ON THE ANALYSIS OF BRINE TRANSPORT IN POROUS MEDIA

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1. Introduction

In this paper we analyse a model for the transport of brine through a porous medium. Brine is water which is strongly impregnated with salt, so much so that the volume of the fluid is affected. Therefore the corresponding fluid flow can no longer be considered as divergence free. This, in contrast to flows with low concentrations of salt, such as sea water, where one normally assumes that the fluid is incompressible.

We shall focuss on the aspect that the flow is not divergence free and we shall investigate how this influences the transport of the salt by the moving fluid. We do this for a two-dimensional flow through a homogeneous and isotropic porous medium. The fluid is assumed to have constant dynamic viscosity μ and a variable density ρ, and the porous medium is characterised by a constant porosity $\Phi \in (0,1)$ and a constant permeability κ. We shall follow Hassanizadeh & Leijnse [HL] in their formulation of the flow of brine fluids. However, we deviate from their description in the sense that we shall use the classical form of Darcy's law for the momentum balance equation for the fluid and Fick's law for the dispersion of the salt. Moreover, in the equation of state, we shall ignore the influence of pressure variations. The equations describing the brine flow then result from the following basic laws:

mass balance of the fluid

$$\Phi \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{q}) = 0 \tag{1.1}$$

mass balance of the salt

$$\Phi \frac{\partial \rho\omega}{\partial t} + \operatorname{div}(\rho\omega\mathbf{q} - D\rho \operatorname{grad}\omega) = 0, \tag{1.2}$$

Darcy's law

$$\frac{\mu}{\kappa}\mathbf{q} + \operatorname{grad}p - \rho\mathbf{g} = 0, \tag{1.3}$$

equation of state

$$\rho = \rho_f e^{\gamma\omega}. \tag{1.4}$$

In these equations $\mathbf{q} = (q_y, q_z)$ denotes the specific discharge of the fluid, p the fluid pressure and $\mathbf{g} = (g_y, g_z)$ the acceleration of gravity. In equation (1.2) and throughout the analysis we consider

$$D = \Phi D_{\text{mol}},$$

where D_{mol} denotes the molecular diffusion, which we assume to be constant. However, in the concluding section we shall also discuss the more realistic situation where D depends on the velocity (see for instance [B] or [HL]). In the equation of state, ρ_f denotes the mass density of the fresh water, $\gamma = 0.6923$ is a constant and ω is the mass fraction of the salt in the brine, (i.e. ω = mass concentration of the salt component of the brine / mass density ρ of the brine).

We shall study two specific flow problems, which we shall refer to as Problems I and II.

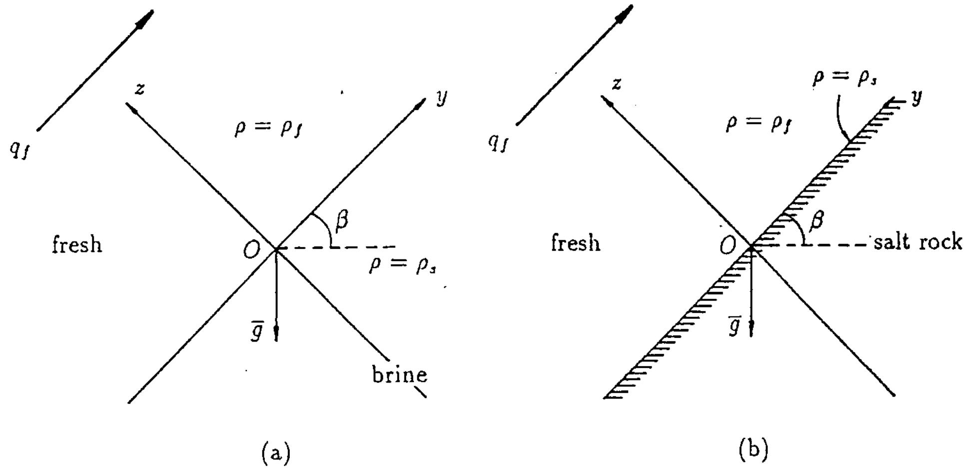

Fig. 1. Initial salt distributions

In Problem I - see Figure 1(a) - the flow domain is unbounded above and below and initially, say at $t = 0$, the region above the plane $\{z = 0\}$ is filled with fresh water and the brine, with mass density $\rho_s > \rho_f$ fills the region below it. In Problem II - see Figure 1(b) - the flow domain consists of the upper half space $\{z > 0\}$ and is bounded below by an impermeable salt rock. Again, at $t = 0$, fresh water occupies the region $\{z > 0\}$ while the salt from the rock ensures that $\rho = \rho_s$ (mass density of saturated brine) along the boundary $\{z = 0\}$.

Since in both problems the y-coordinate ranges from $-\infty$ to $+\infty$ we may look for a density ρ and specific discharge $\mathbf{q}$ of the form

$$\rho = \rho(z,t) \quad \text{and} \quad \mathbf{q} = \mathbf{q}(z,t). \tag{1.5}$$

Under this assumption we obtain a linear relation between the y-component of the specific discharge and the fluid density (see also [dJdJvD]).

To derive the governing equations we first eliminate the pressure from Darcy's law by taking the curl. This results in

$$\frac{\partial}{\partial z}\{q_y - \frac{\kappa}{\mu}\rho g_y\} - \frac{\partial}{\partial y}\{q_z - \frac{\kappa}{\mu}\rho g_z\} = 0.$$

With (1.5) this gives

$$q_y + \frac{\kappa}{\mu}\rho g \sin\beta = \text{constant}, \tag{1.6}$$

where β is the inclination of the original fresh-brine interface in Problem I and the salt boundary in Problem II, and $g = |\mathbf{g}|$. Next we impose the boundary conditions from

Figure 1 to determine the constant in (1.6). Since

$$q_y(z,t) \to q_f \quad \text{as} \quad z \to \infty \quad \text{for all} \quad t > 0, \tag{1.7}$$

in which $q_f \in \mathbf{R}$ is given, and

$$\rho(z,t) \to \rho_f \quad \text{as} \quad z \to \infty \quad \text{for all} \quad t > 0, \tag{1.8}$$

we obtain the relation

$$q_y = q_f - \frac{\kappa}{\mu}(\rho - \rho_f) g \sin\beta. \tag{1.9}$$

Using (1.5) in the equations (1.1) and (1.2) we obtain

$$\Phi\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho q_z) = 0 \tag{1.10}$$

and

$$\Phi\frac{\partial \rho\omega}{\partial t} + \frac{\partial}{\partial z}(\rho\omega q_z - D\rho\frac{\partial \omega}{\partial z}) = 0.$$

Combining these equations yields

$$\Phi\rho\frac{\partial \omega}{\partial t} + \rho q_z\frac{\partial \omega}{\partial z} = \frac{\partial}{\partial z}\Big(\rho D\frac{\partial w}{\partial z}\Big)$$

and with (1.4)

$$\Phi\frac{\partial \rho}{\partial t} + q_z\frac{\partial \rho}{\partial z} = D\frac{\partial^2 \rho}{\partial z^2}. \tag{1.11}$$

In Problem I, equations (1.10) and (1.11) have to be solved for $(z,t) \in \mathbf{R}\times\mathbf{R}^+$ subject to the initial conditions

$$\rho(z,0) = \begin{cases} \rho_f & z > 0 \\ \rho_s & z < 0. \end{cases} \tag{1.12a}$$

At one of the boundaries $z = -\infty$ or $z = +\infty$ we have to specify q_z. Here we choose

$$q_z(z,t) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for all} \quad t > 0. \tag{1.12b}$$

In Problem II we solve the differential equations in the quarter plane $\mathbf{R}^+\times\mathbf{R}^+$ subject to the initial condition

$$\rho(z,0) = \rho_f, \qquad z > 0 \tag{1.13a}$$

and the boundary condition

$$\rho(0,t) = \rho_s, \qquad t > 0. \tag{1.13b}$$

In addition we now require (see [HL])

$$q_z(0,t) = -\frac{D}{\gamma\rho_s(1-\omega_s)}\frac{\partial \rho}{\partial z}(0,t) \quad \text{for all} \quad t > 0. \tag{1.13c}$$

Here $\rho_s = \rho_f \exp(\gamma\omega_s)$, where ω_s is the salt mass fraction of saturated brine.

At this point it is important to note that the differential eqautions (1.10) and (1.11) only involve as unknowns the velocity component q_z and the density ρ. Thus solving these equations subject to the appropriate boundary conditions and using ρ in (1.9) determines a solution $(\mathbf{q}, \rho)$ of the original problem.

To simplify the equations we introduce dimensionless variables, and set

$$\begin{aligned} \rho^* &= \frac{\rho - \rho_f}{\rho_s - \rho_f} & q_z^* &= \frac{q_z}{\frac{k}{\mu}\rho_f g} \\ z^* &= \frac{\frac{k}{\mu}\rho_f g}{D} z, & t^* &= \frac{(\frac{k}{\mu}\rho_f g)^2}{\Phi D} t \end{aligned} \tag{1.14}$$

and

$$\varepsilon = \frac{\rho_s - \rho_f}{\rho_f}. \tag{1.15}$$

In these new variables, (1.10) and (1.11) become (when we omit the asterisks again)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho q_z) + \frac{1}{\varepsilon}\frac{\partial q_z}{\partial z} = 0 \quad \text{in} \quad \Omega \times \mathbf{R}^+ \tag{1.16}$$

$$\frac{\partial \rho}{\partial t} + q_z \frac{\partial \rho}{\partial z} = \frac{\partial^2 \rho}{\partial z^2} \qquad \text{in} \quad \Omega \times \mathbf{R}^+, \tag{1.17}$$

where Ω is either $\mathbf{R}$ or $\mathbf{R}^+$.

In Problem I, the rescaled initial and boundary conditions become

$$\rho(z,0) = \begin{cases} 0 & \text{if} \quad z > 0 \\ 1 & \text{if} \quad z < 0 \end{cases} \tag{1.18a}$$

and

$$q_z(z,t) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for all} \quad t > 0. \tag{1.18b}$$

In Problem II we have

$$\rho(z,0) = 0 \quad \text{for} \quad z > 0 \tag{1.19a}$$

$$\rho(0,t) = 1 \quad \text{for} \quad t > 0 \tag{1.19b}$$

and

$$q_z(0,t) = -K\varepsilon \frac{\partial \rho}{\partial z}(0,t) \quad \text{for} \quad t > 0 \tag{1.19c}$$

in which

$$K = \frac{\rho_f}{\rho_s} \frac{1}{\gamma(1-\omega_s)}.$$

Through ρ_s and ω_s, K is a function of ε; it is given by

$$K(\varepsilon) = \frac{1}{(1+\varepsilon)\{\gamma - \log(1+\varepsilon)\}}, \qquad 0 < \varepsilon < e^{\gamma} - 1. \tag{1.19d}$$

It is readily verified that K is strictly increasing, and that $K(\varepsilon) > 1/\gamma$ if $\varepsilon > 0$.

