2-1-1979

On the Analysis of the Timoshenko Beam Theory With and Without Internal Damping

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ON THE ANALYSIS OF THE TIMOSHENKO BEAM THEORY

WITH AND WITHOUT INTERNAL DAMPING

by

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A Thesis Submitted

in

Partial Fulfillment

of the

Requirements for the Degree of

MASTER OF SCIENCE

in

Mechanical Engineering

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FEBRUARY, 1979

ACKNOWLEDGEMENTS

The author gratefully acknowledges the assistance of his graduate committee and all others who contributed to the completion of this work. In particular, he wishes to express his sincere appreciation to Professor Richard B. Hetnarski for suggesting the problem and for his generous consultation during the course of the study, and to Professor Newburg, chairman of the mathematics department, for his suggestions regarding the solution of non-homogeneous differential equation.

<u>ABSTRACT</u>

Doshi, Chinubhai S.; M.S., Rochester Institute of Technology, Feb. '79. 'On the Analysis of the Timoshenko Beam Theory With and Without Internal Damping.' <u>Advisor</u>: R. B. Hetnarski

The Timoshenko beam equation in terms of variable 'w_B' is derived where 'w_B' is the deflection due to the bending of a beam. The equation is used to analyze an infinite beam loaded with (i) a concentrated transverse load and (ii) an impulse. It is also shown that the rotatory damping in the equation eliminates the increasing amplitude in the propagation of the bending moment when an impulse is applied to an infinite beam. Also the general procedure for the anlysis of a non-homogeneous equation is explained.

List of Symbols

A = cross-sectional area of a beam, a constant

A_i = arbitrary functions of p

C, c_1, c_2 = arbitrary constants

E = modulus of elasticity

G = modulus of rigidity

h = height of a beam

I = moment of inertia of a beam cross-section with respect to the neutral axis of bending

k = constant, depends on the shape of the cross-section of a beam

ℓ = length of a beam

m_i = roots of characteristic equation

$M(x,t)$ = bending moment

p = Laplace transform parameter

P = external load applied to a beam

Q = shearing force

$s(0,t)$ = shear force due to an external load

t = time variable

w = total deflection (both due to bending and shear)

w_B = deflection due to bending only

w_s = deflection due to shear only

x = variable along the axis of a beam

Y, Z = arbitrary constants

β_0 = constant in linear damping term

β_1 = constant in rotatory damping term

List of Symbols (Continued)

γ = slope due to shear

ρ = mass density of a beam material

τ_1, τ_2 = surface shearing stresses

ϕ = slope due to the bending

Bar (-) over a quantity indicates Laplace transform.

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CHAPTER I

Introduction

While studying papers devoted to modified beam theory, i.e. the Timoshenko beam theory [1], some interesting facts were observed which prompted the undertaking of this work.

The Timoshenko beam theory is a modification of Euler's beam theory. Euler's beam theory does not take into account the correction for rotatory inertia or the correction for shear. In the Timoshenko beam theory, Timoshenko has taken into account corrections both for rotatory inertia and for shear. Also Timoshenko has shown that the correction for shear is approximately four times greater than the correction for rotatory inertia.

The modified theory is useful in performing dynamic analysis of a beam such as a vibration analysis, stress analysis and the wave propagation analysis.

Analysis of the Timoshenko beam theory is done in two ways. (i) Use of two equations, one in rotational motion and the other in translatory motion, (ii) use of only one equation obtained by combining equations for rotational motion and translatory motion.

The conclusion was reached after studying the work of Miklowitz [2], Dangler [3] and Eringen [4], that the one equation derived by Eringen [4] can be used both in Miklowitz's [2] and Dangler's [3] work if some changes are made as required by the problem. This is shown in Chapter V, part 1 and part 2a.

It was also noticed that in some problems where particular types of load, such as impulse load and random load, were applied to a beam, and when stress analysis was performed using Timoshenko beam theory, the results obtained led to conclusions which were erroneous and not compatible with the physical expectations.

Eringen [4] showed that in the case of random load application to a beam, it was necessary to modify the Timoshenko beam theory by the introduction of some type of internal damping. Eringen [4] obtained satisfactory results by the introduction of linear damping into the translatory motion, and of rotatory damping into the rotatory motion.

In case of impulse load application it was predicted by Dangler [3] that satisfactory results would be obtained by the use of internal damping. It was not indicated which type of damping would be necessary to obtain satisfactory results.

The same problem as the one Dangler [3] had, is solved in Chapter V, part 2b by using rotatory damping. The result obtained is satisfactory and compatible with the experimental expectations.

A method to solve the Timoshenko beam equation is explained, using variation of parameters, in Chapter V, part 3. The method explained is applicable to any differential equation in general.

CHAPTER II

Review of the Literature

S. P. Timoshenko [1], a pioneer in strength of materials, developed a theory in 1921 which is a modification of Euler's beam theory. The theory takes into account corrections for shear and rotatory inertia neglected in Euler's beam theory. The modified theory is called the 'Timoshenko beam theory.'

Since the development of this theory in 1921, many researchers have used it in various problems. Uflyand [5], in 1948, used the theory to determine the propagation of waves in the transverse vibrations of bars and plates. The Laplace transform was used to obtain the solution of the Timoshenko beam equation.

In 1951, M. A. Danger and M. Gonald [3] used the theory to determine results of impact loading on long beams. The non-dimensional form of the equation in conjunction with the Laplace transform and the Fourier transform was used in the analysis of the theory. Also in 1951, Traill-Nash and Collar [6] used the theory to determine the effects of shear flexibility and the rotatory inertia on the lateral vibration of a beam.

In 1953, both Miklowitz [2] and Anderson [7] used the Timoshenko beam theory. Miklowitz [2] used the theory to obtain the flexural wave solution of an infinite beam using the Laplace transform. Anderson [7] used the theory to find flexural vibrations in a uniform beam using a general series solution method as it is used in the analysis of Euler's beam theory.

B.A. Boley [8,9,10] in 1955-56 showed an approximate analysis of Timoshenko beams under dynamic loads on the basis of a traveling-wave approach and the principle of virtual work. He also showed the use of the Fourier sine transforms in the analysis of the Timoshenko beam under impact loads.

In 1958 Eringen and Samuel [4] introduced the internal damping concepts into the Timoshenko beam theory. The equation was modified by introducing linear and rotatory damping. The modified theory is used in the stress analysis of a simply supported beam with the application of a random load. The Laplace transform, Fourier transform, contour integration and perturbation procedure are used to obtain the solution.

CHAPTER III

Statement of the Problem

The intention of this work is to shed light on the following points:

(i) To show that the porblems considered by both Miklowitz [2] and Dangler [3], can be solved using a simple equation in 'w_B', where 'w_B' is the deflection of beam due to bending only. This equation has the form

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \rho A \frac{\partial^2 w_B}{\partial t^2} = 0 .$$

Miklowitz [2] considered the case of an infinite beam with a concentrated transverse load applied at $x = 0$. To arrive at the solution he used the Timoshenko beam represented by two coupled equations in terms of 'w_B' and 'w_s', where 'w_B' is as described above and 'w_s' is the deflection due to shear only. These equations are:

$$EI \frac{\partial^3 w_B}{\partial x^3} + kAG \frac{\partial w_s}{\partial x} - I\rho \frac{\partial^3 w_B}{\partial x \partial t^2} = 0$$

$$\rho A \frac{\partial^2 w_B}{\partial t^2} + \rho A \frac{\partial^2 w_s}{\partial t^2} - kAG \frac{\partial^2 w_s}{\partial x^2} = 0$$

Dangler [3] considered the case of an infinite beam with an impulse load applied at $x = 0$. To arrive at the solution, he used the Timoshenko equation in 'w', where 'w' is the total deflection of a beam due to both bending and shear. The equation is

$$EI \frac{\partial^4 w}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w}{\partial t^4} + \rho A \frac{\partial^2 w}{\partial t^2}$$

$$= P + \frac{1}{kGA}(\rho I \frac{\partial^2 P}{\partial t^2} - EI \frac{\partial^2 P}{\partial x^2})$$

where P is an external load applied to a beam.

(ii) To show that increasing amplitude appears in wave propagation of the bending moment obtained using the equation without any damping, i.e.,

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \rho A \frac{\partial^2 w_B}{\partial t^2} = 0.$$

can be eliminated by using the equation derived with the rotatory damping, i.e.,

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} - \beta_1 \frac{\partial^3 w_B}{\partial x^2 \partial t} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4}$$

$$+ \frac{\rho \beta_1}{kG} \frac{\partial^3 w_B}{\partial t^3} + \rho A \frac{\partial^2 w_B}{\partial t^2} = 0$$

where β_1 is a constant introduced in the rotatory damping mechanism.

(iii) To show how to use the variation of parameter method for a non-homogeneous Timoshenko equation when other methods used for particular cases are not applicable.

CHAPTER IV

Derivation of Equations

Timoshenko Beam Theory:

S. P. Timoshenko was the first to introduce correction for shear and rotatory inertia in the simple beam theory in 1921 [1]. This is why the equation derived after the introduction of shear correction and rotatory inertia is called the "Timoshenko Beam Theory."

