the group of motion in a six-dimensional Riemannian space locally isomorphic to $T_t \otimes SO(5)$ or $T_t \otimes S^4 \otimes SO(4)$ or $T_t \otimes S^4 \otimes SO(3) \otimes SO(3)$. Taking the line element in this space as

$$ds^2 = -dt^2 - R_T^2\, d\alpha^2 + R_I^2\, d\beta^2 + R_I^2 \sin^2\beta\, d\gamma^2 + R_J^2\, d\delta^2 + R_J^2 \sin^2\delta\, d\varphi^2 \tag{32}$$

the Klein-Gordon equation obtained from the Laplace-Beltrami operator in this space is:

$$\left[-\frac{\partial^2}{\partial t^2} - \frac{1}{R_T^2}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{R_I^2}\frac{1}{\sin\beta}\left(\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin\beta}\frac{\partial^2}{\partial\gamma^2}\right) + \frac{1}{R_J^2}\frac{1}{\sin\delta}\left(\frac{\partial}{\partial\delta}\sin\delta\frac{\partial}{\partial\delta} + \frac{1}{\sin\delta}\frac{\partial^2}{\partial\varphi^2}\right)\right]\psi = -\mathcal{M}_o^2\,\psi \tag{32'}$$

with orthonormal solutions:

$$\psi = e^{imt} e^{is\alpha} Y_{I, I_3}(\beta, \gamma) Y_{J, J_3}(\delta, \varphi) \tag{33}$$

where the eigenvalue $\overline{m}_o$ is contained in m .

Since we are interested in the non-relativistic theory, the corresponding Schrödinger equation which has the same symmetry group is

$$-i\frac{\partial}{\partial t}\psi = \Big\{\overline{m}_o - \frac{1}{2\overline{m}_o}\Big[-\frac{1}{R_T^2}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{R_I^2}\frac{1}{\sin\beta}\Big(\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin\beta}\frac{\partial^2}{\partial\gamma^2}\Big) + \frac{1}{R_J^2}\frac{1}{\sin\delta}\Big(\frac{\partial}{\partial\delta}\sin\delta\frac{\partial}{\partial\delta} + \frac{1}{\sin\delta}\frac{\partial^2}{\partial\varphi^2}\Big)\Big]\Big\}\psi \tag{34}$$

(33) is still the eigenfunction of the non-relativistic Eq. (34) with eigenvalues

$$m = \overline{m}_o + \frac{1}{2\overline{m}_o}\Big[\frac{1}{R_I^2} I(I+1) + \frac{1}{R_J^2} J(J+1) - \frac{1}{R_T^2} S^2\Big]$$

Putting $S = Y + B$* with Y hypercharge we have

$$m = \overline{m}_o - \frac{1}{2\overline{m}_o R_T^2}B^2 + \frac{1}{2\overline{m}_o}\Big[\frac{1}{R_I^2} I(I+1) + \frac{1}{R_J^2} J(J+1) - \frac{1}{R_T^2} Y^2\Big] - \frac{1}{R_T^2\overline{m}_o} BY \tag{35}$$

which is identical to (30), with the substitutions

$$m_o = \overline{m}_o - \frac{1}{2\overline{m}_o R_T^2}B^2 \qquad R_T^2 = 4$$

$$\alpha = \frac{1}{2\overline{m}_o R_I^2} \qquad \frac{R_I^2}{R_J^2} = \frac{5}{3}$$

$$\beta = -\frac{1}{R_T^2\overline{m}_o}B$$

In this formulation I and J can only take integer values but one can note that in order to obtain half integer values of I and J as well as integer ones it is sufficient to take, instead of the Riemannian space $V_6 \approx T_1 \otimes S^1 \otimes S^2 \otimes S^2$ with the group of motion $T^1 \otimes SO(2) \otimes SO(3) \otimes SO(3)$, the Riemannian manifold $V_8 \approx T_1 \otimes S^1 \otimes S^3 \otimes S^3$ with the same isometry group.

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* This substitution was suggested by Rączka.

From this, one can easily obtain the Green function which is connected with the baryon propagator in the static limit.

For the mesons one can try to reduce Sp(6, R) with respect to the non-maximal degenerate representations of SU(6) or to go to higher groups. As an example of this approach one could formulate a model having a closer resemblance to the Elliott model for nuclei also introducing indices into the creation-annihilation operators referring to the space degree of freedom.