Remark. We note that in the formal limit as $\varepsilon \to 0$, equation (1.16) reduces to $\partial q_z/\partial z = 0$. Together with condition (1.18b), or with (1.19c), this implies that in this limit $q_z = 0$. For ρ then remains the linear heat equation. In the literature this is known as the Boussinesq approximation, in which the flow satisfies $\text{div}\mathbf{q} = 0$. For Problem I the Boussinesq approximation was studied in [dJdJvD], in particular when the dispersion depends on the velocity.

We solve Problems I and II by looking for self-similar solutions. We set

$$x = z/\sqrt{t} \tag{1.20}$$

and

$$\rho(z,t) = u(x) \quad \text{and} \quad q_z(z,t) = \frac{1}{\sqrt{t}}v(x). \tag{1.21}$$

This results in the differential equations

$$(uv)' + \frac{1}{\varepsilon}v' - \frac{1}{2}xu' = 0 \qquad \text{in} \quad \Omega \tag{1.22}$$

$$u'v - \frac{1}{2}xu' = u'' \qquad \text{in} \quad \Omega. \tag{1.23}$$

In Problem I the boundary conditions are

$$u(-\infty) = 1 \quad \text{and} \quad u(+\infty) = 0 \tag{1.24a}$$

and

$$v(\infty) = 0, \tag{1.24b}$$

and in Problem II

$$u(0) = 1 \quad \text{and} \quad u(+\infty) = 0 \tag{1.25a}$$

and

$$v(0) = -K\varepsilon u'(0). \tag{1.25b}$$

Throughout we shall restrict the analysis to solutions (u,v) such that $u > 0$ in Ω.

We analyze Problems I and II using ideas from a previous study of a fresh - salt ground water problem [vDP3], which also involved a third order differential equation.

In Sections 2 and 3 we study the self similar solutions related to Problem I. In Section 2 we prove that for every $\varepsilon > 0$, there exists a unique pair of smooth functions (u,v) which solves (1.22) - (1.24) (see Figures 2 and 3). The function u is strictly decreasing on $\mathbf{R}$ and has precisely one inflection point at some $x_0 > 0$, such that $u'' < 0$ on $(-\infty, x_0)$ and $u'' > 0$ on (x_0, ∞). The function v is positive on $\mathbf{R}$, strictly increasing on $(-\infty, x_0)$ and strictly decreasing on (x_0, ∞). Moreover v satisfies

$$v(-\infty) = \varepsilon^2 \int_0^1 \frac{y(u)}{(1+\varepsilon u)^2}\,du. \tag{1.26}$$

where y is the solution of the boundary value problem (2.8), (2.9).

In Section 3 we investigate the Boussinesq limit as $\varepsilon \to 0$. Denoting for $\varepsilon > 0$ the solutions of (1.22)-(1.24) by $(u_\varepsilon, v_\varepsilon)$ we show that

$$v_\varepsilon(x) = -u_0'(x)\varepsilon + \{E_1(x) + o(1)\}\varepsilon^2 \tag{1.27a}$$

and

$$u_\varepsilon(x) = u_0(x) + \frac{1}{2}u_0(x)\{1 - u_0(x)\}\varepsilon + \{E_2(x) + o(1)\}\varepsilon^2 \tag{1.27b}$$

as $\varepsilon \to 0$, uniformly on $\mathbf{R}$.

Here u_0 is the Boussinesq limit

$$u_0(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{2}\right), \qquad x \in \mathbf{R}, \tag{1.28}$$

which can be obtained formally by setting $v = 0$ in (1.23). The functions E_1 and E_2 can be explictly calculated. This is outlined in Lemmas 3.6 and 3.7.

In Section 4 we turn to Problem II where we discuss for $0 < \varepsilon < e^\gamma - 1$ the existence and uniqueness of solutions (u, v) of (1.22), (1.23) and (1.25) (see also Figures 5 and 6). The upper bound for ε is a consequence of formula (1.19d) and is for all practical purposes no restriction. Brines with high salt concentrations have ε in the order of 0.2 (see [HL]). For salt water in coastal aquifers one often uses $\varepsilon = 0.025$ (see [B]).

Here we show that u_0 is strictly decreasing on $\mathbf{R}^+$ and has precisely one inflection point at some $x_0 > 0$: $u'' < 0$ on $(0, x_0)$ and $u'' > 0$ or (x_0, ∞). The function v satisfies $v' > 0$ on $(0, x_0)$ and $v' < 0$ on (x_0, ∞). Notice that

$$v(0) > 0 \quad \text{and} \quad v(\infty) = O(\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{1.29a}$$

whereas we deduce from (1.26) that in Problem I

$$v(-\infty) = O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0. \tag{1.29b}$$

In Section 5 we investigate again the limit $\varepsilon \to 0$. From the results obtained there we find

$$v_\varepsilon(x) = -u_0'(x)\varepsilon + O(\varepsilon^2) \tag{1.30a}$$

and

$$\begin{aligned} u_\varepsilon(x) =& u_0(x) + \Big[-\frac{2(1-\gamma)}{\pi\gamma}\{u_0(x) + \sqrt{\pi}u_0'(x)\} \\ &+ \frac{1}{2}u_0(x)\{1 - u_0(x)\}\Big]\varepsilon + O(\varepsilon^2), \end{aligned} \tag{1.30b}$$

as $\varepsilon \to 0$, uniformly on $\mathbf{R}$. Here u_0 is the Boussinesq limit

$$u_0(x) = \operatorname{erfc}\left(\frac{x}{2}\right), \qquad x \in \mathbf{R}^+. \tag{1.31}$$

In Problem I we have $v(-\infty) > 0$ for any $\varepsilon > 0$ and in Problem II $v(+\infty) > 0$ for ε sufficiently small. This implies for the velocity $\overline{q}(z,t)$ that

$$\lim_{t\downarrow 0} q_z(z,t) = \infty \quad \text{for each } z < 0 \text{ in Problem I}$$

and

$$\lim_{t\downarrow 0} q_z(z,t) = \infty \quad \text{for each } z > 0 \text{ in Problem II.}$$

These unbounded velocities at $t = 0^+$ are caused by the discontinuity in the initial density profile at $z = 0$ which gives rise to an unbounded density gradient. This problem will not occur when the profile is smooth: for instance, if we apply for any $\delta > 0$ the time shift $t \mapsto t + \delta$ and consider as initial distribution

$$\rho(z,0) = \rho_f + (\rho_s - \rho_f)u(z/\sqrt{\delta}),$$

then the velocity will be bounded down to $t = 0$.

In Section 6 we present some numerical results. We computed solutions (u,v) of Problems I and II for the cases $\varepsilon = 0.025$ and $\varepsilon = 0.5$. The case $\varepsilon = 0.025$ corresponds to a relatively small density difference as encountered in coastal aquifers, where fresh and salt water are present. The second value $\varepsilon = 0.5$ is unrealistically high, but it is chosen here to show up in the effect of volume changes caused by high salt concentrations on the distribution of brine.

Finally, in the concluding Section 7, we make some observations about brine transport when the dispersion depends on velocity. We present a von Mises-type transformation which reduces equations (1.10) and (1.11) to a single nonlinear diffusion equation involving ρ only. This allows us to study more general initial value problems and leads to the conjecture that the self-similar solutions considered in this paper are the large time asymptotic profiles of solutions emanating from more general initial distributions. We leave the proof of this conjecture to a future paper.

2. Self-similar solutions on R

In this section we shall study the self-similar solutions (u,v) of equations (1.22) and (1.23) which satisfy the required conditions at $x = \pm\infty$. Here we put the equations into the form

$$\begin{cases} u'' + (\frac{1}{2}x - v)u' = 0 & \qquad (2.1) \\ & -\infty < x < \infty \\ v' = -\dfrac{u''}{u + \frac{1}{\varepsilon}}, \quad u > 0 & \qquad (2.2) \end{cases}$$

We recall that the boundary conditions are

$$u(-\infty) = 1 \quad \text{and} \quad u(+\infty) = 0 \tag{2.3}$$

and

$$v(+\infty) = 0. \tag{2.4}$$

As a preliminary observation we note that u and v' are invariant under shifts. Specifically, set

$$\overline{x} = x - a, \quad \overline{u}(\overline{x}) = u(x), \quad \overline{v}(\overline{x}) = v(x) - \frac{1}{2}a, \tag{2.5}$$

where a is some arbitrary number. Hence, if $(u(x), v(x))$ is a solution of (2.1) - (2.3), then so is $(\overline{u}(\overline{x}), \overline{v}(\overline{x}))$ but with x in (2.1) replaced by $\overline{x}$. Thus, if $(u(x), v(x))$ is a solution and $v(\pm\infty)$ exists, then by shifting x over a suitable distance a, we can reach any prescribed limiting value of $v(x)$, either at $x = -\infty$ or at $x = +\infty$. Thus it follows from (2.5) that by choosing $a = 2v(\infty)$, we can ensure that the boundary condition (2.4) is satisfied.

We begin by establishing a few qualitative properties of solutions of (2.1) - (2.4).

Lemma 2.1. *Let (u, v) be a solution of (2.1) - (2.4). Then*

(a) $$u'(x) < 0 \quad \textit{for all} \quad x \in \mathbf{R}.$$

(b) *There exists a number $x_0 > 0$ such that*

$$u''(x) > 0 \ (<) \quad \textit{if} \quad x > x_0 \ (<)$$
$$v'(x) < 0 \ (>) \quad \textit{if} \quad x > x_0 \ (<).$$

(c) $$u'(x) \to 0 \quad \text{as} \quad x \to \pm\infty.$$

Proof. (a) Suppose to the contrary that there exists a point $\xi \in \mathbf{R}$ such that $u'(\xi) = 0$. Then, by standard uniqueness theory applied to equation (2.1), $u'(x) = 0$ for all $x \in \mathbf{R}$ so that the conditions at $x = \pm\infty$ cannot both be satisfied.

(b) In view of the boundary conditions, u'' must have at least one zero. Let x_0 be such a zero. Then by the equations (2.1) and (2.2),

$$v(x_0) = \frac{1}{2}x_0 \quad \text{and} \quad v'(x_0) = 0. \tag{2.6}$$

Differentiating (2.1) we obain

$$u'''(x_0) = -\{\frac{1}{2} - v'(x_0)\}u'(x_0) - \{\frac{1}{2} - v(x_0)\}u''(x_0)$$

and so

$$u'''(x_0) = -\frac{1}{2}\,u'(x_0) > 0.$$

Thus, at every zero, u'' passes from negative to positive, which means that there can be only one zero. Since $v(\infty) = 0$ and $v'(x) < 0$ for $x > x_0$, we conclude that $v(x_0) > 0$ which implies by (2.1) that $x_0 > 0$.

(c) Because $u'(x) < 0$ and $u''(x) > 0$ for x large enough, $\ell = \lim_{x\to\infty} u'(x)$ exists. Since $u(x) \to 0$, ℓ can only be zero. That $u'(x) \to 0$ as $x \to -\infty$ is proved the same way.

