ON THE ALMOST-COMPLEX STRUCTURE OF TANGENT BUNDLES OF RIEMANNIAN SPACES

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Recently we can see several papers concerning almost-Kählerian spaces, but it seems for the authors that there does not exist a non-Kählerian global example of such a space. In this paper we shall show that the tangent bundle space $T(M^n)$ of any non-flat Riemannian space M^n always admits an almost-Kählerian structure which is not Kählerian. This is done by making use of the almost-complex structure of $T(M^n)$ owing to T. Nagano [1][1)] and of the Riemannian metric of $T(M^n)$ owing to S.Sasaki [2]. By virtue of this structure we shall also see that an infinitesimal affine transformation has an almost-analytic property in a certain sense.

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1. Almost-Kählerian spaces. Let us consider a $2n$-dimensional differentiable manifold admitting a tensor field $\varphi_\kappa{}^\lambda$ such that $\varphi_\lambda{}^\alpha\varphi_\alpha{}^\kappa = -\delta_\lambda^\kappa$ [2)]. Such a manifold is called an almost-complex space and it is said that the tensor field assigns to the manifold an almost-complex structure. An almost-complex structure is called to be integrable if the tensor field defined by

$$N_{\mu\lambda}^{\kappa} = \varphi_\mu{}^\alpha(\partial_\alpha\varphi_\lambda{}^\kappa - \partial_\lambda\varphi_\alpha{}^\kappa) - \varphi_\lambda{}^\alpha(\partial_\alpha\varphi_\mu{}^\kappa - \partial_\mu\varphi_\alpha{}^\kappa)$$

vanishes identically.

An infinitesimal transformation V^κ of an almost-complex space is called to be almost-analytic [3] if it satisfies $\underset{V}{\pounds}\,\varphi_\lambda{}^\kappa = 0$, where $\underset{V}{\pounds}$ means the operator of Lie derivation.

An almost-complex space always admits a Riemannian metric $G_{\mu\lambda}$ such that

$$G_{\beta\alpha}\varphi_\mu{}^\beta\varphi_\lambda{}^\alpha = G_{\mu\lambda} \tag{1. 1}$$

which is equivalent to the fact that $\varphi_{\mu\lambda}$ defined by $\varphi_{\mu\lambda} = \varphi_\mu{}^\alpha G_{\lambda\alpha}$ is skew-symmetric or that $G_{\mu\lambda}$ is hybrid [3].

An almost-complex space with such a Riemannian metric is called an almost-Hermitian space and the differential form $\varphi = (1/2)\varphi_{\mu\lambda}dx^\mu \wedge dx^\lambda$ is called the fundamental form. If the form is closed, the almost-Hermitian

1) The number in brackets refers to Bibliography at the end of the paper.

2) $\lambda, \mu, \nu, \alpha, \cdots\cdots = 1, 2, \cdots\cdots, 2n.$

space is called an almost-Kählerian space. An almost-Hermitian space satisfying $\nabla_\nu\varphi_{\mu\lambda} = 0$ is nothing but Kählerian, where ∇_ν means the operator of Riemannian covariant derivation.

It is known that the almost-complex structure of an almost-Kählerian space is integrable if and only if the space is Kählerian.

2. Tangent bundles. Let M^n be an n-dimensional differentiable manifold and $T(M^n)$ be its tangent bundle space. $T(M^n)$ is a $2n$-dimensional differentiable manifold with the natural structure.

Let x^i [3] be local coordinates of a point P of M^n, then a tangent vector y at P, which is an element of $T(M^n)$, is expressible in the form (x^i, y^i), where y^i are components of y with respect to the natural frame $\partial_i = \partial/\partial x^i$. We may consider (x^i, y^i) local coordinates of $T(M^n)$. To a transformation of local coordinates of M^n

$$x^{i'} = x^{i'}(x^1, \cdots\cdots, x^n)$$

there corresponds in $T(M^n)$ the coordinate transformation

$$(2.\ 1) \qquad x^{i'} = x^{i'}(x^1, \cdots\cdots, x^n), \qquad y^{i'} = y^r\ \partial_r x^{i'}.$$

If we put

$$x^{i^*} = y^i, \qquad x^{i'^*} = y^{i'},^{4)}$$

then we may write (2. 1) as

$$(2.\ 2) \qquad x^{\kappa'} = x^{\kappa'}(x^1, \cdots\cdots, x^{2n}).$$

The Jacobian matrix of (2. 1) or (2. 2) is given by

$$\begin{pmatrix} \partial_j x^{i'} & y^r \partial_{rj} x^{i'} \\ 0 & \partial_j x^{i'} \end{pmatrix}, \qquad \partial_{rj} = \partial^2/\partial x^r \partial x^j.$$

For an infinitesimal transformation or a contravariant vector field v^i on M^n, if we define V^λ by

$$(2.\ 3) \qquad V^i = v^i, \qquad V^{i^*} = y^r \partial_r v^i = x^{r^*} \partial_r v^i\ {}^{5)},$$

then it is a contravariant vector field on $T(M^n)$ or it defines an infinitesimal transformation of $T(M^n)$. V^λ is called the extension of v^i.

