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On the algebraic Riccati equation

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In this note the matrix equation $A + WB + B^TW + WCW = 0$ is considered. A monotoneity result and an inertia theorem on the location of the eigenvalues of W and $B + CW$ are proved.

1. A monotoneity result

We study the algebraic Riccati equation

$$A + WB + B^TW + WCW = 0 , \tag{1}$$

where all matrices are $n \times n$ and real, and A, C , and W are symmetric. We assume

$$C \leq 0 \quad \text{(negative semidefinite)} \tag{2}$$

and

$$(B, C) \quad \text{controllable.} \tag{3}$$

[See the next section for the definition of controllability.] Coppel [4] has given a comprehensive algebraic theory of (1) on which we base our note.

We recall the following results from [4]. If (1) is solvable, then there exists a maximal solution W_+ ; that is, $W_+ \geq \tilde{W}$ for each solution $\tilde{W}$ of (1). If M ,

$$M = \begin{pmatrix} B & C \\ -A & -B^T \end{pmatrix} ,$$

has no eigenvalues on the imaginary axis, then there exists a solution of

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(1) and $B + CW$ is a stable matrix (that is with all eigenvalues in the left half plane), if and only if $W = W_+$. Set

$$H = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} .$$

Then (1) can be written as

$$(I \ W)H\binom{I}{W} = 0 .$$

THEOREM 1. *Let* M *have no eigenvalues on the imaginary axis and let* W_+ *be the maximal solution of* (1). *If* W_1 *is a solution of*

$$A_1 + W_1B_1 + B_1^TW_1 + W_1C_1W_1 = 0 , \tag{4}$$

and

$$H = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \geq H_1 = \begin{pmatrix} A_1 & B_1^T \\ B_1 & C_1 \end{pmatrix} ,$$

then $W_+ \geq W_1$.

Proof. H_1 can be written as $H_1 = H - \tilde{H}$, $\tilde{H} \geq 0$. Thus (4) is equivalent to

$$A + W_1B + B^TW_1 + W_1CW_1 = (I \ W_1)\tilde{H}\begin{pmatrix} I \\ W_1 \end{pmatrix} = R \geq 0 .$$

For $D = W_+ - W_1$ we obtain

$$D(B+CW_+) + (B+CW_+)^TD = DCD - R$$

which is a Ljapunov matrix equation with $B + CW_+$ stable and $DCD - R \leq 0$. Hence $D \geq 0$. This theorem generalizes a result in [7] where monotoneity with respect to A was proved.

2. An inertia theorem

We shall need the following lemmas and definitions. The pair (F, G) , $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{n\times m}$, is called controllable (Hautus [5]), if

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(5) $$\operatorname{rank}\left(G,\ FG,\ \ldots,\ F^{n-1}G\right) = n\ .$$

The pair $\binom{F}{K}$, $K \in \mathbb{C}^{p\times n}$, is called observable [2], if $\left(F^T,\ K^T\right)$ is controllable.

LEMMA 1 (Hautus [5]). *The pair* $\binom{F}{K}$ *is observable, if and only if for* $\lambda \in \mathbb{C}$, $y \in \mathbb{C}^n$,

$$Fy = \lambda y\ ,\quad Ky = 0 \Rightarrow y = 0\ .$$

For a complex $n \times n$ matrix F the inertia, in F, of F is defined [6] as the triple

$$\operatorname{in} F = \{\pi(F),\ \nu(F),\ \delta(F)\}$$

where $\pi(F)$, $\nu(F)$, and $\delta(F)$ are respectively the number of eigenvalues of F with positive, negative, and vanishing real part.

LEMMA 2 ([3], [8]). *Let* F, W, *and* S *be real* $n \times n$ *matrices,* S *and* W *symmetric. If* $S \le 0$ *and* $\binom{F}{S}$ *is observable and*

$$WF + F^T W = S\ ,$$

then

(a) $\operatorname{in} F = \operatorname{in}(-W)$ *and* $\delta(F) = \delta(W) = 0$,

and especially

(b) F *is stable, if and only if* $W > 0$.

The following inertia result is obtained under additional assumptions on the coefficient matrices of (1).

THEOREM 2. *Assume* $C \le 0$, $(B,\ C)$ *controllable,*

(6) $$A \ge 0$$

and

(7) $$\binom{B}{A}\ \text{observable.}$$

Then

(a) there exists a solution of (1), *and*

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(b) for each solution W *,*

$$\mathrm{in}(B+CW) = \mathrm{in}(-W) \quad and \quad \delta(B+CW) = \delta(W) = 0$$

holds.

Proof. *(a)* We show that $\delta(M) = 0$, which implies the existence of a solution. Suppose $i\alpha$, α real, is an eigenvalue of M with eigenvector $\binom{r}{s}$; that is,

$$M\binom{r}{s} = i\alpha\binom{r}{s}, \quad \binom{r}{s} \neq \binom{0}{0}. \tag{8}$$

Premultiplying both sides of (8) by (s^*, r^*), we get

$$s^*Cs - r^*Ar + \left(s^*Br-r^*B^Ts\right) = i\alpha(s^*r+r^*s),$$

and, separating real and imaginary parts, we obtain

$$s^*Cs - r^*Ar = 0.$$

(2) and (6) imply $Cs = 0$ and $Ar = 0$. Now (8) yields $Br = i\alpha r$ and $B^Ts = -i\alpha s$. Thus $\binom{B-i\alpha I}{A}r = 0$, and since $\binom{B}{A}$ is assumed to be observable, we deduce from Lemma 1 that $r = 0$. Similarly $s = 0$. Therefore $i\alpha$ can not be an eigenvalue of M. A different proof that there exists a solution W relies on optimal control theory and can be found in [2].

(b) Let W be a solution of (1). Then

$$W(B+CW) + (B+CW)^TW = WCW - A.$$

From $(WCW-A)q = 0$ we get $CWq = 0$ and $Aq = 0$. Therefore $(B+CW)q = \lambda q$ and $(WCW-A)q = 0$ implies

$$Bq = \lambda q, \quad Aq = 0. \tag{9}$$

Because of (7) and (9) we have $q = 0$ and the pair $\binom{B+CW}{WCW-A}$ is also observable. The conditions of Lemma 2 are satisfied and the statement of the inertia follows.

One of the problems which lead to (1) with coefficients satisfying (3), (4), (6), and (7) is the output regulator problem over an infinite time interval (see [1]). This close connection between optimal control

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theory and the algebraic Riccati equation has been used to establish the following result (see [2]), for which an algebraic proof is now immediate.

THEOREM 3 ([2]). *Let* (1) *be given together with* (3), (4), (6), *and* (7). *Then* W_+ *is the only positive definite solution of* (1).

Proof. Since $B + CW_+$ is stable, $W_+ > 0$ follows from Theorem 2 *(b)*. Conversely, if $W > 0$, then by the same theorem $A + CW$ is stable, which is only possible for $W = W_+$.

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