Remark. It is interesting to note that due to property (2.6), it is possible to shift the pair (u, v) by such an amount that we obtain

$$\overline{v}(0) = 0 \quad \text{as well as} \quad \overline{v}(\overline{x}) < 0 \quad \text{if} \quad \overline{x} \neq 0.$$

For the velocity field $\mathbf{q}$ this means that $q_z(0, t) = 0$ for all $t > 0$ so that there is no flux across the plane $\{z = 0\}$.

We now turn to proving existence and uniqueness of a solution (u, v) of (2.1) - (2.4). We proceed in two steps. First we replace the condition (2.4) on v at infinity by a third condition on u at the origin:

$$u(0) = \frac{1}{2} \tag{2.7}$$

and we establish existence and uniqueness for this modified problem (2.1) - (2.3), (2.7). We then show that the limit $v(\infty)$ exists. By means of the transformation (2.5) we finally obtain the desired solution of the original problem. Because the shift $a = 2v(\infty)$ is uniquely determined by the (unique) solution if the modified problem, the resulting solution is unique.

We transform the system (2.1), (2.2) to a single well known second order differential equation by introducing u as an independent variable. Since any solution u is strictly decreasing by Lemma 2.1, this is possible. Thus, we define

$$x = x(u) \quad \text{and} \quad y(u) = -u'(x(u)). \tag{2.8}$$

Then we find after an elementary computation that $y(u)$ is a solution of the equation

$$y\{(1 + \varepsilon u)y'\}' = -\frac{1}{2}(1 + \varepsilon u), \quad 0 < u < 1, \tag{2.9}$$

whilst the boundary conditions (2.3) become by Lemma 2.1

$$y(0) = 0 \quad \text{and} \quad y(1) = 0. \tag{2.10}$$

To bring (2.9) really into standard form we carry out one more transformation, setting

$$t = \frac{\log(1 + \varepsilon u)}{\log(1 + \varepsilon)} \quad \text{and} \quad z(t) = \frac{\varepsilon}{\log(1 + \varepsilon)}\, y(u). \tag{2.11}$$

This yields the problem

$$(\mathrm{P}_1) \begin{cases} -zz'' = \dfrac{1}{2} e^{2t\log(1+\varepsilon)}, & z > 0 \quad \text{for} \quad 0 < t < 1 \\ z(0) = 0, \quad z(1) = 0. \end{cases}$$

Problem (P_1) has been studied in detail in [AP, vDGZ, vDP1,T]. It has a unique solution $z \in C^\infty(0,1) \cap C([0,1])$ which is concave and satisfies $z'(0^+) = \infty$ and $z'(1^-) = -\infty$. Plainly, this implies the existence, uniqueness and positivity of the solution $y(u)$ of (2.9), (2.10).

We return to the original variables x, u and v by means of

$$x(u) = \int_u^{\frac{1}{2}} \frac{ds}{y(s)}, \tag{2.12}$$

which follows from (2.8), and

$$v(x) = \frac{1}{2}x - y'(u(x)), \tag{2.13}$$

which follows from (2.1) and (2.8). The boundary conditions (2.3) follow from the fact that $1/y$ is not integrable near $u = 0$ and $u = 1$.

Lemma 2.2. *Let (u, v) be a solution of (2.1) - (2.3), (2.7). Then*

(a) $$v(\pm\infty) = \lim_{x\to\pm\infty} v(x) \textit{ exists}$$

and $v(-\infty)$ as well as $v(+\infty)$ are finite.

(b) $$v(-\infty) > v(+\infty),$$

(c) $$v(x) > v(+\infty) \quad \textit{for all} \quad x \in \mathbf{R}.$$

Proof. (a) That the limits exist follows at once from the monotonicity of v for large positive and large negative values of x. Thus, it remains to show that both limits are finite.

If we integrate (2.2) over $(0, x)$ we obtain

$$\begin{aligned} v(x) &= v(0) - \varepsilon \int_0^x \frac{u''(s)}{1+\varepsilon u(s)}\, ds \\ &= v(0) - \varepsilon \frac{u'(s)}{1+\varepsilon u(s)}\Big|_0^x - \varepsilon^2 \int_0^x \frac{u'^2(s)}{\{1+\varepsilon u(s)\}^2}\, ds \\ &= v(0) + \varepsilon \frac{y(u(x))}{1+\varepsilon u(x)} - \varepsilon \frac{y(\frac{1}{2})}{1+\frac{\varepsilon}{2}} - \varepsilon^2 \int_{u(x)}^{\frac{1}{2}} \frac{y(u)}{(1+\varepsilon u)^2}\, du. \end{aligned} \tag{2.14}$$

Thus, letting $x \to \infty$ we obtain

$$v(\infty) = v(0) - \varepsilon \frac{y(\frac{1}{2})}{1+\frac{\varepsilon}{2}} + \varepsilon^2 \int_0^{\frac{1}{2}} \frac{y(u)}{(1+\varepsilon u)^2}\, du. \tag{2.15}$$

That $v(-\infty)$ is finite can be proved in a similar way.

(b) and (c) We integrate (2.2) over (x, ∞). This yields, after some manipulations similar to those carried out to derive (2.14),

$$v(x) = v(\infty) + \varepsilon \frac{y(u(x))}{1+\varepsilon u(x)} + \varepsilon^2 \int_0^{u(x)} \frac{y(u)}{(1+\varepsilon u)^2}\, du, \tag{2.16}$$

from which the assertions follow at once.

Remark. If we let $x \to -\infty$ we obtain a relation between between the limits of v:

$$v(-\infty) - v(\infty) = \varepsilon^2 \int_0^1 \frac{y(u)}{(1+\varepsilon u)^2}\, du, \tag{2.17}$$

Having proved that $v(\infty)$ exists and is finite, we can complete the proof of the existence and uniqueness of a solution of (2.1) - (2.4) by an appropriate shift, as indicated at the beginning of this section.

Theorem 2.3. *For each $\varepsilon > 0$, there exists one and only one pair of functions $(u(x), v(x))$ which satisfies (2.1) - (2.4). The function v is related to u through the relation*

$$v(x) = -\varepsilon \frac{u'(x)}{1+\varepsilon u(x)} + \varepsilon^2 \int_x^{\infty} \frac{u'^2(s)}{\{1+\varepsilon u(s)\}^2}\, ds. \tag{2.18}$$

We conclude this section with asymptotic estimates for $u(x)$ and $v(x)$ as $x \to -\infty$ and $x \to +\infty$.

Lemma 2.4. *Let $(u(x), v(x))$ be the solution of (2.1) - (2.4). Then there exist positive constants A_+ and A_- such that*

$$\left.\begin{aligned} u(x) &< A_+ x^{-1} e^{-x^2/4} \\ v(x) &< A_+ \varepsilon x^{-1} e^{-x^2/4} \end{aligned}\right\} \quad \textit{if} \quad x > 0$$

and

$$\left.\begin{aligned} 1 - u(x) &< A_- x^{-1} e^{-(x-a)^2/4} \\ v(x) - v(-\infty) &< A_- \varepsilon x^{-1} e^{-(x-a)^2/4} \end{aligned}\right\} \quad \textit{if} \quad x < 0,$$

where $a = 2v(-\infty)$.

Proof. For convenience we write

$$V(x) = \int_0^x v(s)\, ds.$$

Then equation (2.1) can be integrated once to yield

$$u'(x) = u'(0) \exp\Big(-\frac{1}{4}x^2 + V(x)\Big) \quad \text{for} \quad x \in \mathbf{R}. \tag{2.19}$$

By Lemma 2.2 there exists a constant K such that

$$v(x) < K \quad \text{for} \quad x \in \mathbf{R}. \tag{2.20}$$

Hence $V(x) < Kx$ for $x > 0$ and

$$|u'(x)| \leq |u'(0)| e^{K^2} e^{-(x-2K)^2/4} \quad \text{for} \quad x > 0. \tag{2.21}$$

Using this bound in (2.18) we conclude that v is integrable on $(0, \infty)$ so that $V(\infty) < \infty$ and

$$|u'(x)| \leq |u'(0)| e^{V(\infty)} e^{-x^2/4} \quad \text{for} \quad x > 0. \tag{2.22}$$

The estimate for $u(x)$ now follows upon integration; the one for $v(x)$ follows when we use (2.22) in (2.18).

Next we let $x \to -\infty$. Because $v(x) \to v(-\infty) > 0$ if $\varepsilon > 0$

$$V(x) = v(-\infty)x - \int_x^0 \{v(s) - v(-\infty)\}\, ds.$$

As before we find that the integral converges as $x \to -\infty$, so that

$$|u'(x)| \leq |u'(0)| e^{a^2/4} e^{-(x-a)^2/4} \quad \text{for} \quad x < 0, \tag{2.23}$$

where $a = 2v(-\infty)$. The estimates for $x \to -\infty$ now follow as they did for $x \to +\infty$.

3. The limit $\varepsilon \to 0$

In this section we shall study the dependence of the solution of (2.1) - (2.4) on ε for small values of ε. Thus, to emphasize the dependence on ε, we shall denote the solution by $(u_\varepsilon(x), v_\varepsilon(x))$.

Lemma 3.1. *There exist positive constants C_1 and C_2, which do not depend on ε such that*

$$|u_\varepsilon'(x)| < C_1 \quad \textit{and} \quad 0 < v_\varepsilon(x) < C_2\varepsilon$$

for all $\varepsilon \in (0,1)$.

Proof. Observe that

$$e^{2t\log(1+\varepsilon)} \leq 4 \quad \text{for} \quad 0 \leq t \leq 1, \quad 0 < \varepsilon < 1.$$

Hence, by the monotone dependence of z_ε on the right hand side of its differential equation [vDP], it follows that

$$z_\varepsilon(t) < \overline{z}(t) \quad \text{for} \quad 0 \le t \le 1,$$

where $\overline{z}$ is the positive solution of Problem (P) with right hand side equal to 2. Therefore z_ε, and thereby also y_ε, are uniformly bounded for $0 < \varepsilon < 1$. This means that $|u'_\varepsilon(x)|$ is uniformly bounded on $\mathbf{R}$.

The bound for v_ε follows at once from (2.16).

Corollary 3.2. (a) *The constants A_+ and A_- in Lemma 2.4 can be chosen independent of ε.*

(b) *We have*

$$u'_\varepsilon(x) \to 0 \ \ \textit{and} \ \ u''_\varepsilon(x) \to 0 \quad \textit{as} \quad |x| \to \infty,$$

uniformly with respect to $\varepsilon \in (0,1)$.

Proof. (a) By Lemma 3.1, the constant K in (2.20) can be chosen independent of ε if $\varepsilon < 1$. This implies, putting (2.21) into (2.18), that $V(\infty)$ can also be estimated independent of ε. Thus, using Lemma 3.1 again, the constant in (2.22) does not depend on ε and so A_+ does not depend on ε.

That A_- is independent of ε follows at once from (2.23) and Lemma 3.1.

Part (b) follows immediately from (2.19) and Lemma 3.1 as well as the differential equation (2.1).