Derivation:

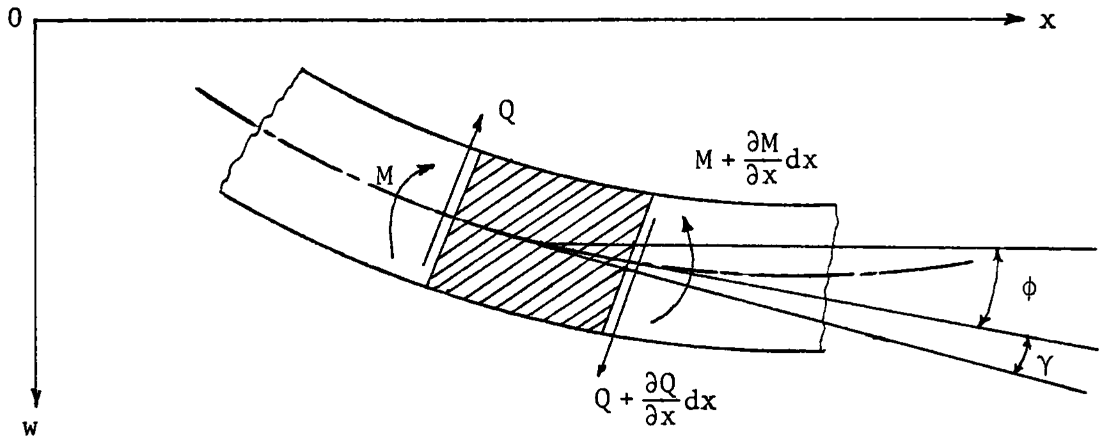


Figure 1. Element of the beam.

In the above Figure 1, the bending moment is denoted by M and shearing force is denoted by Q. Let angle ϕ be due to bending and γ be due to shear. Deflection is w.

For very small deflections

$$\frac{\partial w}{\partial x} = \phi + \gamma \tag{4.1}$$

and from simple beam theory

$$M = -EI \frac{\partial \phi}{\partial x} ; \quad Q = kAG\gamma \tag{4.2}$$

where EI is flexural rigidity; k is a constant depending on the shape of cross-section of a beam; A is area of cross-section and G is modulus of rigidity.

The equations of motion are:

For the rotation -

$$- \frac{\partial M}{\partial x} dx + Qdx = \rho I \frac{\partial^2 \phi}{\partial t^2} dx \tag{4.3}$$

where ρ is density of the material.

For translation in the direction of w -

$$\frac{\partial Q}{\partial x} dx = \rho A \frac{\partial^2 w}{\partial t^2} dx \tag{4.4}$$

Now if the value of Q from equation (4.2) is substituted into equations (4.3) and (4.4) we obtain,

$$- \frac{\partial M}{\partial x} + kAG\gamma = \rho I \frac{\partial^2 \phi}{\partial t^2} \tag{4.5}$$

$$\frac{\partial (kAG\gamma)}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2} \tag{4.6}$$

Substituting for $\gamma = \frac{\partial w}{\partial x} - \phi$ from equation (4.1) and $M = -EI \frac{\partial \phi}{\partial x}$ from equation (4.2) into equations (4.5) and (4.6) we obtain,

$$EI \frac{\partial^2\phi}{\partial x^2} + kAG(\frac{\partial w}{\partial x} - \phi) - \rho I \frac{\partial^2\phi}{\partial t^2} = 0 \tag{4.7}$$

$$\rho A \frac{\partial^2 w}{\partial t^2} - kAG(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\phi}{\partial x}) = 0 \ . \tag{4.8}$$

To eliminate ϕ from equations (4.7) and (4.8), we rearrange (4.8) to read

$$\frac{\partial\phi}{\partial x} = - \frac{\rho A}{kAG} \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} \ .$$

Now differentiating equation (4.7) with respect to x and substituting for $\frac{\partial\phi}{\partial x}$ we get

$$EI \frac{\partial^2}{\partial x^2} [-\frac{\rho A}{kAG} \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2}] + kAG[\frac{\partial^2 w}{\partial x^2} + \frac{\rho A}{kAG} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2}]$$

$$- \rho I \frac{\partial^2}{\partial t^2} [- \frac{\rho A}{kAG} \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2}] = 0 \ .$$

Simplifying the above expression we obtain,

$$- \frac{EI\rho}{kG} \frac{\partial^4 w}{\partial x^2 \partial t^2} + EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w}{\partial t^4} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \ .$$

Therefore,

$$EI \frac{\partial^4 w}{\partial x^4} - \rho I(1 + \frac{E}{kG}) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 w}{\partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w}{\partial t^4} = 0 \ . \tag{4.9}$$

This is called the "Timoshenko" equation.

In equation (4.9) rotatory inertia is represented by $-\rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$ and correction due to shear by $- \frac{\rho IE}{kG} \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w}{\partial t^4}$.

The Euler's Equation is obtained from the Timoshenko equation by eliminating the corrections due to both shear and rotatory inertia.

$$EI \frac{\partial^4 w}{\partial x^4} + \rho I \frac{\partial^2 w}{\partial t^2} = 0 \ . \tag{4.10}$$

The Timoshenko beam theory can be considered either in the form of two equations, such as (4.7) and (4.8) or in the form of the one equation, as (4.9).

Various researchers have used these equations in different ways, according to their goals.

In the later part of this thesis, it will be shown that only one equation in terms of 'w_B' is sufficient to solve most engineering problems where stress is to be found. Here w_B is deflection due to bending.

Also the equation will be used with internal damping. The most general equation considering linear and rotatory damping is derived by Eringen [4]. From the general equation, we can obtain the simple Timoshenko equation or an equation with either only rotatory damping or only linear damping.

Derivation of equation in 'w_B' with linear and rotatory damping:

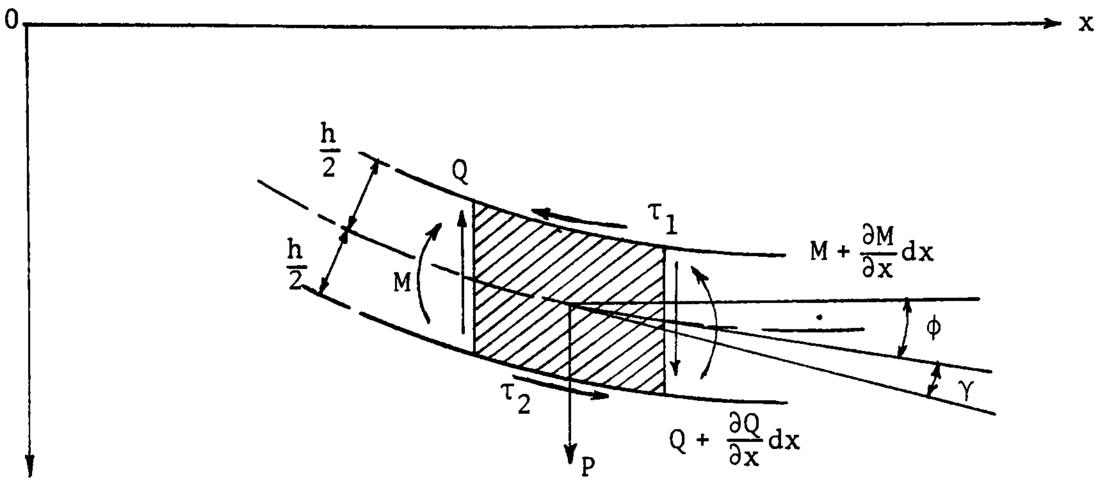


Figure 2. Element of the beam.

In these equations, linear damping is introduced in the translatory motion and rotatory damping is introduced in the rotatory motion.

The equation for rotatory motion from Figure 2 is,

$$- \frac{\partial M}{\partial x} dx + Qdx - \frac{h}{2}(\tau_1 + \tau_2) dx = \rho I \frac{\partial^2 \phi}{\partial t^2} dx$$

which yields,

$$- \frac{\partial M}{\partial x} + Q - \frac{h}{2}(\tau_1 + \tau_2) = \rho I \frac{\partial^2 \phi}{\partial t^2} . \tag{4.11}$$

The equation for translatory motion in direction of w is,

$$\frac{\partial Q}{\partial x} dx + Pdx + (\tau_2 - \tau_1) \frac{\partial w}{\partial x} dx = \rho A \frac{\partial^2 w}{\partial t^2} dx + \beta_0 \frac{\partial w}{\partial t} dx$$

which yields,

$$\frac{\partial Q}{\partial x} + P + (\tau_2 - \tau_1)\frac{\partial w}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2} + \beta_0 \frac{\partial w}{\partial t}\,. \tag{4.12}$$

In equations (4.11) and (4.12) M, Q, P are the bending moment, vertical shearing force, and vertical applied load respectively. τ_1, τ_2 are surface shearing stresses which will play the role of rotatory damping; h, A, I are the thickness, the cross-section area and the moment of inertia respectively about the neutral axis; ρ is the mass density, ϕ is the angle due to bending, w is the deflection and t is the time.