Then the operators are

$$a_i^\alpha, \; a_i^{\alpha\dagger} \qquad \begin{matrix} \alpha = 1, \dots 6 \\ i = 1, \dots 3 \end{matrix}$$

where the upper indices refer to unitary and ordinary spin and the lower to spatial degrees of freedom. The algebra is now U(18) which can be broken first to U(6) x U(3) and then to U(6) x O(3) when a quadrupole term of interaction is added to the elastically bound quarks; the mass formula will then contain a term $\alpha\ell\ (\ell+1)$ with ℓ orbital quantum number. Starting from a bigger group, one obviously obtains mixing of representations when the subgroup is considered. In this particular example in the 1140 representation of SU(18), the baryon appears as a mixing of 56, 70 and 20 representations of SU(6).

IV. CHARGE ALGEBRA

Referring to the models constructed with the SU(6) group, let us now consider the six states

$$a_i^\dagger \, |0\rangle \qquad i = 1, \dots 6$$

which form the basis for the irreducible representation (10000) of SU(6), in cases of both broken and unbroken symmetry. We can consider them as the six components ψ_i of a six-dimensional basic spinor $|\psi\rangle$. Then defining in the usual way the corresponding bra $\langle\psi|$ it follows that the charges

(36) $$\langle \psi | a_i^\dagger a_j | \psi \rangle = C^i_j$$

transform as the generators E^i_j under an SU(6) transformation.

If we introduce a second quantization of this theory it is clear that, corresponding to the "charges", (36) would be integral non-commuting operators which would obey the equal-time commutation relations

(37) $$[C^i_j, C^k_l] = \delta_j^k C^i_l - \delta^i_l C_j^k$$

These commutation relations are exact and the theory builds up the SU(6) algebra despite the fact that the symmetry is broken from SU(6) to SU(3) x SU(3) x U(1) or to SU(2) x U(1) x SU(2).

The breaking of the symmetry will determine the fact that, of the 35 SU(6) charges C^i_j, only the 17 belonging to SU(3) x SU(3) x U(1) or the 7 of SU(2) x U(1) x SU(2) will be time independent; that is, they will commute with the mass operator m.

In each particular case considered it will be possible to calculate the matrix elements of C^i_j between particular eigenstates of m which, in the baryon model, will be linear combinations of the type

$$|\psi_M\rangle = \sum_\alpha c_\alpha \frac{(a_1^\dagger)^{\alpha_1} \cdots (a_6^\dagger)^{\alpha_6}}{\sqrt{\alpha_1! \cdots \alpha_6!}} |0\rangle$$

but the matrix element of the charges (36) will be different from zero only between states of the same multiplet of m_o since they all commute with m_o.

The time-dependent charges will now have zero matrix element between states belonging to different representations of SU(3) x SU(3) x U(1) (or SU(2) x U(1) x SU(2)) and these matrix elements will be proportional to the parameter multiplying the term in the mass operator which reduces the symmetry from SU(6) to SU(3) x SU(3) x U(1) (or SU(2) x U(1) x (SU(2)). But the matrix element of the time-independent charges of the reduced symmetry (the 17 generators of SU(3) x SU(3) x U(2) or the ones of SU(2) x U(1) x SU(2) will be different from zero only between states belonging to the same representation of SU(3) x SU(3) x U(1) or SU(2) x U(1) x SU(2).

Once the states are determined it will be possible from (37) to determine sum rules etc.

V. CONCLUSION AND OUTLOOK

We have shown that once the labelling of the states of a system according to a given symmetry group is assigned and the sequence of the multiplets of this group is known, one can find a non-compact (eventually compact) spectrum-generating algebra which generates the given states spectrum.

From the Casimir operator of the non-compact group one can then obtain exact solutions for the mass spectrum of the system, even in cases of so-called broken symmetry, in terms of the conserved quantum numbers. In corresponding quantized field theories the "charges" generate the unbroken symmetry algebra but only some of them commute with the mass operator and are constants of the motion.

There are still a number of open questions. One is the identification of the proper group. It is clear that the choice of the spectrum-generating algebra is not unambiguous. And this choice determines both the number of arbitrary parameters in the mass formula and the multiplet mixing of the assigned symmetry. Another important problem is the generalization of such an approach to a moving relativistic system and it is to be expected that the relativistic covariance requirements will bring about restrictions on the symmetry breaking of the original group as obviously happens in the case of the hydrogen atom when relativistic covariance reduces the symmetry in a well defined way from SO(4) to SO(3).

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