Our first estimate states that u_ε converges to the limit profile u_0 which solves (2.1), (2.3) when we set $v = 0$. Thus

$$u_0(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{2}\right) = \frac{1}{\sqrt{\pi}}\int_{x/2}^{\infty} e^{-t^2}\, dt. \tag{3.1}$$

Lemma 3.3. *We have*

$$u_\varepsilon(x) \to u_0(x), \quad u'_\varepsilon(x) \to u'_0(x), \quad u''_\varepsilon(x) \to u''_0(x) \quad \textit{as} \quad \varepsilon \to 0$$

uniformly on $\mathbf{R}$.

Proof. By Lemma 3.1, u'_ε and v_ε are uniformly bounded. Hence by equations (2.1) and (2.2), u''_ε and v'_ε are uniformly bounded on bounded sets. Finally, by differentiating (2.1) we find that u'''_ε is also bounded on bounded sets. Therefore by Arzela-Ascoli's theorem there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, and a function U such that

$$u_\varepsilon \to U, \quad u'_\varepsilon \to U', \quad u''_\varepsilon \to U'' \quad \text{as} \quad \varepsilon \to 0$$

along the subsequence $\{\varepsilon_n\}$, uniformly on bounded intervals. Taking the limit in (2.1) and remembering that $v_\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0$, we find that

$$U'' + \frac{1}{2}xU' = 0 \quad \text{for} \quad x \in (-\infty, \infty).$$

By the asymptotic estimates of Lemma 2.4, which are uniform with respect to ε by Corollary 3.2(b), we conclude that

$$U(-\infty) = 1 \quad \text{and} \quad U(+\infty) = 0.$$

Therefore $U = u_0$ and the entire family $\{u_\varepsilon\}$ converges to u_0 as $\varepsilon \to 0$ uniformly on $\mathbf{R}$. The uniform convergence of the derivatives u'_ε and u''_ε follows from Corollary 3.2(b).

Lemma 3.4. *We have*

$$\lim_{\varepsilon \to 0} \frac{v_\varepsilon(x)}{\varepsilon} = -u'_0(x) \quad \textit{uniformly on } \mathbf{R}.$$

Proof. By (2.18) we have

$$\frac{v_\varepsilon(x)}{\varepsilon} = -\frac{u'_\varepsilon(x)}{1 + \varepsilon u_\varepsilon(x)} + \varepsilon \int_0^{u_\varepsilon(x)} \frac{y_\varepsilon(u)}{(1+\varepsilon u)^2}\, du. \tag{3.2}$$

Hence, because the integral is uniformly bounded, the limit follows from Lemma 3.3.

Lemma 3.5. *We have*

$$\lim_{\varepsilon \to 0} \frac{u_\varepsilon(x) - u_0(x)}{\varepsilon} = \frac{1}{2} u_0(x)\{1 - u_0(x)\} \quad \textit{uniformly on } \mathbf{R}.$$

Proof. Set

$$w_\varepsilon = \frac{u_\varepsilon - u_0}{\varepsilon}.$$

Then, subtracting the equation for u_0 from the one for u_ε and dividing the difference by ε, we find that w_ε is a solution of the problem

$$\begin{cases} w'' + \frac{1}{2} x w' = -f_\varepsilon(x) & x \in (-\infty, \infty) \\ w(-\infty) = 0, \qquad w(\infty) = 0, \end{cases}$$

where

$$f_\varepsilon(x) = -\frac{v_\varepsilon(x)}{\varepsilon}\, u'_\varepsilon(x). \tag{3.3}$$

Note that by Lemmas 3.3 and 3.4

$$f_0(x) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} f_\varepsilon(x) = \{u'_0(x)\}^2 \quad \text{uniformly on } \mathbf{R}.$$

The solution w_ε can be written explicitly in terms of f_ε by means of

$$w_\varepsilon(x) = \int_x^\infty e^{-t^2/4}\, dt \int_0^t e^{s^2/4} f_\varepsilon(s)\, ds - A_\varepsilon \int_x^\infty e^{-t^2/4}\, dt \tag{3.5a}$$

where

$$A_\varepsilon = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2/4}\, dt \int_0^t e^{s^2/4} f_\varepsilon(s)\, ds. \tag{3.5b}$$

By Lemma 2.4 and Corollary 3.2 there exists a constant C, which does not depend on ε such that

$$f_\varepsilon(x) \le Ce^{-x^2/4} \qquad \text{for} \quad -\infty < x < \infty.$$

Hence, we may let $\varepsilon \to 0$ in (3.5). Using the symmetry of $f_0(x)$ with respect to $x = 0$, we conclude that $\lim_{\varepsilon\to 0} A_\varepsilon = 0$ and that

$$\lim_{\varepsilon \to 0} w_\varepsilon(x) = w_0(x) \quad \text{uniformly on } \mathbf{R},$$

where

$$w_0(x) = \int_x^{\infty} e^{-t^2/4}\, dt \int_0^t e^{s^2/4} \{u_0'(s)\}^2\, ds.$$

Remembering that $u_0(x)$ is given explicitly in (3.1) we can write $w_0(x)$ as

$$w_0(x) = \frac{1}{8}\Big[1 - \Big\{\operatorname{erf}\Big(\frac{x}{2}\Big)\Big\}^2\Big]. \tag{3.6}$$

Again, using the expression for u_0 we arrive at the desired result.

Next we estimate the second order terms in the expansions for u_ε and v_ε with respect to ε. For v_ε we find from (3.2) for $\varepsilon > 0$ and $x \in \mathbf{R}$

$$\begin{aligned} \frac{1}{2}\Big\{\frac{v_\varepsilon(x)}{\varepsilon} + u_0'(x)\Big\} &= \frac{1}{2}\Big\{u_0'(x) - \frac{u_\varepsilon'(x)}{1+\varepsilon u_\varepsilon(x)}\Big\} + \int_0^{u_\varepsilon(x)} \frac{y_\varepsilon(u)}{(1+\varepsilon u)^2}\, du \\ &= -w_\varepsilon'(x) + \frac{u_\varepsilon(x)u_\varepsilon'(x)}{1+\varepsilon u_\varepsilon(x)} + \int_0^{u_\varepsilon(x)} \frac{y_\varepsilon(u)}{(1+\varepsilon u)^2}\, du \end{aligned} \tag{3.7}$$

We inspect the convergence of the three terms on the right hand side of (3.7) in succession.

By differentiating (3.5a) we obtain

$$w_\varepsilon(x) = -e^{-x^2/4} \int_0^x e^{s^2/4} f_\varepsilon(s)\, ds - A_\varepsilon e^{-x^2/4}.$$

Using the arguments from the proof of Lemma 3.5 we conclude that

$$\lim_{\varepsilon \to 0} w_\varepsilon'(x) = -e^{-x^2/4} \int_0^x e^{s^2/4} f_0(s)\, ds = w_0'(x), \tag{3.8}$$

uniformly in $\mathbf{R}$.

From Lemma 3.3 we deduce that

$$\lim_{\varepsilon \to 0} \frac{u_\varepsilon(x)u_\varepsilon'(x)}{1+\varepsilon u_\varepsilon(x)} = u_0(x)u_0'(x) \quad \text{uniformly in } \mathbf{R} \tag{3.9}$$

To pass to the limit in the third term, we first consider the convergence of the functions z_ε. Because the right hand side of the differential equation for z_ε varies monotonically with ε we have from [vDGZ] that

$$\varepsilon_1 > \varepsilon_2 > 0 \quad \Rightarrow \quad z_{\varepsilon_1}(t) > z_{\varepsilon_2}(t) > z_0(t) \quad \text{for} \quad 0 < t < 1.$$

Hence there exists a function $\tilde{z}(t) \geq z_0(t)$ such that

$$\lim_{\varepsilon \to 0} z_\varepsilon(t) = \tilde{z}(t) \quad \text{pointwise in} \quad [0,1].$$

Again, using an argument from [vDGZ, proof of Proposition 2.4] we find that $\tilde{z}(t) = z_0(t)$ for $0 \leq t \leq 1$. Thus, we have the situation where a sequence of smooth functions $\{z_\varepsilon\}$ converges monotonically to a continuous function z_0. Then Dini's theorem implies that

$$\lim_{\varepsilon \to 0} z_\varepsilon(t) = z_0(t) \quad \text{uniformly on} \quad [0,1],$$

and so, by (2.11),

$$\lim_{\varepsilon \to 0} y_\varepsilon(u) = y_0(u) \quad \text{uniformly on} \quad [0,1]. \tag{3.10}$$

Using (3.8) - (3.10) in (3.7) we obtain the desired limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left\{\frac{v_\varepsilon(x)}{\varepsilon} + u_0'(x)\right\} = E_1(x),$$

where

$$\begin{aligned} E_1(x) &= -w_0'(x) + u_0(x)u_0'(x) + \int_0^{u_0(x)} y_0(u)\,du \\ &= 2u_0(x)u_0'(x) - \frac{1}{2}u_0'(x) + \int_x^\infty \{u_0'(s)\}^2\,ds. \end{aligned} \tag{3.11}$$

Thus we have proved

Lemma 3.6. *We have*

$$v_\varepsilon(x) = -u_0'(x)\varepsilon + \{E_1(x) + o(1)\}\varepsilon^2 \quad \text{as} \quad \varepsilon \to 0,$$

where E_1 is given by (3.11), uniformly in **R**.

To improve the estimate for u_ε given by Lemma 3.5, we investigate the limiting behaviour of the function

$$D_\varepsilon w \stackrel{\text{def}}{=} \frac{1}{\varepsilon}(w_\varepsilon - w_0) \quad \text{as} \quad \varepsilon \to 0.$$

Thus, remembering (3.5) we need to inverstigate

$$D_\varepsilon f = \frac{1}{\varepsilon}(f_\varepsilon - f_0) \quad \text{as} \quad \varepsilon \to 0.$$

By (3.3) and Lemma 3.6

$$\begin{aligned} D_\varepsilon f(x) &= \frac{1}{\varepsilon}\Big\{-\frac{v_\varepsilon(x)}{\varepsilon}u_\varepsilon'(x) - \{u_0'(x)\}^2\Big\} \\ &= \frac{1}{\varepsilon}\big[\{u_0'(x) - (E_1(x)+o(1))\varepsilon\}u_\varepsilon'(x) - \{u_0'(x)\}^2\big] \\ &= u_0'(x)w_\varepsilon'(x) - (E_1(x)+o(1))u_\varepsilon'(x) \end{aligned}$$

Thus, by (3.8)

$$\lim_{\varepsilon\to 0} D_\varepsilon f(x) \text{ exists} = D_0 f(x) \quad \text{uniformly in } \mathbf{R},$$

where

$$D_0 f(x) = \{1 - 3u_0(x)\}\{u_0'(x)\}^2 - u_0'(x)\int_x^\infty \{u_0'(s)\}^2\, ds. \tag{3.12}$$

Moreover, by Lemma 3.1 and Corollary 3.2 there exists a constant $C > 0$, which does not depend on ε, such that

$$D_\varepsilon f(x) \le Ce^{-x^2/4} \qquad \text{for} \quad -\infty < x < \infty.$$

Therefore we may pass to the limit in the expression for $D_\varepsilon w(x)$. This leads to

Lemma 3.7. *We have*

$$u_\varepsilon(x) = u_0(x) + \frac{1}{2}u_0(x)\{1 - u_0(x)\}\varepsilon + \{E_2(x) + o(1)\}\varepsilon^2 \quad \text{as} \quad \varepsilon \to 0,$$

uniformly in $\mathbf{R}$. *Here* E_2 *is given by (3.5) in which* f_ε *is replaced by* $D_0 f$ *which is defined in (3.12).*

4. Self-similar solutions on $\mathbf{R}^+$

As in Sections 2 and 3, we study solutions (u, v) of the system of equations

$$\begin{cases} u'' + (\frac{1}{2}x - v)u' = 0 & \qquad (4.1) \\ & 0 < x < \infty \\ v' = -\dfrac{u''}{u + \frac{1}{\varepsilon}}, & \qquad (4.2) \end{cases}$$

but now subject to the boundary conditions

$$u(0) = 1 \quad \text{and} \quad u(\infty) = 0 \tag{4.3}$$

and

$$v(0) = -\varepsilon K(\varepsilon)u'(0), \tag{4.4}$$

where K is the positive constant which has been defined in (1.19d). Note that due to the boundary conditions at $x = 0$, the functions u and v' are no longer invariant under shifts.