In equation (4.12) $\beta_0 \frac{\partial w}{\partial t}$ is the linear damping where β_0 is a constant.

The term $(\tau_2 - \tau_1)\frac{\partial w}{\partial x}$ comes from vertical component of τ_2 and τ_1 where small deflection theory is assumed.

Again, consider the following relations,

$$\frac{\partial w}{\partial x} = \phi + \gamma\ ; \quad \text{where} \quad \phi \cong \frac{\partial w_B}{\partial x} \tag{4.13}$$

$$M = -EI\frac{\partial \phi}{\partial x} \cong -EI\frac{\partial^2 w_B}{\partial x^2} \tag{4.14}$$

$$Q = kGA\gamma \tag{4.15}$$

where 'w_B' is the deflection due to bending.

Now to introduce rotatory damping we set,

$$\tau_1 = \tau_2 = \frac{1}{h}\beta_1 \frac{\partial\phi}{\partial t} \tag{4.16}$$

where β_1 is a constant. Substituting for τ_1 and τ_2 from equation (4.16) into equations (4.11) and (4.12) we obtain,

$$-\frac{\partial M}{\partial x} + Q - \frac{h}{2}\left(\frac{2}{h}\beta_1 \frac{\partial\phi}{\partial t}\right) = \rho I \frac{\partial^2\phi}{\partial t^2} \tag{4.17}$$

$$\frac{\partial Q}{\partial x} + P = \rho A \frac{\partial^2 w}{\partial t^2} + \beta_0 \frac{\partial w}{\partial t} . \tag{4.18}$$

Substituting for M and Q from equations (4.14) and (4.15) we obtain,

$$EI \frac{\partial^2\phi}{\partial x^2} + kGA\gamma - \beta_1 \frac{\partial\phi}{\partial t} = \rho I \frac{\partial^2\phi}{\partial t^2} \tag{4.19}$$

$$\frac{\partial}{\partial x}(kGA\gamma) + P = \rho A \frac{\partial^2 w}{\partial t^2} + \beta_0 \frac{\partial w}{\partial t} . \tag{4.20}$$

Introducing $\gamma = \frac{\partial w}{\partial x} - \phi$ from equation (4.13) and rearranging the terms we get,

$$EI \frac{\partial^2\phi}{\partial x^2} - kGA\phi + kGA \frac{\partial w}{\partial x} = \rho A \frac{\partial^2\phi}{\partial t^2} + \beta_1 \frac{\partial\phi}{\partial t} \tag{4.21}$$

$$kGA \frac{\partial\phi}{\partial x} = kGA \frac{\partial^2 w}{\partial x^2} + P - \rho A \frac{\partial^2 w}{\partial t^2} - \beta_0 \frac{\partial w}{\partial t} . \tag{4.22}$$

To eliminate ϕ equation (4.21) is differentiated with respect to x and then the value of $\frac{\partial\phi}{\partial x}$ is substituted from equation (4.22) into the differentiated equation (4.21). This results in the following equation.

$$EI \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 w}{\partial x^2} + \frac{P}{kGA} - \frac{\rho A}{kGA}\frac{\partial^2 w}{\partial t^2} - \frac{\beta_0}{kGA}\frac{\partial w}{\partial t}\right] - \left[kGA \frac{\partial^2 w}{\partial x^2} + P - \rho A \frac{\partial^2 w}{\partial t^2} - \beta_0 \frac{\partial w}{\partial t}\right] + kGA \frac{\partial^2 w}{\partial x^2} = \rho I \frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial x^2} + \frac{P}{kGA} - \frac{\rho A}{kGA}\frac{\partial^2 w}{\partial t^2} - \frac{\beta_0}{kGA}\frac{\partial w}{\partial t}\right] + \beta_1 \frac{\partial}{\partial t}\left[\frac{\partial^2 w}{\partial x^2} + \frac{P}{kGA} - \frac{\rho A}{kGA}\frac{\partial^2 w}{\partial t^2} - \frac{\beta_0}{kGA}\frac{\partial w}{\partial t}\right]$$

Simplifying the above expression we obtain,

$$EI \frac{\partial^4 w}{\partial x^4} + \frac{EI}{kGA}\frac{\partial^2 P}{\partial x^2} - \frac{\rho EI}{kG}\frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\beta_0 EI}{kGA}\frac{\partial^3 w}{\partial x^2 \partial t} - kGA \frac{\partial^2 w}{\partial x^2} - P + \rho A \frac{\partial^2 w}{\partial t^2} + \beta_0 \frac{\partial w}{\partial t} + kGA \frac{\partial^2 w}{\partial x^2} = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho I}{kGA}\frac{\partial^2 P}{\partial t^2} - \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} - \frac{\beta_0 \rho I}{kGA}\frac{\partial^3 w}{\partial t^3} + \beta_1 \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\beta_1}{kGA}\frac{\partial P}{\partial t} - \frac{\beta_1 \rho}{kG}\frac{\partial^3 w}{\partial t^3} - \frac{\beta_0 \beta_1}{kGA}\frac{\partial^2 w}{\partial t^2}$$

keeping all the terms in P on one side and rearranging all other terms we get,

$$EI \frac{\partial^4 w}{\partial x^4} - \left(\rho I + \frac{\rho EI}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} - \left(\frac{\beta_0 EI}{kGA} + \beta_1\right)\frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} + \left(\frac{\beta_0 \rho I}{kGA} + \frac{\beta_1 \rho}{kG}\right)\frac{\partial^3 w}{\partial t^3} + \left(\rho A + \frac{\beta_0 \beta_1}{kGA}\right)\frac{\partial^2 w}{\partial t^2} + \beta_0 \frac{\partial w}{\partial t} = P + \frac{\beta_1}{kGA}\frac{\partial P}{\partial t} + \frac{\rho I}{kGA}\frac{\partial^2 P}{\partial t^2} - \frac{EI}{kGA}\frac{\partial^2 P}{\partial x^2}\,. \tag{4.23}$$

Let the left hand side be equal to Lw; where 'L' is an operator.

Then,

$$Lw = P + \frac{1}{kGA} \beta_1 \frac{\partial P}{\partial t} + \rho I \frac{\partial^2 P}{\partial t^2} - EI \frac{\partial^2 P}{\partial x^2} . \tag{4.24}$$

Here to get the simple relation between w and w_B, equation (4.21) is integrated with respect to x after substituting for $\phi \cong \frac{\partial w_B}{\partial x}$ from equation (4.13) into equation (4.21). This gives the following equation,

$$EI \frac{\partial^2 w_B}{\partial x^2} - kGAw_B + kGAw = \rho I \frac{\partial^2 w_B}{\partial t^2} + \beta_1 \frac{\partial w_B}{\partial t} .$$

Rearranging the above terms we get,

$$kGA(w - w_B) = \rho I \frac{\partial^2 w_B}{\partial t^2} + \beta_1 \frac{\partial w_B}{\partial t} - EI \frac{\partial^2 w_B}{\partial x^2} .$$

Then,

$$w_s = (w - w_B) = \frac{1}{kGA} (\rho I \frac{\partial^2 w_B}{\partial t^2} + \beta_1 \frac{\partial w_B}{\partial t} - EI \frac{\partial^2 w_B}{\partial x^2}) \tag{4.25}$$

where w_s is deflection due to shear only. From equation (4.25) we obtain,

$$w = w_B + \frac{1}{kGA} (\rho I \frac{\partial^2 w_B}{\partial t^2} + \beta_1 \frac{\partial w_B}{\partial t} - EI \frac{\partial^2 w_B}{\partial x^2}) . \tag{4.26}$$

Applying operator L to equation (4.26) results in,

$$Lw = Lw_B + \frac{1}{kGA} (\rho I \frac{\partial^2 Lw_B}{\partial t^2} + \beta_1 \frac{\partial Lw_B}{\partial t} - EI \frac{\partial^2 Lw_B}{\partial x^2}) . \tag{4.27}$$

The right hand sides of both equations (4.24) and (4.27) are equal. Therefore,

$$P + \frac{1}{kGA}\left(\rho I \frac{\partial^2 P}{\partial t^2} + \beta_1 \frac{\partial P}{\partial t} - EI \frac{\partial^2 P}{\partial x^2}\right) = Lw_B + \frac{1}{kGA}\left(\rho I \frac{\partial^2 Lw_B}{\partial t^2} + \beta_1 \frac{\partial Lw_B}{\partial t} - EI \frac{\partial^2 Lw_B}{\partial x^2}\right) \tag{4.28}$$

From equation (4.28) we get,

$$Lw_B = P \tag{4.29}$$

Then the equation in $'w_B'$ with linear and rotatory damping is,

$$EI \frac{\partial^4 w_B}{\partial x^4} - \left(\rho I + \frac{\rho EI}{kG}\right) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} - \left(\frac{\beta_0 EI}{kGA} + \beta_1\right) \frac{\partial^3 w_B}{\partial x^2 \partial t} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \left(\frac{\rho I \beta_0}{kGA} + \frac{\rho \beta_1}{kG}\right) \frac{\partial^3 w_B}{\partial t^3} + \left(\rho A + \frac{\beta_0 \beta_1}{kGA}\right) \frac{\partial^2 w_B}{\partial t^2} + \beta_0 \frac{\partial w_B}{\partial t} = P . \tag{4.30}$$

To obtain an equation in $'w_B'$ without any internal damping, put $\beta_0 = 0$ and $\beta_1 = 0$ in equation (4.30). Also take external load $P = 0$.

$$EI \frac{\partial^4 w_B}{\partial x^4} - \left(\rho I + \frac{\rho EI}{kG}\right) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \rho A \frac{\partial^2 w_B}{\partial t^2} = 0 . \tag{4.31}$$

Equation (4.31) is the same as the Timoshenko equation except $'w'$ is replaced by $'w_B'$. This means that the Timoshenko equation can be split up into two equations, one in 'w_B' and the other in 'w_s', without any difficulty.