3. The almost-complex structure of $T(M^n)$. In the following we shall mean by M^n an n-dimensional Riemannian space whose metric tensor is g_{ji}. Following T. Nagano [1] we shall introduce in $T(M^n)$ an almost-complex structure as follows.

Put

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3) $i, j, k, \cdots = 1, 2, \cdots n$.
4) $i^* = n + i$, $i'^* = n + i'$.
5) Of course we adopt the summation convention on r.

$$\Gamma_i^h = \left\{ {h \atop i\ r} \right\} y^r, \tag{3. 1}$$

where $\left\{ {h \atop i\ r} \right\}$ denotes the Christoffel's symbol formed by the Riemannian metric g_{ji}.

If we define φ_λ^κ with respect to each local coordinates (x^i, y^i) of $T(M^n)$ by

$$\begin{aligned} \varphi_i^h &= \Gamma_i^h, & \varphi_i^{h*} &= -\delta_i^h - \Gamma_i^r\Gamma_r^h, \\ \varphi_{i*}^h &= \delta_i^h, & \varphi_{i*}^{h*} &= -\Gamma_i^h, \end{aligned} \tag{3. 2}$$

then we can see that $\varphi_\lambda^\alpha\varphi_\alpha^\kappa = -\delta_\lambda^\kappa$ holds good. On the other hand we can also show that φ_λ^κ defines a tensor field on $T(M^n)$.

Hence the tangent bundle of any Riemannian space is an almost-complex space.

REMARK. The tangent vector space of $T(M^n)$ is spanned by n vertical vectors $e_{i*} = \partial_{i*} = \partial/\partial y^i$ and n horizontal vectors $e_i = \partial_i - \Gamma_i^r\partial_{r*}$. φ_λ^κ defines a transformation on each tangent vector space of $T(M^n)$ and by the transformation a tangent vector X with components (X^i, X^{i*}) with respect to the frame (e_i, e_{i*}) is transformed into a vector with components $(X^{i*}, -X^i)$.

Next we consider under what condition the almost-complex structure of $T(M^n)$ is integerable.

Let $R_{kji}{}^h$ be the Riemannian curvature tensor of M^n, i. e.

$$R_{kji}{}^h = \partial_k \left\{ {h \atop j\ i} \right\} - \partial_j \left\{ {h \atop k\ i} \right\} + \left\{ {h \atop k r} \right\}\left\{ {r \atop j\ i} \right\} - \left\{ {h \atop j\ r} \right\}\left\{ {r \atop k\ i} \right\}$$

and put

$$R_{ji}{}^h = R_{jir}{}^h y^r.$$

After some complicated calculations we have the following equations

$$\begin{aligned} N_{ji}{}^h &= \Gamma_j^r R_{ri}{}^h - \Gamma_i^r R_{rj}{}^h, \\ N_{ji}{}^{h*} &= R_{ji}{}^h - \Gamma_j^s\Gamma_i^r R_{sr}{}^h + \Gamma_s^h(-\Gamma_j^r R_{ri}{}^s + \Gamma_i^r R_{rj}{}^s), \\ N_{ji*}{}^h &= R_{ji}{}^h, \\ N_{j*i*}{}^h &= 0, \\ N_{ji*}{}^{h*} &= -\Gamma_j^r R_{ri}{}^h - \Gamma_r^h R_{ji}{}^r, \\ N_{j*i*}{}^{h*} &= -R_{ji}{}^h. \end{aligned}$$

From these equations we have

THEOREM 1.[6] *In order that the almost-complex structure of* $T(M^n)$ *is*

6) The analogous theorem has been obtained by T. Nagano [1], but his almost-complex structure of $T(M^n)$ is slightly different from ours.

integrable, it is necessary and sufficient that the Riemannian space M^n is flat. (C. J. Hsu [4])

Now let V^λ be the extension of an infinitesimal transformation v^i. If we denote by $\underset{V}{\pounds}$ the operator of Lie derivation with respect to V^λ, then we have

$$\underset{V}{\pounds}\, \varphi_\lambda{}^\kappa = V^\alpha \partial_\alpha \varphi_\lambda{}^\kappa - \varphi_\lambda{}^\alpha \partial_\alpha V^\kappa + \varphi_\alpha{}^\kappa \partial_\lambda V^\alpha.$$