We begin again with some qualitative properties of u and v.

Lemma 4.1. *Let (u, v) be a solution of (4.1) - (4.4). Then*

(a)
$$u'(x) < 0 \quad \text{for all} \quad x \in [0, \infty).$$

(b) *There exists a number $x_0 > 0$ such that*

$$\begin{aligned} u''(x) > 0 \quad (<) \quad &\text{if} \quad x > x_0 \ (<) \\ v'(x) < 0 \quad (>) \quad &\text{if} \quad x > x_0 \ (<). \end{aligned}$$

(c)
$$u'(x) \to 0 \quad \text{as} \quad x \to \infty.$$

Proof. We shall only prove Part (b); for the proofs of the other parts we refer to the proof of Lemma 2.1.

Because by Part (a) $u'(0) < 0$, we conclude from (4.4) that $v(0) > 0$. Using this in equation (4.1) we find that $u'' < 0$ in a neighbourhood of $x = 0$. By the boundary condition at infinity, u'' has to change sign at least once. Arguing as in the proof of Lemma 2.1(b) we show that this can happen no more than once.

To prove the existence and uniqueness of a solution (u, v), we proceed as in Section 2 and introduce the variables y and z as in (2.8) and (2.11) respectively. This leads for z to a problem of the form

$$(\mathrm{P}_2) \begin{cases} -zz'' = \dfrac{1}{2} e^{2t\log(1+\varepsilon)}, \quad z > 0 \quad \text{for} \quad 0 < t < 1 \\ z(0) = 0, \qquad z'(1) = -Lz(1), \end{cases}$$

in which L is a constant. If we choose

$$L = K(\varepsilon)(1+\varepsilon)\log(1+\varepsilon) = \frac{\log(1+\varepsilon)}{\gamma - \log(1+\varepsilon)} \tag{4.5}$$

then, returning to the original variables we find that $u(x)$ and $v(x)$ satisfy the boundary condition (4.4) at $x = 0$.

Theorem 4.2. *For any $L > -1$, Problem (P_2) has a unique solution $z \in C^\infty((0, 1]) \cap C([0, 1])$.*

Proof. Instead of Problem (P_2) we consider the problem

$$(\mathrm{P}_\alpha) \begin{cases} -ww'' = \dfrac{1}{2} e^{2t\log(1+\varepsilon)} \quad w > 0 \quad \text{for} \quad 0 < t < 1 \\ w(0) = 0, \qquad w(1) = \alpha, \end{cases}$$

where $\alpha > 0$ is an arbitrary positive number.

From [vDGZ], where further references are given, we know that for each $\alpha > 0$, Problem (P_α) has a unique, concave solution $w_\alpha \in C^\infty((0,1]) \cap C([0,1])$.

It will be convenient to fix the boundary conditions. Thus we normalize w_α and write

$$r_\alpha = \alpha^{-1} w_\alpha.$$

The functions r_α then satisfy

$$-rr'' = \frac{1}{2\alpha^2} e^{2t\log(1+\varepsilon)}, \quad r > 0 \quad \text{for} \quad 0 < t < 1 \tag{4.6}$$

and

$$r(0) = 0, \qquad r(1) = 1.$$

For every $\alpha > 0$ we can define

$$\beta(\alpha) = r'_\alpha(1). \tag{4.7}$$

It then remains to show that the equation

$$\beta(\alpha) = -L \tag{4.8}$$

has one and only one solution if $L > -1$. In the following lemma we derive some properties of the function β which together ensure that this is indeed the case.

Lemma 4.3. *We have*

(a)
$$\alpha_1 < \alpha_2 \quad \Rightarrow \quad \beta(\alpha_1) < \beta(\alpha_2)$$

(b) $\beta \in C(0,\infty)$ *and*

$$\lim_{\alpha \to \infty} \beta(\alpha) = 1 \quad \textit{and} \quad \lim_{\alpha \to 0} \beta(\alpha) = -\infty.$$

Proof. (a) Because the right hand side of (4.6) decreases with increasing α, we apply again a comparison argument and find

$$\alpha_1 < \alpha_2 \qquad \Rightarrow \qquad r_{\alpha_1} > r_{\alpha_2} \qquad \text{on} \quad (0,1)$$

and so

$$r'_{\alpha_1}(1) \le r'_{\alpha_2}(1). \tag{4.9}$$

In fact, using an argument from [vDP1, Theorem 4], we can show that strict inequality holds in (4.9):

$$\alpha_1 < \alpha_2 \quad \Rightarrow \quad r'_{\alpha_1}(1) < r'_{\alpha_2}(1).$$

To prove Part (b) we use a compactness argument. Let $\overline{\alpha} > 0$ and let α_n be an increasing sequence such that

$$\alpha_n \nearrow \overline{\alpha} \qquad \text{as} \quad n \to \infty.$$

Then $\{r_{\alpha_n}\}$ is a decreasing sequence, bounded below by $r_{\overline{\alpha}}$. Hence, there exists a function $\overline{r} \geq r_{\overline{\alpha}}$ such that for every $t \in [0,1]$,

$$r_{\alpha_n}(t) \searrow \overline{r}(t) \qquad \text{as} \quad n \to \infty.$$

Because the sequence $\{r_{\alpha_n}\}$ is bounded away from zero on any interval $[\delta, 1]$, where $\delta \in (0,1)$, the differential equation implies that $r'_{\alpha_n}, r''_{\alpha_n}$ and r'''_{α_n} are all bounded on $[\delta, 1]$, uniformly with respect to n. Hence, there exists a subsequence $\{r_{\alpha_{n'}}\}$ such that

$$r_{\alpha_{n'}} \to \overline{r}, \quad r'_{\alpha_{n'}} \to \overline{r}', \quad r''_{\alpha_{n'}} \to \overline{r}'' \qquad n \to \infty, \tag{4.10}$$

uniformly on $[\delta, 1]$. By taking the limit in the equation we conclude that

$$-\overline{r}\,\overline{r}'' = \frac{1}{2\overline{\alpha}^2} e^{2t\log(1+\varepsilon)}, \quad \overline{r} > 0 \qquad \text{for} \quad 0 < t < 1,$$

because we may choose δ arbitrary small. Since

$$r_{\overline{\alpha}} \leq \overline{r} < r_{\alpha_n} \quad \text{for} \quad 0 \leq t \leq 1,$$

we also have

$$\overline{r}(0) = 0 \quad \text{and} \quad \overline{r}(1) = 1.$$

Thus, $\overline{r} = r_{\overline{\alpha}}$ and the limit is uniquely determined by $\overline{\alpha}$. Therefore (4.10) holds, with $\overline{r}$ replaced by $r_{\overline{\alpha}}$, for the entire sequence. This means in particular that

$$r'_{\alpha_n}(1) \to r'_{\overline{\alpha}}(1) \qquad \text{as} \quad n \to \infty,$$

or

$$\beta(\alpha_n) \to \beta(\overline{\alpha}) \qquad \text{as} \quad n \to \infty.$$

A similar argument can be given for a decreasing sequence. This proves the continuity of β at $\overline{\alpha}$.

Thus, for any given $L > -1$, there exists a unique number α_L such that

$$\beta(\alpha_L) = -L$$

and (4.8) is satisfied. This completes the proof of Theorem 4.2.

We return to the original variables x, u and v by means of

$$x(u) = \int_u^1 \frac{ds}{y(s)} \quad \text{for} \quad 0 < u \leq 1$$

and

$$v(x) = \frac{1}{2}x - y'(u(x)) \quad \text{for} \quad 0 \leq x < \infty.$$

By an elementary computation we then verify that the pair of functions $(u(x), v(x))$ satisfies (4.1) - (4.4). Thus we have proved

Theorem 4.4. *For any $0 < \varepsilon < e^{\gamma} - 1$ there exists a unique pair of functions (u, v) which satisfies (4.1) - (4.4).*

In the next lemma we list some properties of $v(x)$.

Lemma 4.5. *Let (u, v) be the solution of (4.1) - (4.4). Then*

(a)
$$v(\infty) = \lim_{x \to \infty} v(x) \quad \textit{exists}$$

and is finite;

(b)
$$v(x) > v(\infty) \quad \text{for} \quad 0 \leq x < \infty.$$

(c)
$$v(x) = \varepsilon\Big(K(\varepsilon) - \frac{1}{1+\varepsilon}\Big)y(1) + \varepsilon\frac{y(u(x))}{1+\varepsilon u(x)} - \varepsilon^2 \int_{u(x)}^{1} \frac{y(u)}{(1+\varepsilon u)^2}\, du.$$

(d)
$$v(\infty) > 0 \quad \textit{for } \varepsilon \textit{ small enough.}$$

Proof. The proofs of Parts (a) - (c) use arguments very similar to those used in the proof of Lemma 2.2. We therefore omit them.

To prove Part (d) we observe that by (c),

$$v(\infty) = \varepsilon\Big(K(\varepsilon) - \frac{1}{1+\varepsilon}\Big)y(1) - \varepsilon^2 \int_0^1 \frac{y(u)}{(1+\varepsilon u)^2}\, du. \tag{4.11}$$

By (1.19d),

$$K(\varepsilon) - \frac{1}{1+\varepsilon} = \frac{1}{1+\varepsilon}\Big(\frac{1}{\gamma - \log(1+\varepsilon)} - 1\Big) > \frac{1}{\gamma} - 1 > 0 \tag{4.12}$$

and $y(1) = -u'(0) > 0$ Thus, since for small values of ε the first term on the right hand side of (4.11) dominates the second term, the assertion is proved.

In Problem I we required that $v(\infty) = 0$. Equation (2.16) then implies that $v(-\infty) > 0$ and that $v(-\infty) = O(\varepsilon^2)$ as $\varepsilon \to 0$. Here we find that $v(\infty)$ is not so small.