The equation with only rotatory damping is obtained by setting $\beta_0 = 0$ in equation (4.30). Also take external load $P = 0$.

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} - \beta_1 \frac{\partial^3 w_B}{\partial x^2 \partial t} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \frac{\rho \beta_1}{kG} \frac{\partial^3 w_B}{\partial t^3}$$

$$+ \rho A \frac{\partial^2 w_B}{\partial t^2} = 0 . \tag{4.32}$$

Equations (4.30) to (4.32) are used in the next chapter for the purpose of conducting the analysis of an infinite beam.

CHAPTER V

Theoretical Results

In this chapter equations (4.30) to (4.32) are used for the purpose of obtaining various results. Chapter V is divided into three parts.

Part 1.

Showing that the same results as Miklowitz [2] are obtained by using the equation (4.31).

Part 2.

(a) Showing that the same results as Dangler and Gonald [3] are obtained by using the equation (4.31).

(b) Showing that the problem of increasing amplitude of bending moment encountered in Part 2a can be eliminated by using rotatory damping in the analysis via equation (4.32).

Part 3.

Explaining the alternate method of solution, using matrices and variation of parameters, for the analysis of a non-homogeneous differential equation (i.e., Timoshenko equation with external load P).

Part 1. Analysis of an Infinite Beam with a Concentrated Traverse Load Applied.

Showing that the same result is obtained as in Miklowitz [2] by using equation (4.31) instead of two equations used in [2].

Here a simple case of an uniform infinite beam is considered with a load applied at $x = 0$.

A concentrated transverse load is applied at the center of the beam as shown in Figure 3.

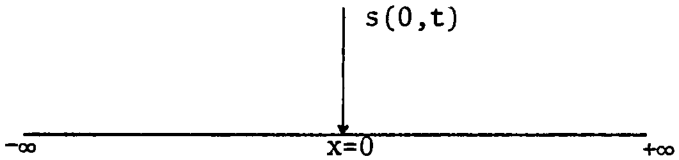


Figure 3. An infinite beam with load applied at $x = 0$.

where $s(0,t)$ is half of the load, applied at $x = 0$. Boundary conditions at $|x| = \infty$ for all t are as follows:

$$w_B = 0 \quad ; \quad \frac{\partial w_B}{\partial x} = 0 \; ; \quad \frac{\partial^2 w_B}{\partial x^2} = 0 \; .$$

Now consider equation (4.31) as the equation of motion. That is,

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \rho A \frac{\partial^2 w_B}{\partial t^2} = 0 \qquad (5.1.1)$$

Taking the Laplace transform of the equation (5.1.1) we obtain,

$$EI \frac{\partial^4 \bar{w}_B}{\partial x^4} - p^2(\rho I + \frac{\rho EI}{kG}) \frac{\partial^2 \bar{w}_B}{\partial x^2} + p^4 \frac{\rho^2 I}{kG} \bar{w}_B + p^2 \rho A \bar{w}_B = 0 \qquad (5.1.2)$$

where p is a transformation variable. Now assume the form of the solution for differential equation (5.1.2) as follows,

$$\bar{w}_B = e^{-mx} . \qquad (5.1.3)$$

Then, substituting equation (5.1.3) into equation (5.1.2) we get,

$$m^4 EIe^{-mx} - p^2 m^2 (\rho I + \frac{\rho EI}{kG}) e^{-mx} + (p^4 \frac{\rho^2 I}{kG} + p^2 \rho A) e^{-mx} = 0$$

$$m^4 EI - p^2 m^2 (\rho I + \frac{\rho EI}{kG}) + (p^4 \frac{\rho^2 I}{kG} + p^2 \rho A) = 0 . \qquad (5.1.4)$$

Let $\sqrt{\frac{E}{\rho}} = c_1$; $\sqrt{\frac{kG}{\rho}} = c_2$; $\frac{\rho A}{EI} = C$.

Substituting the above into equation (5.1.4) we obtain

$$m^4 - (\frac{1}{c_1^2} + \frac{1}{c_2^2}) p^2 m^2 + p^2 (\frac{p^2}{c_1^2 c_2^2} + C) = 0 . \qquad (5.1.5)$$

Equation (5.1.5) is the same as equation (7a) in Miklowitz [2]. The roots of the biquadratic equation (5.1.5) are,

$$(m_i)^2 = \frac{1}{2}[(\frac{1}{c_1^2} + \frac{1}{c_2^2}) p^2 \pm \{(\frac{1}{c_1^2} + \frac{1}{c_2^2})^2 p^4 - 4p^2 (\frac{p^2}{c_1^2 c_2^2} + c)\}^{\frac{1}{2}}]$$

simplifying the above,

$$(m_i)^2 = \frac{1}{2}[(\frac{1}{c_1^2} + \frac{1}{c_2^2})p^2 \pm \{(\frac{1}{c_1^2} - \frac{1}{c_2^2})^2 - 4p^2C\}^{\frac{1}{2}}]$$

Therefore,

$$m_i = \pm \frac{1}{\sqrt{2}}[(\frac{1}{c_1^2} + \frac{1}{c_2^2})p^2 \pm \{(\frac{1}{c_1^2} - \frac{1}{c_2^2})^2 - 4p^2C\}^{\frac{1}{2}}]^{\frac{1}{2}} .$$

To satisfy boundary conditions at infinity we take only positive value of m into consideration. Therefore,

$$m_{1,2} = \frac{1}{\sqrt{2}}[(\frac{1}{c_1^2} + \frac{1}{c_2^2})p^2 \pm \{(\frac{1}{c_1^2} - \frac{1}{c_2^2})^2 - 4p^2C\}^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{5.1.6}$$

Then equation (5.1.3) becomes,

$$\bar{w}_B = A_1e^{-m_1x} + A_2e^{-m_2x} . \tag{5.1.7}$$

To solve for A_1 and A_2 we need boundary conditions at $x = 0$.

Boundary conditions at $x = 0$ are,

(i) $$\frac{\partial w_B(0,t)}{\partial x} = 0 \tag{5.1.8}$$

(ii) $$\frac{\partial w_s(0,t)}{\partial x} = - \frac{s(0,t)}{kAG} . \tag{5.1.9}$$

Boundary condition (ii) needs to be converted into the terms of 'w_B' as equation is in terms of 'w_B' only. From elementary theory

$$s(0,t) = - EI \frac{\partial^3 w(0,t)}{\partial x^3} .$$

Substituting $w = w_B + w_s$ in the above equation we get,

$$s(0,t) = - EI \left[\frac{\partial^3 w_B(0,t)}{\partial x^3} + \frac{\partial^3 w_s(0,t)}{\partial x^3}\right].$$

From equation (5.1.9) $\dfrac{\partial\ w_s(0,t)}{\partial x^3} = 0$ as $s(o,t)$ is constant, therefore,

$$s(0,t) = - EI \frac{\partial^3 w_B(0,t)}{\partial x^3} ,$$

rearranging the equation we obtain,

$$\frac{\partial^3 w_B(0,t)}{\partial x^3} = - \frac{s(0,t)}{EI} \qquad (5.1.10)$$

Therefore the boundary conditions are as follows:

(i) $$\frac{\partial w_B(0,t)}{\partial x} = 0 \qquad (5.1.8)$$

(ii) $$\frac{\partial^3 w_B(0,t)}{\partial x^3} = - \frac{s(0,t)}{EI} \qquad (5.1.10)$$

Taking the Laplace transform of equations (5.1.8) and (5.1.10) for boundary conditions we acquire,

(i) $$\frac{\partial \bar{w}_B(0,p)}{\partial x} = 0 \tag{5.1.11}$$

(ii) $$\frac{\partial^3 \bar{w}_B(0,p)}{\partial x^3} = -\frac{\bar{s}(0,p)}{EI} \tag{5.1.12}$$

Now applying boundary conditions (5.1.11) and (5.1.12) to equation (5.1.7) we get,

$$-m_1A_1 - m_2A_2 = 0 \tag{5.1.13}$$

$$-m_1^3A_1 - m_2^3A_2 = -\frac{\bar{s}(0,p)}{EI}. \tag{5.1.14}$$

Substituting $A_1 = -\frac{m_2}{m_1}A_2$ from equation (5.1.13) into equation (5.1.14) results in the following,

$$-m_1^3\left(-\frac{m_2}{m_1}A_2\right) - m_2^3A_2 = -\frac{\bar{s}(0,p)}{EI}.$$

Simplifying the above equation we obtain,

$$m_2A_2(m_2^2 - m_1^2) = -\frac{\bar{s}(0,p)}{EI}$$

which gives,

$$A_2 = -\frac{\bar{s}(0,p)}{EIm_2(m_2^2 - m_1^2)}. \tag{5.1.15}$$

Also

$$A_1 = \frac{\bar{s}(0,p)}{EIm_1(m_2^2 - m_1^2)} \tag{5.1.16}$$

The values of A_1 and A_2 are the same as in the Miklowitz [2] equation (16).