On taking account of (2. 3) and (3. 2) we have after some calculations

$$\underset{V}{\pounds}\, \varphi_j{}^h = -\underset{V}{\pounds}\, \varphi_{j^*}{}^{h^*} = y^r t_{rj}{}^h,$$

$$\underset{V}{\pounds}\, \varphi_{j^*}{}^h = 0,$$

$$\underset{V}{\pounds}\, \varphi_j{}^{h^*} = -y^r(\Gamma_j{}^s t_{rs}{}^h + \Gamma_s{}^h t_{rj}{}^s),$$

where $t_{ji}{}^h$ is given by

$$\begin{aligned} t_{ji}{}^h &= \nabla_j \nabla_i v^h + v^r R_{rji}{}^h \\ &= \partial_{ji} v^h + v^r \partial_r \left\{ {h \atop j\, i} \right\} + \left\{ {h \atop j\, r} \right\} \partial_i v^r + \left\{ {h \atop i\, r} \right\} \partial_j v^r - \left\{ {r \atop j\, i} \right\} \partial_r v^h, \end{aligned}$$

∇_j being the operator of Riemannian covariant derivation in M^n.

On the other hand we have known that an infinitesimal transformation v^i of M^n is called affine if its $t_{ji}{}^h$ vanishes.

Thus we have the following

THEOREM 2. *In order that an infinitesimal transformation of a Riemannian space M^n is affine, it is necessary and sufficient that its extension in $T(M^n)$ is almost-analytic.*

4. The almost-Hermitian structure of $T(M^n)$. Following S.Sasaki [2] we shall introduce a Riemannian metric into $T(M^n)$. This is done by defining a line element of $T(M^n)$ such as

$$d\sigma^2 = g_{ji} dx^j dx^i + g_{ji} Dy^j Dy^i, \tag{4. 1}$$

where Dy^i are n differential forms on $T(M^n)$ given by

$$Dy^i = dy^i + y^r \left\{ {i \atop r\, s} \right\} dx^s.$$

If we write (4. 1) in the form

$$d\sigma^2 = G_{\mu\lambda} dx^\mu dx^\lambda,$$

the Riemannian metric $G_{\mu\lambda}$ of $T(M^n)$ is given by

$$G_{ji} = g_{ji} + \Gamma_j{}^r \Gamma_{ir},$$

$$G_{ji^*} = \Gamma_{ji},$$

$$G_{j^*i^*} = g_{ji},$$

where $\Gamma_{ji} = \Gamma_j{}^r g_{ri}$.

Computing $\varphi_{\mu\lambda}=\varphi_\mu{}^\alpha G_{\alpha\lambda}$ we get

$$
\begin{aligned}
&\varphi_{ji} = \Gamma_{ji} - \Gamma_{ij} = y^r(\partial_j g_{ir} - \partial_i g_{jr}),\\
(4.\ 2)\qquad &\varphi_{j^*i} = -\varphi_{ij^*} = g_{ji},\\
&\varphi_{j^*i^*} = 0.
\end{aligned}
$$

From these equations we know that $\varphi_{\mu\lambda}$ is skew-symmetric. Hence $G_{\mu\lambda}$ and $\varphi_\lambda{}^\kappa$ satisfy (1.1) and $T(M^n)$ is an almost-Hermitian space by virtue of this sturcture.

Now we define a covariant vector field η_λ in $T(M^n)$ by

$$
(4.\ 3)\qquad \eta_i = g_{ir}y^r, \qquad \eta_{i^*} = 0,
$$

then the differential form $\eta = \eta_\lambda dx^\lambda$ is defined globally on $T(M^n)$.

As we obtain $\varphi = d\eta$ by virtue of (4. 2) and (4. 3), the fundamental form of $T(M^n)$ is derived. Thus we get

THEOREM 3. *The tangent bundle space of any Riemannian space admits an almost-Kählerian structure.*

The form η is called the homogeneous contact form of $T(M^n)$.[7]

Consider an infinitesimal transformation v^i on M^n and its extension V^λ. Sincc we have by definition

$$
\underset{V}{\pounds}\ \eta_\lambda = V^\alpha \partial_\alpha \eta_\lambda + \eta_\alpha \partial_\lambda V^\alpha,
$$

we get the following equations

$$
\begin{aligned}
&\underset{V}{\pounds}\ \eta_i = y^s(v^r\partial_r g_{si} + g_{sr}\partial_i v^r + g_{ir}\partial_s v^r),\\
&\underset{V}{\pounds}\ \eta_{i^*} = 0.
\end{aligned}
$$

Thus we have

THEOREM 4. *In order that an infinitesimal transformation of a Riemannian space M^n is an isometry, it is necessary and sufficient that its extension leaves invariant the homogeneous contact form of $T(M^n)$.*

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