Lemma 4.6. *We have*

$$v(\infty) = O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.$$

Proof. Writing (4.11) in terms of z we obtain

$$v(\infty) = \log(1+\varepsilon)z(1)\Big\{K(\varepsilon) - \frac{1}{1+\varepsilon} - \log(1+\varepsilon)\int_0^1 \frac{z(t)}{z(1)}e^{-t\log(1+\varepsilon)}\,dt\Big\}. \qquad (4.13)$$

Thus, it remains to bound $z(t)$ from above and below, uniformly with respect to ε. To that end we introduce two auxiliary functions $\underline{z}$ and $\overline{z}$, which will serve as lower and upper bound respectively.

– For $\underline{z}$ we choose the solution of the problem

$$\begin{cases} -zz'' = \dfrac{1}{2}, \quad z > 0 \qquad \text{for} \quad 0 < t < 1 \\ z(0) = 0, \quad z'(1) = -Lz(1), \end{cases}$$

in which we choose $L = \Lambda(\varepsilon_0)$, where (see (4.11)),

$$\Lambda(\varepsilon) \stackrel{\text{def}}{=} (1+\varepsilon)\log(1+\varepsilon)K(\varepsilon) \quad \text{and} \quad \varepsilon_0 \in (0, e^{\gamma}-1).$$

Note that Λ is an increasing function.

– For $\overline{z}$ we choose the solution of the problem

$$\begin{cases} -zz'' = \dfrac{1}{2}(1+\varepsilon_0), \quad z > 0 \qquad \text{for} \quad 0 < t < 1 \\ z(0) = 0, \quad z'(1) = 0, \end{cases}$$

Lemma 4.7. *Let $z_\varepsilon(t)$ be the solution of Problem P_2 in which $L = \Lambda(\varepsilon)$, and let $\underline{z}$ and $\overline{z}$ be as defined above. Then for $\varepsilon \in (0, \varepsilon_0)$, where $0 < \varepsilon_0 < e^{\gamma} - 1$,*

$$\underline{z}(t) < z_\varepsilon(t) < \overline{z}(t) \quad \text{for} \quad 0 < t < 1.$$

Proof. We shall only prove the lower bound; the upper bound is proved in the same way. We argue by contradiction. Thus, suppose there exist numbers $0 \le a < b \le 1$ such that

$$w(t) = z(t) - \underline{z}(t) < 0 \quad \text{for} \quad a < t < b$$

and

$$w(a) = 0.$$

If $b \le 1$ and $w(b) = 0$ we obtain a contradiction as in Theorem 2.1 of [vDGZ]. On the other hand, if $b = 1$ and $w(1) < 0$ it follows from the boundary condition at $t = 1$ that

$$w'(1) = -\Lambda(\varepsilon)z(1) + \Lambda(\varepsilon_0)\underline{z}(1) > \{\Lambda(\varepsilon_0) - \Lambda(\varepsilon)\}z(1) > 0, \qquad (4.14)$$

where we have used the monotonicity of $\Lambda(\varepsilon)$. From the differential equations for z and $\underline{z}$ we finally deduce that

$$
\begin{aligned}
w''(t) &= \frac{1}{2\underline{z}} - \frac{1}{2z} e^{2t\log(1+\varepsilon)} \\
&< \frac{1}{2z}\left(1 - e^{2t\log(1+\varepsilon)}\right) < 0.
\end{aligned} \tag{4.15}
$$

It follows from (4.14) and (4.15) that

$$w'(t) > 0 \quad \text{for} \quad a \leq t \leq 1.$$

Since $w(a) = 0$ this means that

$$w(t) > 0 \quad \text{for} \quad a \leq t \leq 1$$

which contradicts our assumption. This proves the lower bound.

We conclude this section with the observation that the asymptotic behaviour of the solution $(u(x), v(x))$ as $x \to \infty$ is similar to what we found in Lemma 2.4. Here we have for $x > 0$

$$u(x) < A_+ x^{-1} e^{-x^2/4} \tag{4.16a}$$

and

$$v(x) - v(\infty) < A_+ x^{-1} e^{-x^2/4}, \tag{4.16b}$$

where A_+ ia a positive constant, which does not depend on ε.

5. The limit $\varepsilon \to 0$ for Problem II

Having established the existence and uniqueness of a solution $(u_\varepsilon(x), v_\varepsilon(x))$ of (4.1)-(4.4), we now determine its limiting behaviour as $\varepsilon \to 0$.

Lemma 5.1. *There exist positive constants C_1 and C_2, which do not depend on ε such that*

$$|u'_\varepsilon(x)| < C_1 \quad \text{and} \quad 0 < v_\varepsilon(x) < C_2\varepsilon$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. By Lemma 4.7, $z_\varepsilon(t)$ is uniformly bounded and thus, so is $y_\varepsilon(u)$. This proves the first bound.

The second bound follows at once from Lemma 4.5(c).

Let u_0 be the solution of the problem

$$
\begin{cases}
u'' + \dfrac{1}{2} x u' = 0, & 0 < x < \infty \\
u(0) = 1, \quad u(\infty) = 0.
\end{cases}
$$

It can be written down explicitly and is found to be

$$u_0(x) = \operatorname{erfc}\Big(\frac{x}{2}\Big). \tag{5.1}$$

As in Section 3, we can establish the following limits.

Lemma 5.2. *We have*

$$u_\varepsilon(x) \to u_0(x), \quad u'_\varepsilon(x) \to u'_0(x), \quad u''_\varepsilon(x) \to u''_0(x) \qquad \text{as} \quad \varepsilon \to 0$$

uniformly on $[0, \infty)$.

To obtain an estimate for $v_\varepsilon(x)$ we use Lemma 4.5(c). With the limits from Lemma 5.2 we find that

$$\frac{v_\varepsilon(x)}{\varepsilon} \to -\{K(0) - 1\}u'_0(0) - u'_0(x) \qquad \text{as} \quad \varepsilon \to 0$$

uniformly on $[0, \infty)$. Remembering the definition (1.19d) of $K(\varepsilon)$ we arrive at the following limit

Lemma 5.3. *We have*

$$\lim_{\varepsilon\to 0} \frac{v_\varepsilon(x)}{\varepsilon} = \frac{1-\gamma}{\sqrt{\pi}\,\gamma} - u'_0(x) \quad \textit{uniformly on} \quad [0, \infty).$$

To improve the estimate for $u_\varepsilon(x)$ we consider again the function

$$w_\varepsilon = \frac{u_\varepsilon - u_0}{\varepsilon}$$

which now is the solution of the problem

$$\begin{cases} w'' + \dfrac{1}{2}xw' = -f_\varepsilon(x) & x \in (0, \infty) \\ w(0) = 0, \qquad w(\infty) = 0, \end{cases}$$

where

$$f_\varepsilon(x) = -\frac{v_\varepsilon(x)}{\varepsilon}\, u'_\varepsilon(x). \tag{5.2}$$

Note that by (4.16a) and the boundedness of $v_\varepsilon(x)$,

$$|f_\varepsilon(x)| \le Ce^{-x^2/4} \qquad \text{for} \quad x \ge 0, \tag{5.3}$$

where C may be chosen independent of ε. In addition, by Lemmas 5.1 and 5.2

$$\lim_{\varepsilon\to 0} f_\varepsilon(x) = -\frac{1-\gamma}{\sqrt{\pi}\,\gamma} u'_0(x) + \{u'_0(x)\}^2 \quad \text{uniformly on} \quad [0, \infty).$$

We find that

$$w_\varepsilon(x) = A_\varepsilon \int_0^x e^{-t^2/4}\,dt - \int_0^x e^{-t^2/4}\,dt \int_0^t e^{s^2/4} f_\varepsilon(s)\,ds \tag{5.4}$$

and the constant A_ε is chosen so that $w_\varepsilon(\infty) = 0$:

$$\sqrt{\pi} A_\varepsilon = \int_0^\infty e^{-t^2/4}\,dt \int_0^t e^{s^2/4} f_\varepsilon(s)\,ds. \tag{5.5}$$

It follows from (5.3) that the integral on the right hand side of (5.5) exists and is uniformly convergent with respect to ε. Therefore we may let $\varepsilon \to 0$ in (5.4) and (5.5) to obtain the required limit. In this manner we find:

Lemma 5.4. *We have*

$$\lim_{\varepsilon\to 0} \frac{u_\varepsilon(x) - u_0(x)}{\varepsilon} = -\frac{2(1-\gamma)}{\pi\gamma}\{u_0(x) + \sqrt{\pi} u_0'(x)\} + \frac{1}{2} u_0(x)\{1 - u_0(x)\}$$

uniformly with respect to $x \in [0, \infty)$.

Further terms in the asymptotic expansions for u_ε and v_ε as $\varepsilon \to 0$ may be obtained in exactly the same way as was done for Problem I in Section 3. We leave that to the interested reader.

6. Numerical results

Problem I

To solve equations (1.22) and (1.23) on $\mathbf{R}$ we first put them in the form of three first order equations by setting

$$p = u, \quad q = u' \quad \text{and} \quad r = v \tag{6.1}$$

This gives

$$p' = q \tag{6.2a}$$

$$q' = q(r - \frac{1}{2}x) \tag{6.2b}$$

$$r' = -\frac{q(r - \frac{1}{2}x)}{p + \frac{1}{\varepsilon}}, \tag{6.2c}$$

We solve this first order system with a Runge-Kutta method in the regions $\{x < 0\}$ and $\{x > 0\}$, subject to the initial conditions imposed on $x = 0$:

$$p(0) = u(0) = \frac{1}{2}, \quad q(0) = u'(0) = q_0, \quad r(0) = v(0) = r_0. \tag{6.3}$$

The values of q_0 and r_0 have to be selected so that

$$p(\infty) = u(\infty) = 0 \quad \text{and} \quad p(-\infty) = u(-\infty) = 1.$$

This is done by solving Problem (P_1) for z. This equation is discretised with $\Delta t = 10^{-3}$ and the nonlinear system is solved using a multi dimensional Newton method with an accuracy of 10^{-8}. In this manner we compute

$$y\left(\frac{1}{2}\right) = \frac{\log(1+\varepsilon)}{\varepsilon}\, z\left(\frac{\log(1+\frac{\varepsilon}{2})}{\log(1+\varepsilon)}\right)$$

and

$$y'\left(\frac{1}{2}\right) = \frac{1}{1+\frac{\varepsilon}{2}}\, z'\left(\frac{\log(1+\frac{\varepsilon}{2})}{\log(1+\varepsilon)}\right).$$

By (2.8) and (2.13) we have

$$q_0 = -y\left(\frac{1}{2}\right) \text{ and } r_0 = -y'\left(\frac{1}{2}\right).$$

This yields a solution of (1.22), (1.23) and (1.24a) which satisfies $u(0) = \frac{1}{2}$. In particular it gives $v(\infty) = r(\infty)$, so that we can apply the shift (2.5) to obtain the desired solution.

Remark. In principle we could have constructed a solution (u, v) directly from the function $z(t)$ or $y(u)$, as was done theoretically in Section 2. However this does not yield smooth asymptotic behaviour near $u = 0$ as x becomes large and near $u = 0$ as x becomes small. Therefore we have chosen to work here with the Runge Kutta method for the system (6.2). This procedure gives smooth decay at large values of $|x|$. Moreover it serves as a check for the accuracy with which $y(\frac{1}{2})$ and $y'(\frac{1}{2})$ are determined.