Part 2a. Analysis of an Infinite Beam with an Impulse Applied.

To show that the application of the equation (4.31) to an uniform infinite beam with an impulse $A\delta(t)$ applied at $x=0$ can lead to the result which is the same as presented in Dangler and Gonald [3]. The function $A\delta(t)$ is chosen arbitrarily for convenience, where $\delta(t)$ is a 'Dirac' function and A is a constant.

A single equation in terms of total deflection 'w' is used by Dangler and Gonald [3].

Now equation (4.31) from Chapter IV is,

$$EI\,\frac{\partial^4 w_B}{\partial x^4} - \left(\rho I + \frac{\rho EI}{kG}\right)\frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\,\frac{\partial^4 w_B}{\partial t^4} + \rho A\,\frac{\partial^2 w_B}{\partial t^2} = 0\,. \qquad (5.2.1)$$

Dividing equation (5.2.1) by ρA and using notations $\frac{I}{A} = r^2$ and $\frac{1}{kG} = c$, where r = radius of gyration, we obtain,

$$\frac{E}{\rho}\,r^2\,\frac{\partial^4 w_B}{\partial x^4} - r^2(1 + Ec)\,\frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \rho r^2 c\,\frac{\partial^4 w_B}{\partial t^4} + \frac{\partial^2 w_B}{\partial t^2} = 0\,. \qquad (5.2.2)$$

To simplify the mathematical process, equation (5.2.2) is transformed into non-dimensional form. The same method is used as in Dangler and Gonald [3]. A quantity is divided by itself to get unity, i.e. $\frac{E}{E} = 1$ and $\frac{\rho}{\rho} = 1$. Equation (5.2.2) then becomes,

$$r^2\,\frac{\partial^4 w_B}{\partial x^4} - r^2(1 + c)\,\frac{\partial^4 w_B}{\partial x^2 \partial t^2} + r^2 c\,\frac{\partial^4 w_B}{\partial t^4} + \frac{\partial^2 w_B}{\partial t^2} = 0 \qquad (5.2.3)$$

Here the example with the simple case c=1 is considered, although in reality generally $c > 1$ for a beam. This is necessary to compare the results with those received by Dangler and Gonald [3].

When $c = 1$ equation (5.2.3) becomes

$$r^2 \frac{\partial^4 w_B}{\partial x^4} - 2r^2 \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + r^2 \frac{\partial^4 w_B}{\partial t^4} + \frac{\partial^2 w_B}{\partial t^2} = 0 . \tag{5.2.4}$$

Now taking Laplace transform of equation (5.2.4) we obtain,

$$r^2 \frac{\partial^4 \bar{w}_B}{\partial x^4} - 2r^2 p^2 \frac{\partial^2 \bar{w}_B}{\partial x^2} + (r^2 p^4 + p^2)\, \bar{w}_B = 0 . \tag{5.2.5}$$

Boundary conditions are the same as in Part 1; both at infinity and at $x = 0$ (equations 5.1.11 and 5.1.12). That is,

(i) $$\frac{\partial \bar{w}_B(0,p)}{\partial x} = 0 \tag{5.2.6}$$

(ii) $$\frac{\partial^3 \bar{w}_B(0,p)}{\partial x^3} = -\frac{\bar{s}(0,p)}{EI} \tag{5.2.7}$$

Then the solution is also the same as in Part 1,

$$\bar{w}_B = A_1 e^{-m_1 x} + A_2 e^{-m_2 x} \tag{5.2.8}$$

where $A_1 = \dfrac{\bar{s}(0,p)}{EI m_1 (m_2^2 - m_1^2)}$

and $A_2 = - \dfrac{\bar{s}(0,p)}{EI\, m_2(m_2^2 - m_1^2)}$.

Here $\bar{s}(0,p)$ depends on the applied load and m_1 and m_2 are determined from equation (5.2.5). Now the applied load is $A\delta(t)$ at $x=0$, where $\delta(t)$ is 'Dirac' function and A is a constant. Therefore,

$$s(0,t) = \frac{A}{2}\,\delta(t) \; .$$

The Laplace transform of $s(0,t)$ gives

$$\bar{s}(0,p) = \frac{A}{2} \text{ (as Laplace transform of } '\delta(t)' \text{ is } '1'.)$$

Substituting $\bar{s}(0,p)$ into equations for A_1 and A_2 we obtain,

$$A_1 = \frac{A}{2EIm_1(m_2^2 - m_1^2)} \tag{5.2.9}$$

$$A_2 = - \frac{A}{2EIm_2(m_2^2 - m_1^2)} \tag{5.2.10}$$

From equation (5.2.5) we obtain, (substituting $\bar{w}_B = e^{-mx}$.)

$$r^2m^4 - 2r^2p^2m^2 + (r^2p^4 + p^2) = 0 \tag{5.2.11}$$

Then the roots of the biquadratic equation (5.2.11) are,

$$(m_i)^2 = \frac{2r^2p^2 \pm \{4r^4p^4 - 4(r^2p^4 + p^2)r^2\}^{\frac{1}{2}}}{2r^2} \; .$$

Therefore,

$$m_i = \pm \left[\frac{2r^2p^2 \pm \{4r^4p^4 - 4(r^2p^4 + p^2)r^2\}^{\frac{1}{2}}}{2r^2}\right]^{\frac{1}{2}}$$

Here only positive values of m are considered to satisfy the boundary conditions at infinity. Therefore,

$$m_{1,2} = \left[\frac{2r^2p^2 \pm 2irp}{2r^2}\right]^{\frac{1}{2}}$$

$$m_{1,2} = [p^2 \pm \frac{ip}{r}]^{\frac{1}{2}}$$

which gives,

$$m_1 = p^{\frac{1}{2}}[p + \frac{i}{r}]^{\frac{1}{2}} \quad \text{and} \quad m_2 = p^{\frac{1}{2}}[p - \frac{i}{r}]^{\frac{1}{2}} \tag{5.2.12}$$

Therefore,

$$m_2^2 - m_1^2 = p[p - \frac{i}{r}] - p[p + \frac{i}{r}]$$

Simplifying the above equation gives

$$m_2^2 - m_1^2 = -2p\frac{i}{r} \tag{5.2.13}$$

Substituting equations (5.2.9) and (5.2.10) into equation (5.2.8) results into the following equation,

$$\bar{w}_B = \frac{Ae^{-m_1 x}}{2EIm_1(m_2^2 - m_1^2)} - \frac{Ae^{-m_2 x}}{2EIm_2(m_2^2 - m_1^2)} .$$

Therefore,

$$\bar{w}_B = \frac{A}{2EI(m_2^2 - m_1^2)}\left[\frac{1}{m_1} e^{-m_1 x} - \frac{1}{m_2} e^{-m_2 x}\right] . \tag{5.2.14}$$

Substituting the value of $(m_2^2 - m_1^2)$ from equation (5.2.13) we get,

$$\bar{w}_B = \frac{A}{2EI(-2p\frac{i}{r})}\left[\frac{1}{m_1} e^{-m_1 x} - \frac{1}{m_2} e^{-m_2 x}\right] .$$

Simplifying the above equation we obtain,

$$\bar{w}_B = \frac{iAr}{4EIp}\left[\frac{1}{m_1} e^{-m_1 x} - \frac{1}{m_2} e^{-m_2 x}\right] \tag{5.2.15}$$

The following relation is known from the basic theory.

$$M(x,t) = -EI \frac{\partial^2 w_B}{\partial x^2} .$$

Taking the Laplace transform, we obtain,

$$\bar{M}(x,p) = -EI \frac{\partial^2 \bar{w}_B}{\partial x^2} . \tag{5.2.16}$$

Substituting for $\bar{w}_B$ from equation (5.2.15) into the equation (5.2.16) we get,

$$\overline{M}(x,p) = (-EI)\ \frac{iAr}{4EIp}\left[\frac{m_1^2}{m_1}\ e^{-m_1x} - \frac{m_2^2}{m_2}\ e^{-m_2x}\right].$$

Simplifying the above equation we get,

$$\overline{M}(x,p) = \frac{iAr}{4p}\left[m_2e^{-m_2x} - m_1e^{-m_1x}\right] \tag{5.2.17}$$

Substituting value of m_1 and m_2 from equation (5.2.12) into equation (5.2.17) and simplifying we obtain,

$$\overline{M}(x,p) = \frac{iAr}{4}\left[(1 - \frac{i}{rp})^{\frac{1}{2}}e^{-p^{\frac{1}{2}}(p - \frac{i}{r})^{\frac{1}{2}}x} - (1 + \frac{i}{rp})^{\frac{1}{2}}e^{-p^{\frac{1}{2}}(p + \frac{i}{r})^{\frac{1}{2}}x}\right].$$

This equation is the same as the equation (18) in Dangler and Gonald [3].