We carried out the computations for a small $\varepsilon(\varepsilon = 0.025)$ and for a large value of $\varepsilon(\varepsilon = 0.5)$. The latter value was chosen in order to emphasize the differences. We find

$$\begin{aligned} \varepsilon = 0.025 \quad &: \quad \text{shift} \quad a = 2v(\infty) \quad = \quad -1.84210^{-2}; \\ \epsilon = 0.5 \quad &: \quad \text{shift} \quad a = 2v(\infty) \quad = \quad -0.2198. \end{aligned}$$

Figure 2 shows the scaled density distribution u for the two values of ε. The difference between the Boussinesq limit u_0, see expression (1.28), and the solution u for $\varepsilon = 0.025$ is so small that they would coincide in Figure 2.

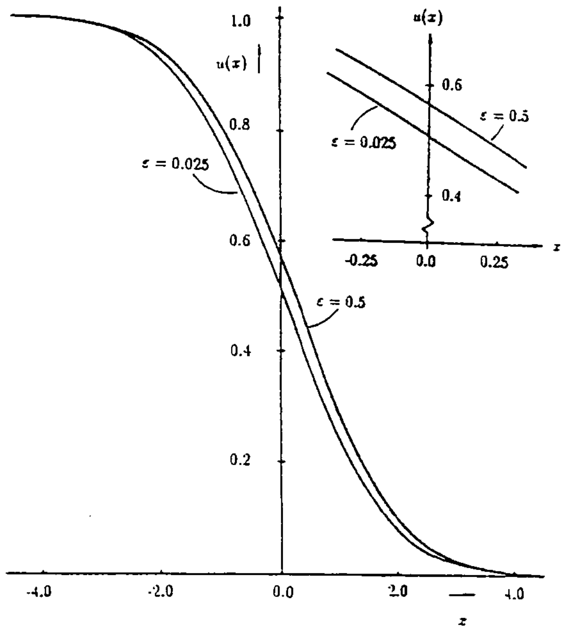

Fig. 2. Computed scaled density distributions u on **R**.

Figure 3 shows the scaled velocity $v = q_z\sqrt{t}$ induced by the density effects. Here the Boussinesq limit corresponds to $v = 0$.

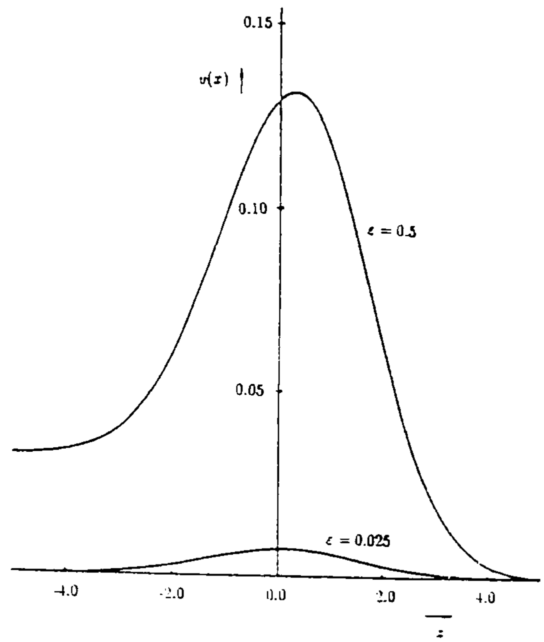

Fig. 3. Computed scaled velocities $v = q_z\sqrt{t}$ on **R**.

Observe that the results represented in both figures correspond qualitatively to the analysis in Sections 2 and 3.

Problem II

Here we solve the system (6.2) for $x > 0$, subject to the initial conditions

$$p(0) = u(0) = 1 \tag{6.4a}$$

$$q(0) = u'(0) = q_0 \tag{6.4b}$$

$$r(0) = v(0) = -\varepsilon K(\varepsilon) q_0 \tag{6.4c}$$

The special form of (6.4c) results from the boundary condition (1.25b). Again we apply a Runge Kutta procedure, this time with q_0 as a shooting parameter. Its value is obtained as follows:
(i) Using Newton's method we solve numerically, for each given $\varepsilon > 0$, the sequence of Problems (P_α) where we let α vary from 0 to a sufficiently large number $\overline{\alpha}$ in steps $\Delta\alpha = 10^{-3}$. This leads to graphs of the functions $\beta(\alpha)$, shown in Figure 4, with the properties of Lemma 4.3

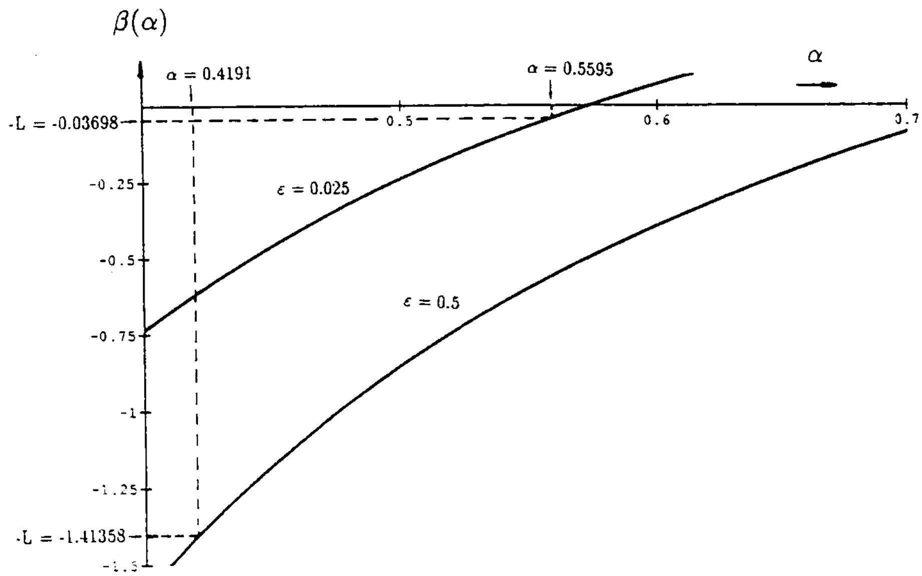

Fig. 4. Representation of the function $\beta(\alpha)$ defined by (4.7).

(ii) Given the constant $L > 0$ from (4.5), we determine the value α_L from the graph $\beta(\alpha)$ such that

$$\beta(\alpha_L) = -L.$$

(iii) Having obtained $\alpha_L = z(1)$ in Problem (P_2), we apply the transformation (2.10) to obtain

$$q_0 = -y(1) = -\frac{\log(1+\varepsilon)}{\varepsilon}\alpha_L. \tag{6.5}$$

With this value of q_0 we solve the initial value problem (6.2), (6.4) and check if $r(x)$ is sufficiently close to zero for large x. For example, with $\varepsilon = 0.5$, we obtain from interpolation of points on the $\beta(\alpha)$ curve

$$\alpha_L = 0.4191 \quad \text{giving} \quad 0 < r(\infty) < 10^{-4}.$$

Figures 5 and 6 give the results of the computations for the functions u and v, respectively, using the values $\varepsilon = 0.025$ and $\varepsilon = 0.5$.

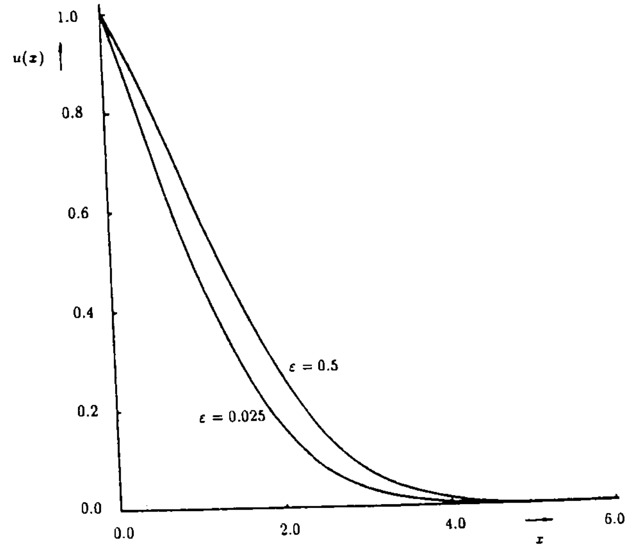

Fig. 5. Computed scaled density distributions on $\mathbf{R}^+$.

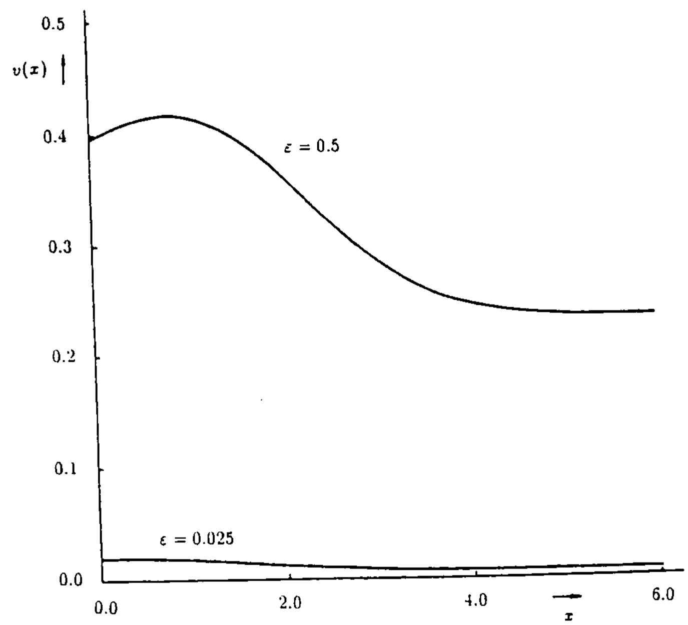

Fig. 6. Computed scaled velocities $v = q_z\sqrt{t}$ on $\mathbf{R}^+$.

Again observe that these results correspond to the qualitative behaviour as analysed in Sections 4 and 5.

From the analysis and the computations we conclude that the influence of q_z on the salt distribution is relatively small in all cases and even negligible for the value $\varepsilon = 0.025$ (as to be expected). Therefore in many applications one may disregard the influence of volume changes due to variations in the salt concentration and use the Boussinesq limit

$$\mathrm{div}\mathbf{q} = 0, \tag{6.6}$$

$$\Phi\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{q}\rho - D\mathrm{grad}\ \rho) = 0, \tag{6.7}$$

$$\frac{\mu}{\kappa}\mathbf{q} + \mathrm{grad}p - \rho\mathbf{g} = 0. \tag{6.8}$$

7. Velocity dependent dispersion

In a more realistic description of the transport of solutes through porous media, a velocity dependent dispersion matrix has to be introduced into equation (1.2). For the specific flow problems considered in this paper this means that, instead of equation (1.11), we are led to consider the transport equation

$$\Phi\frac{\partial \rho}{\partial t} + q_z\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial z}\left(D(|\mathbf{q}|)\frac{\partial \rho}{\partial z}\right), \tag{7.1}$$

where

$$D(|\mathbf{q}|) = \Phi D_{\mathrm{mol}} + a_T|\mathbf{q}| + (a_L - a_T)\frac{q_z^2}{|\mathbf{q}|}. \tag{7.2}$$

Here a_L and a_T are positive numbers, called the longitudinal and transversal dispersion length, respectively and by $|\mathbf{q}|$ we denote the length of the vector $\mathbf{q}$.

When $|q_z| << |q_y|$, the last term in (7.2) may be neglected and we can write D, with the help of (1.9) as

$$\begin{aligned} D &= \Phi D_{\mathrm{mol}} + a_T|q_y| \\ &= \Phi D_{\mathrm{mol}} + a_T\left|q_f - \frac{\kappa}{\mu}(\rho - \rho_f)g\sin\beta\right|. \end{aligned} \tag{7.3}$$

Thus, in this approximation, D is a function of ρ.

Scaling the variables in equations (1.10), (7.1) and (7.3) according to

$$\rho^* = \frac{\rho - \rho_f}{\rho_s - \rho_f} \quad , \quad q_z^* = \frac{q_z}{\hat{q}},$$

$$z^* = \frac{z}{a_T} \quad , \quad t^* = \frac{\hat{q}}{\Phi a_T}t \quad \text{and} \quad D^* = \frac{D}{a_T\hat{q}},$$

where $\hat{q} = \frac{\kappa}{\mu}(\rho_s - \rho_f)g\sin\beta$, and dropping the asterisks again, we find

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho q_z) + \frac{1}{\varepsilon}\frac{\partial q_z}{\partial z} = 0 \qquad \text{in } \Omega \times \mathbf{R}^+ \tag{7.4}$$

$$\frac{\partial \rho}{\partial t} + q_z\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial z}\left(D(\rho)\frac{\partial \rho}{\partial z}\right) \qquad \text{in } \Omega \times \mathbf{R}^+ \tag{7.5}$$

in which

$$D(\rho) = \lambda + |U - \rho|. \tag{7.6}$$

The constant ε in (7.4) is given again by (1.15) and in (7.6) we have

$$\lambda = \frac{\Phi D_{\mathrm{mol}}}{a_T \hat{q}} \quad \text{and} \quad U = \frac{q_f}{\hat{q}}.$$

Observe that the scaling proposed here is different from the scaling applied in (1.14). It allows us to consider the limit of small molecular diffusion with respect to the transversal dispersion, i.e. $\lambda = 0$, a mathematically interesting limit because it may lead to degenerate diffusion at points where $\rho = U$.

Applying the similarity transformation (1.20) and (1.21) yields the equations

$$(uv)' + \frac{1}{\varepsilon}v' - \frac{1}{2}xu' = 0 \qquad \text{in } \Omega \tag{7.7}$$

$$u'v - \frac{1}{2}xu' = (D(u)u')' \qquad \text{in } \Omega \tag{7.8}$$

In the nondegenerate case, i.e. $\lambda > 0$, or when $\lambda = 0$ and either $U < 0$ or $U > 1$, the qualitative properties as obtained in Sections 2 and 4 can be obtained with minor modifications of the proofs. Only the degenerate case $\lambda = 0$ and $0 \leq U \leq 1$, requires extra attention. Also, for all choices of λ and U, the asymptotic analysis, as $\varepsilon \to 0$, becomes more involved and less explicit. We shall leave these questions to a future paper. Formally the $\varepsilon = 0$ limit gives the equations

$$v = 0 \quad \text{in } \Omega \tag{7.9}$$

$$(D(u)u')' + \frac{1}{2}xu' = 0 \quad \text{in } \Omega \tag{7.10}$$

Self similar solutions of (7.10) were studied in detail in [vDP1, dJdJvD].

The von Mises transformation

We return to equations (1.10) and (1.11). Absorbing the porosity into the time coordinate and confining ourselves to flow situations as described in Problem I, we obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho q_z) = 0 \qquad \text{in } \mathbf{R} \times \mathbf{R}^+ \tag{7.11}$$

$$\frac{\partial \rho}{\partial t} + q_z \frac{\partial \rho}{\partial z} = \frac{\partial}{\partial z}\Big(D(\rho)\frac{\partial \rho}{dz}\Big) \quad \text{in } \mathbf{R} \times \mathbf{R}^+ \tag{7.12}$$

with, for example, $D(\rho)$ given by (7.3).

We shall apply a coordinate transformation which is a variant of the *von Mises transformation*, well known in the theory of the Prandtl boundary layer equations, (see for instance [O,vMF]). Considering (7.11) as the divergence operator in the (t, z)-plane, acting on the vector $(\rho, \rho q_z)$, we introduce the *stream function* $\psi = \psi(z, t)$ which satisfies

$$\rho = \frac{\partial \psi}{\partial z} \quad \text{and} \quad \rho q_z = -\frac{\partial \psi}{\partial t}. \tag{7.13}$$

Next we set

$$\hat{\rho}(\psi,t) = \hat{\rho}(\psi(z,t),t) = \rho(z,t) \tag{7.14}$$

and obtain

$$\frac{\partial \rho}{\partial t} = \frac{\partial \hat{\rho}}{\partial \psi} \cdot \frac{\partial \psi}{\partial t} + \frac{\partial \hat{\rho}}{\partial t} = -\rho q_z \frac{\partial \hat{\rho}}{\partial \psi} + \frac{\partial \hat{\rho}}{\partial t},$$
$$\frac{\partial \rho}{\partial z} = \frac{\partial \hat{\rho}}{\partial \psi} \cdot \frac{\partial \psi}{\partial z} = \rho \frac{\partial \hat{\rho}}{\partial \psi}.$$

Hence

$$\frac{\partial \hat{\rho}}{\partial t} = \hat{\rho} \frac{\partial}{\partial \psi}\Big(D(\hat{\rho})\hat{\rho}\frac{d\hat{\rho}}{d\psi}\Big). \tag{7.15}$$

In view of (7.13) we also consider this equation on the domain $\{(\psi,t) : -\infty < \psi < \infty,\ t > 0\}$.

Now suppose that the initial salt distribution, in the original coordinates, is given by

$$\rho(z,0) = \rho_0(z), \qquad -\infty < z < \infty, \tag{7.16}$$

where the function ρ_0 satisfies $\rho_f \leq \rho_0 \leq \rho_s$ such that

$$\lim_{z\to\infty} \rho_0(z) = \rho_f \quad \text{and} \quad \lim_{z\to-\infty} \rho_0(z) = \rho_s.$$

Then we define the function $\tilde{z} : \mathbf{R} \to \mathbf{R}$ by

$$\int_0^{\tilde{z}(\psi)} \rho_0(s)ds = \psi, \quad -\infty < \psi < \infty,$$

and we take as initial distribution in the (ψ, t) coordinate system

$$\hat{\rho}(\psi,0) = \rho_0(\tilde{z}(\psi)) \quad , -\infty < \psi < \infty. \tag{7.17}$$

After having obtained the solution $\hat{\rho} = \hat{\rho}(\psi,t)$, of the initial value problem (7.15), (7.17), we return to the original coordinates as follows. First define the *stream function* ψ : $\mathbf{R} \times \overline{\mathbf{R}}^+ \to \mathbf{R}$ by

$$z = \int_0^{\psi(z,t)} \frac{1}{\hat{\rho}(s,t)} ds + h(t), \tag{7.18}$$

where $h(t)$ will be chosen later in relation to the boundary conditions on q_z. Note that because by the maximum principle $0 < \rho_f \leq \hat{\rho}(\psi,t) \leq \rho_s$, $z \to \pm\infty$ implies that $\psi \to \pm\infty$. Next we define

$$\rho(z,t) = \hat{\rho}(\psi,t)$$

and finally

$$q_z(z,t) = h'(t) - \int_0^{\psi(z,t)} \frac{\hat{\rho}_t}{\hat{\rho}^2}(s,t)ds. \tag{7.19}$$

Then the boundary condition $q_z(\infty, t) = 0$ for all $t > 0$ will be satisfied if we choose

$$h(t) = \int_0^t \Big\{ \int_0^\infty \frac{\hat{\rho}_t}{\hat{\rho}^2}(s,\tau) ds \Big\} d\tau. \tag{7.20}$$

under the assumption that the integrals exist. In that case, q_z is given by

$$q_z(z,t) = \int_{\psi(z,t)}^\infty \frac{\hat{\rho}_t}{\hat{\rho}^2}(s,\tau) ds. \tag{7.21}$$

Another condition on q_z could have been: $q_z(0,t) = 0$. In that case we would have chosen $h(t) = 0$.

One easily verifies that the pair (ρ, q_z) solves (7.11), (7.12) and (7.16).

If we take as initial profile the piecewise constant function

$$\rho_0(z) = \begin{cases} \rho_f & z > 0 \\ \rho_s & z < 0 \end{cases}$$

then also

$$\hat{\rho}(\psi, 0) = \begin{cases} \rho_f & \psi > 0 \\ \rho_s & \psi < 0. \end{cases} \tag{7.22}$$

Consequently the solution of (7.15), (7.22) has self similar form:

$$\hat{\rho}(\psi, t) = f(\eta) \quad \text{with } \eta = \psi/\sqrt{t}.$$

Here f is the solution of the boundary value problem

$$\begin{cases} f\{D(f)ff'\}' + \frac{1}{2}\eta f' = 0, & -\infty < \eta < \infty \\ f(-\infty) = \rho_s, \quad f(\infty) = \rho_f. \end{cases}$$

Using the similarity in equation (7.18) we obtain

$$x = \frac{z}{\sqrt{t}} = \int_0^{\psi(z,t)/\sqrt{t}} \frac{d\xi}{f(\xi)} - \int_0^\infty \frac{\xi f'(\xi)}{f^2(\xi)} d\xi \tag{7.23}$$

This implies that $\psi/\sqrt{t}$ only depends on x, i.e.

$$\psi(z,t) = \phi(x)\sqrt{t},$$

and consequently that

$$\rho(z,t) = \hat{\rho}(\psi(z,t),t) = f\Big(\frac{\psi(z,t)}{\sqrt{t}}\Big) = f(\phi(x)) \stackrel{\text{def}}{=} u(x).$$

For q_z we obtain from (7.19)

$$q_z(z,t) = -\frac{1}{2\sqrt{t}} \int_{\phi(x)}^\infty \frac{\xi f'(\xi)}{f^2(\xi)} d\xi \stackrel{\text{def}}{=} \frac{1}{\sqrt{t}} v(x).$$

Much is known about the large time behaviour of solutions of the initial value problem (7.15), (7.17) for fairly general initial functions $\hat{\rho}(\psi, 0)$. In particular sharp estimates were obtained for the rate at which the solutions converge to the self-similar profile $f(\eta)$, (see for instance [vDP2,Z]). The translation of these convergence results to the original, physical, coordinates (z, t) has not yet been studied in detail. We intend to do this in a future study.

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