Inverse Laplace transform of equation (5.2.18) is given in Dangler and Gonald [3] as follows:

$$M(x,t) = 0 \quad \text{for} \quad t < |x| \tag{5.2.19}$$

$$M(x,t) = \frac{A}{4}\left\{\cos\frac{t}{2r}\cdot J_0(\frac{u}{2r}) + \frac{t}{u}\sin\frac{t}{2r}\cdot J_1(\frac{u}{2r})\right\} \quad \text{for} \quad t > |x| \tag{5.2.20}$$

where $u = (t^2 - x^2)^{\frac{1}{2}}$, J_0 and J_1 are Bessel functions. It is noted from equation (5.2.19) and (5.2.20) that bending moment at a station x is zero until $t = x$, and then jumps to the value (obtained as the limit of equation (5.2.20) as $t \to |x|$)

$$\lim_{t \to x} M(x,t) = \frac{A}{4}\left\{ \cos \frac{|x|}{2r} + \frac{|x|}{4r} \sin \frac{|x|}{2r} \right\}. \tag{5.2.21}$$

From the above equation (5.2.21) it is shown in [3] that there is an increasing amplitude when $t = x$ in the equation for $M(x,t)$. This result is not acceptable in reality.

It is predicted that internal damping used in the equation would eliminate the increasing amplitude in the solution for $M(x,t)$.

On the next pages, it is shown how internal damping affects the outcome.

The result is discussed later in the thesis.

(b) Same Analysis as in (a) Including Rotatory Inertia.

In Part 2a it was noted that there is an increasing amplitude in the propagation of the bending moment when $t = x$. This is contrary to the physical expectations. It is suggested by Dangler and Gonald [3] that the increase is attributed to the neglect of the internal damping in the beam.

It is shown in this part that by using internal damping in the rotatory mechanism the increasing amplitude in the propagation of the bending moment is eliminated.

The problem is the same as in Part 2a except that the Timoshenko equation is modified to include the rotatory damping.

Take the equation (4.32)

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} - \beta_1 \frac{\partial^3 w_B}{\partial x^2 \partial t} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \frac{\rho \beta_1}{kG} \frac{\partial^3 w_B}{\partial t^3} + \rho A \frac{\partial^2 w_B}{\partial t^2} = 0 . \qquad (5.2.22)$$

Dividing (5.2.22) by ρA and writing $\frac{I}{A} = r^2$ and $\frac{1}{kG} = c$ we get,

$$\frac{E}{\rho} r^2 \frac{\partial^4 w_B}{\partial x^4} - r^2(1 + Ec) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} - \frac{\beta_1}{\rho A} \frac{\partial^3 w_B}{\partial x^2 \partial t} + \rho c r^2 \frac{\partial^4 w_B}{\partial t^4} + \frac{\beta_1 c}{A} \frac{\partial^3 w_B}{\partial t^3} + \frac{\partial^2 w_B}{\partial t^2} = 0 . \qquad (5.2.23)$$

Now, as it is done in Part 2a, we change the above equation (5.2.23) into non-dimensional form and let $c = 1$. Then we get,

$$r^2 \frac{\partial^4 w_B}{\partial x^4} - 2r^2 \frac{\partial^4 w_B}{\partial x^2 \partial t^2} - \frac{\beta_1}{A} \frac{\partial^3 w_B}{\partial x^2 \partial t} + r^2 \frac{\partial^4 w_B}{\partial t^4} + \frac{\beta_1}{A} \frac{\partial^3 w_B}{\partial t^3} + \frac{\partial^2 w_B}{\partial t^2} = 0.$$

Performing the Laplace transform we obtain,

$$r^2 \frac{\partial^4 \bar{w}_B}{\partial x^4} - 2r^2 p^2 \frac{\partial^2 \bar{w}_B}{\partial x^2} - \frac{\beta_1}{A} p \frac{\partial^2 \bar{w}_B}{\partial x^2} + r^2 p^4 \bar{w}_B + \frac{\beta_1}{A} p^3 \bar{w}_B + p^2 \bar{w}_B = 0.$$

Therefore,

$$r^2 \frac{\partial^4 \bar{w}_B}{\partial x^4} - (2r^2 p^2 + \frac{\beta_1}{A} p) \frac{\partial^2 \bar{w}_B}{\partial x^2} + (r^2 p^4 + \frac{\beta_1}{A} p^3 + p^2) \bar{w}_B = 0 . \tag{5.2.24}$$

Solution of equation (5.2.24) is the same as in Part 2a. That is,

$$\bar{w}_B = A_1 e^{-m_1 x} + A_2 e^{-m_2 x} \tag{5.2.25}$$

with

$$A_1 = \frac{A}{2EIm_1 (m_2^2 - m_1^2)}$$

$$A_2 = -\frac{A}{2EIm_2 (m_2^2 - m_1^2)}$$

where m_1 and m_2 are different and obtained by substituting $\bar{w}_B = e^{-mx}$ in equation (5.2.24)

$$r^2m^4 - (2r^2p^2 + \frac{\beta_1}{A}p)m^2 + (r^2p^4 + \frac{\beta_1}{A}p^3 + p^2) = 0 . \qquad (5.2.26)$$

Roots of the biquadratic equation (5.2.26) are,

$$(m_i)^2 = \frac{(2r^2p^2 + \frac{\beta_1}{A}p) \pm \{(2r^2p^2 + \frac{\beta_1}{A}p)^2 - 4r^2(r^2p^4 + \frac{\beta_1}{A}p^3 + p^2)\}^{\frac{1}{2}}}{2r^2} \qquad (5.2.27)$$

Simplify the terms under the root sign as follows:

$$(2r^2p^2 + \frac{\beta_1}{A}p)^2 - 4r^2(r^2p^4 + \frac{\beta_1}{A}p^3 + p^2)$$

$$= 4r^4p^4 + (\frac{\beta_1}{A})^2p^2 + 4r^2 \cdot \frac{\beta_1}{A}p^3 - 4r^4p^4 - 4r^2\frac{\beta_1}{A}p^3 - 4r^2p^2$$

$$= (\frac{\beta_1}{A})^2p^2 - 4r^2p^2$$

$$= p^2 [(\frac{\beta_1}{A})^2 - 4r^2] .$$

Substituting back into equation (5.2.27) we get,

$$(m_i)^2 = \frac{2r^2p^2 + \frac{\beta_1}{A}p \pm p[(\frac{\beta_1}{A})^2 - 4r^2]^{\frac{1}{2}}}{2r^2} .$$

Therefore,

$$m_{1,2} = \pm p^{\frac{1}{2}} \left[p + \frac{\frac{\beta_1}{A} \pm \{(\frac{\beta_1}{A})^2 - 4r^2\}^{\frac{1}{2}}}{2r^2} \right]^{\frac{1}{2}}$$

Only positive values of m are considered, to satisfy boundary conditions at infinity. Therefore,

$$m_1 = p^{\frac{1}{2}}\left[p + \frac{\frac{\beta_1}{A} + \{(\frac{\beta_1}{A})^2 - 4r^2\}^{\frac{1}{2}}}{2r^2}\right]^{\frac{1}{2}} \qquad (5.2.28)$$

and

$$m_2 = p^{\frac{1}{2}}\left[p + \frac{\frac{\beta_1}{A} - \{(\frac{\beta_1}{A})^2 - 4r^2\}^{\frac{1}{2}}}{2r^2}\right]^{\frac{1}{2}}. \qquad (5.2.29)$$

Let

$$Y = \frac{\frac{\beta_1}{A} + \{(\frac{\beta_1}{A})^2 - 4r^2\}^{\frac{1}{2}}}{2r^2}$$

in equation (5.2.28) and

$$Z = \frac{\frac{\beta_1}{A} - \{(\frac{\beta_1}{A})^2 - 4r^2\}^{\frac{1}{2}}}{2r^2}$$

in equation (5.2.29) where $(\frac{\beta_1}{A})^2 > 4r^2$ or $\beta_1 > 2(IA)^{\frac{1}{2}}$ to get Y and Z real and not complex. Then,

$$m_1 = p^{\frac{1}{2}}[p + Y]^{\frac{1}{2}} \qquad (5.2.30)$$

$$m_2 = p^{\frac{1}{2}}[p + Z]^{\frac{1}{2}} \qquad (5.2.31)$$

From the above equations (5.2.30) and (5.2.31) we get,

$$m_2^2 - m_1^2 = p[p+Y] - p[p+Z] = p(Z-Y) \ . \qquad (5.2.32)$$

Substituting $(m_2^2 - m_1^2)$ from equation (5.2.32) into the equations for A_1 and A_2 we obtain,

$$A_1 = \frac{A}{2EIm_1p(Z-Y)} \quad \text{and} \quad A_2 = -\frac{A}{2EIm_2p(Z-Y)}$$

where Z and Y are real and not equal. Substituting for A_1 and A_2 into equation (5.2.25) we get,

$$\bar{w}_B = \frac{Ae^{-m_1x}}{2EIm_1p(Z-Y)} - \frac{Ae^{-m_2x}}{2EIm_2p(Z-Y)} \ .$$

Therefore,

$$\bar{w}_B = \frac{A}{2EIp(Z-Y)}\left\{\frac{1}{m_1}e^{-m_1x} - \frac{1}{m_2}e^{-m_2x}\right\} . \qquad (5.2.33)$$

Using $\overline{M}(x,p) = -EI\,\dfrac{\partial^2\bar{w}_B}{\partial x^2}$ from Part 2a and substituting for $\bar{w}_B$ from equation (5.2.33) we obtain,

$$\overline{M}(x,p) = (-EI)\,\frac{A}{2EIp(Z-Y)}\left\{\frac{m_1^2}{m_1}e^{-m_1x} - \frac{m_2^2}{m_2}e^{-m_2x}\right\} .$$

which gives,

$$\overline{M}(x,p) = \frac{A}{2p(Z-Y)}\left\{m_2e^{-m_2x} - m_1e^{-m_1x}\right\} \ . \qquad (5.2.34)$$

Substituting m_1 and m_2 from equations (5.2.30) and (5.2.31) respectively into the equation (5.2.34) we obtain,

$$\overline{M}(x,p) = \frac{A}{2p(Z-Y)}\{p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}e^{-p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}x} - p^{\frac{1}{2}}(p+Y)^{\frac{1}{2}}e^{-p^{\frac{1}{2}}(p+Y)^{\frac{1}{2}}x}\}$$

which gives,

$$\overline{M}(x,p) = \frac{A}{2(Z-Y)}\left\{\frac{(p+Z)^{\frac{1}{2}}}{p^{\frac{1}{2}}}\,e^{-p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}x} - \frac{(p+Y)^{\frac{1}{2}}}{p^{\frac{1}{2}}}\,e^{-p^{\frac{1}{2}}(p+Y)^{\frac{1}{2}}x}\right\} \tag{5.2.35}$$

For the inverse transform of $\overline{M}(x,p)$ we use the following formulae from Roberts and Kaufman [11].

Inverse transforms of

(1) $g_1(p)g_2(p)$ is $\int_0^t f_1(t-u)f_2(u)\,du$

(2) p is $\delta'(t-u)$

(3) $\dfrac{e^{-p^{\frac{1}{2}}(p+c)^{\frac{1}{2}}x}}{p^{\frac{1}{2}}(p+c)^{\frac{1}{2}}}$ is (i) 0 for $t<x$

(ii) $e^{-\frac{c}{2}t}\, I_0[\frac{c}{2}(t^2-x^2)^{\frac{1}{2}}]$ for $t > x$ (5.2.36)

where I_0 is a modified Bessel function. Rewriting equation (5.2.35) we get,

$$\overline{M}(x,p) = \frac{A}{2(Z-Y)}\left[\left\{(p+Z)\,\frac{e^{-p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}x}}{p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}}\right\} - (p+Y)\left\{\frac{e^{-p^{\frac{1}{2}}(p+Y))^{\frac{1}{2}}x}}{p^{\frac{1}{2}}(p+Y)^{\frac{1}{2}}}\right\}\right]. \tag{5.2.37}$$

Part of the equation (5.2.37) is considered below for inverse transformation

$$(p+Z)\left\{\frac{e^{-p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}x}}{p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}}\right\} .$$

Rearranging the above expression we obtain,

$$p\left\{\frac{e^{-p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}x}}{p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}}\right\} + Z\left\{\frac{e^{-p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}x}}{p^{\frac{1}{2}}(p+Z)^{\frac{1}{2}}}\right\} \tag{5.2.38}$$

Using formulae from equation (5.2.36) to get inverse transform of equation (5.2.38) we obtain,

$$\int_0^t \delta'(t-u)\{e^{-\frac{Z}{2}u} I_0[\tfrac{Z}{2}(u^2-x^2)^{\frac{1}{2}}]\}du + Z\{e^{-\frac{Z}{2}t} I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}]\}$$

$$\text{for} \quad u,t > x \tag{5.2.39}$$

and 0 for $t < x$.

Now $\int_{-\infty}^{\infty} \delta'(x-\alpha)G(\alpha)d\alpha = G'(x)$ from Jones [12]. Application of this formula to equation (5.2.39) gives,

$$\frac{\partial}{\partial t}\{e^{-\frac{Z}{2}t} I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}]\} + Z\{e^{-\frac{Z}{2}t} I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}]\} .$$

Simplifying the above expression gives,

$$-\frac{Z}{2}\, e^{-\frac{Z}{2}t}\, I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}] + e^{-\frac{Z}{2}t}\, I_0'[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}] \cdot \frac{Z}{2} \cdot \frac{1}{2}(t^2-x^2)^{-\frac{1}{2}} \cdot 2t$$

$$+ Z\{e^{-\frac{Z}{2}t}\, I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}]\} .$$

That is,

$$\frac{Z}{2}\, e^{-\frac{Z}{2}t}\, I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}] + 2t\, \frac{Z}{4}(t^2-x^2)^{-\frac{1}{2}} e^{-\frac{Z}{2}t}\, I_0'[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}] \qquad (5.2.40)$$

To simplify the second part of the equation (5.2.40) we use the following formulae from Abromowitz and Stegun [13].

$$\left.\begin{aligned} &\text{(i)} \quad I_0'(x) = I_1(x) \\ &\text{(ii)} \quad I_1'(x) = I_0(x) - \frac{1}{x}\, I_1(x) \end{aligned}\right\} \qquad (5.2.41)$$

These are recurrence relations for modified Bessel functions. Using $I_0'(x) = I_1(x)$ into the equation (5.2.40) we get,

$$\frac{Z}{2}\, e^{-\frac{Z}{2}t}\, I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}] + t\, \frac{Z}{2}\, e^{-\frac{Z}{2}t}\, (t^2-x^2)^{-\frac{1}{2}} I_1[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}] \,. \qquad (5.2.42)$$

Substituting inverse transform equation (5.2.42) of equation (5.2.38) into the equation (5.2.37) we obtain,

$$M(x,t) = \frac{A}{2(Z-Y)}\left[\frac{Z}{2}\,e^{-\frac{Z}{2}t}\{I_0[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}] + t(t^2-x^2)^{-\frac{1}{2}}I_1[\tfrac{Z}{2}(t^2-x^2)^{\frac{1}{2}}]\}\right.$$

$$\left. - \frac{Y}{2}\,e^{-\frac{Y}{2}t}\{I_0[\tfrac{Y}{2}(t^2-x^2)^{\frac{1}{2}}] + t(t^2-x^2)^{-\frac{1}{2}}I_1[\tfrac{Y}{2}(t^2-x^2)^{\frac{1}{2}}]\}\right]$$

$$\text{for} \quad t > x \qquad (5.2.43)$$

$$M(x,t) = 0 \qquad \text{for} \quad t < x \qquad (5.2.44)$$

when the limit of $M(x,t)$ as $t \to x$ is calculated we obtain,

$$\lim_{t\to x} M(x,t) = \frac{A}{2(Z-Y)}\left[\frac{Z}{2}\,e^{-\frac{Z}{2}x}\left\{I_0(0) + \frac{xI_1(0)}{0}\right\}\right.$$

$$\left. - \frac{Y}{2}\,e^{-\frac{Y}{2}x}\left\{I_0(0) + \frac{xI_1(0)}{0}\right\}\right] \qquad (5.2.45)$$

From Abromowitz and Stegun [13] we have,

$$I_0(0) = 1 \qquad \text{and} \qquad I_1(0) = 0 \,.$$

Thus,

$$\lim_{t\to x} M(x,t) = \frac{A}{2(Z-Y)}\left[\frac{Z}{2}\,e^{-\frac{Z}{2}x}\left\{1+\frac{0}{0}\right\} - \frac{Y}{2}\,e^{-\frac{Y}{2}x}\left\{1+\frac{0}{0}\right\}\right] . \qquad (5.2.46)$$

To get the limit of the terms where we have $\frac{0}{0}$ (indeterminate) we use

L'Hospital's rule. Take the term

$$\frac{I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]}{(t^2 - x^2)^{\frac{1}{2}}}$$

applying L'Hospital's rule as $t \to x$ we get,

$$\frac{\frac{\partial}{\partial t} I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]}{\frac{\partial}{\partial t}(t^2 - x^2)^{\frac{1}{2}}} = \frac{I_1'[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]\frac{Z}{2} \cdot \frac{1}{2}(t^2 - x^2)^{-\frac{1}{2}} \cdot 2t}{\frac{1}{2}(t^2 - x^2)^{-\frac{1}{2}} \cdot 2t}$$

$$= \frac{Z}{2} I_1'[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]$$

Using the formula from equation (5.2.41) we obtain,

$$= \frac{Z}{2} \left\{I_0[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}] - \frac{2}{Z} \frac{I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]}{(t^2 - x^2)^{\frac{1}{2}}}\right\} .$$

Therefore after applying L'Hospital's rule when $t \to x$ we get,

$$\lim_{t \to x} \frac{I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]}{(t^2 - x^2)^{\frac{1}{2}}} = \lim_{t \to x} \frac{Z}{2} \left\{I_0[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}] - \frac{2}{Z} \frac{I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]}{(t^2 - x^2)^{\frac{1}{2}}}\right\}$$

which gives,

$$\lim_{t \to x} \frac{I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]}{(t^2 - x^2)^{\frac{1}{2}}} = \lim_{t \to x} \frac{Z}{4} I_0[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}] .$$

Therefore when $t \to x$ we get,

$$\lim_{t \to x} \frac{I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}]}{(t^2 - x^2)^{\frac{1}{2}}} = \frac{Z}{4} .$$

Substituting back into the equation (5.2.45) we obtain,

$$\lim_{t \to x} M(x,t) = \frac{A}{2(Z - Y)} \left[\frac{Z}{2} e^{-\frac{Z}{2}x} \left\{ 1 + x\frac{Z}{4} \right\} - \frac{Y}{2} e^{-\frac{Y}{2}x} \left\{ 1 + x\frac{Y}{4} \right\} \right] \qquad (5.2.47)$$

rewrite equation (5.2.43)

$$M(x,t) = \frac{A}{2(Z - Y)} \left[\frac{Z}{2} e^{-\frac{Z}{2}t} \left\{ I_0[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}] \right. \right.$$

$$\left. + t(t^2 - x^2)^{-\frac{1}{2}} I_1[\frac{Z}{2}(t^2 - x^2)^{\frac{1}{2}}] \right\}$$

$$\left. - \frac{Y}{2} e^{-\frac{Y}{2}t} \left\{ I_0[\frac{Y}{2}(t^2 - x^2)^{\frac{1}{2}}] + t(t^2 - x^2)^{-\frac{1}{2}} I_1[\frac{Y}{2}(t^2 - x^2)^{\frac{1}{2}}] \right\} \right]$$

$$\text{for} \quad t > x \qquad (5.2.43)$$

We want to analyze whether $M(x,t)$ in equation (5.2.43) is convergent or not. In equation (5.2.43) $t > x$ and so if we find $M(x,t)$ for $x = 0$, the result will hold true for any x in general. Therefore substituting $x = 0$ in equation (5.2.43) we get,

$$M(x,t) = \frac{A}{2(Z-Y)}\left[\frac{Z}{2}\, e^{-\frac{Z}{2}t}\left\{I_0\left(\frac{Z}{2}t\right) + I_1\left(\frac{Z}{2}t\right)\right\} - \frac{Y}{2}\, e^{-\frac{Y}{2}t}\left\{I_0\left(\frac{Y}{2}t\right) + I_1\left(\frac{Y}{2}t\right)\right\}\right] \tag{5.2.48}$$

From the tables of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ for different values given in Abromowitz and Stegun [13], it is evident that both $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ are convergent. Therefore $M(x,t)$ in the equation (5.2.48) is convergent. Thus $M(x,t)$ for $t > x$ in the equation (5.2.43) is convergent.

In the equation (5.2.47) there are two basic terms, $e^{-\frac{Z}{2}x}$ and $x\ e^{-\frac{Z}{2}x}$, which are known to be convergent. Therefore $M(x,t)$ is convergent even when $t \to x$. Thus it is seen here that the increasing amplitude in propagation of $M(x,t)$ is eliminated by introduction of internal rotatory damping in the Timoshenko equation.

Part 3. Variation of Parameters Method for a Non-Homogeneous Differential Equation.

After studying the procedure used by Eringen [4] to analyze the simply supported beam with application of a random load, the same procedure was tried to analyze the case of cantilever beam with a random load.

The procedure was not applicable as we could not find a linear solution for the fourth order differential equation (i.e., the Timoshenko equation) that would satisfy boundary conditions, equation itself, and also would be applicable to the procedure shown in [4]. The explanation of this difficulty is as follows.

In the case of the simply supported beam considered in [4], only even derivatives of the deflection entered into the equations for boundary conditions. Therefore, due to the format of applied load, sine terms on both sides of equation (11) in [4] were cancelled.

Because of cancellation of sine terms, the equation became simply and contained only arbitrary constants. These constants became functions of x in the case of a cantilever beam as discussed below.

In the case of a cantilever beam, the boundary conditions are expressed by odd derivatives of the deflection. Because of this and due to the format of applied load, sine and cosine terms enter into the equation. Thus the terms which are constnats in [4] , are functions of x in case of a cantilever beam.

Because of this the format obtained for w_B is not differentiable with respect to x and the moment could not be found, as the moment is the second derivative of w_B with respect to x.

Thus the procedure followed by Eringen [4] is not applicable to a cantilever beam. After trying some other cases it was concluded that the procedure followed by Eringen [4] is only applicable to a particular case of a simply supported beam

Because of the above difficulty the following method is described which is applicable to all kinds of boundary conditions in general. Also it is applicable to any differential equation in general.

Consider the general equation (4.30) from Chapter IV,

$$EI\frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG})\frac{\partial^4 w_B}{\partial x^2 \partial t^2} - (\frac{\beta_0 EI}{kGA} + \beta_1)\frac{\partial^3 w_B}{\partial x^2 \partial t} + \frac{\rho^2 I}{kG}\frac{\partial^4 w_B}{\partial t^4}$$

$$+ (\frac{\rho I \beta_0}{kGA} + \frac{\rho \beta_1}{kG})\frac{\partial^3 w_B}{\partial t^3} + (\rho A + \frac{\beta_0 \beta_1}{kGA})\frac{\partial^2 w_B}{\partial t^2}$$

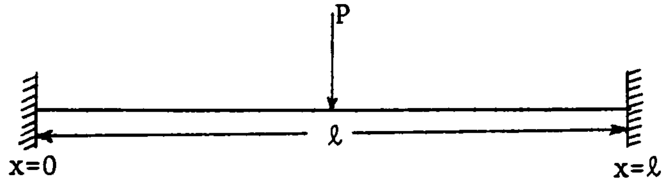
$$+ \beta_0 \frac{\partial w_B}{\partial t} = P \quad . \qquad (5.3.1)$$

A solution can be obtained for equation (5.3.1), but to simplify the process we take $\beta_0 = \beta_1 = 0$. Thus,

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \rho A \frac{\partial^2 w_B}{\partial t^2} = P \tag{5.3.2}$$

where P is an external load and can be a function of only x or both x and t.

Consider the case of a fixed beam as shown in the figure,



Boundary conditions for a fixed beam at $x = 0$ and $x = \ell$ are,

$$\left.\begin{array}{lll} \text{(i)} & w_B(0,t) = w_B(\ell,t) = 0 \\ \text{(ii)} & \frac{\partial w_B}{\partial x}(0,t) = \frac{\partial w_B}{\partial x}(\ell,t) = 0 \end{array}\right\} \tag{5.3.3}$$

Taking the Laplace transform of equations (5.3.2) and (5.3.3) we obtain,

$$EI \frac{\partial^4 \bar{w}_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) p^2 \frac{\partial^2 \bar{w}_B}{\partial x^2} + (\frac{\rho^2 I}{kG} p^4 + \rho A p^2)\bar{w}_B = \bar{P} \tag{5.3.4}$$

$$\left.\begin{aligned} &\text{(i)} \quad \bar{w}_B(0,p) = \bar{w}_B(\ell,p) = 0 \\ &\text{(ii)} \quad \frac{\partial \bar{w}_B}{\partial x}(0,p) = \frac{\partial \bar{w}_B}{\partial x}(\ell,p) = 0 \end{aligned}\right\} \tag{5.3.5}$$

where p is a transformation variable. Equation (5.3.4) is a non-homogeneous differential equation and therefore the solution to the equation is in two parts (i) a complementary solution and (ii) a particular solution. Therefore,

General Solution = Complementary solution + Particular solution.

Complementary solution is for homogeneous part of the equation (5.3.4), i.e.,

$$\text{(i)} \quad EI\frac{\partial^4 \bar{w}_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG})\, p^2 \frac{\partial^2 \bar{w}_B}{\partial x^2} + (\frac{\rho^2 I}{kG}\, p^4 + \rho A p^2)\, \bar{w}_B = 0 \tag{5.3.6}$$

Complementary solution is

$$(\bar{w}_B)_c = \sum_{i=1}^{4} A_i e^{-m_i x} \tag{5.3.7}$$

where A_i are constants and

$$m_i = \pm\left[\frac{(\rho I + \frac{\rho EI}{kG})\, p^2 \pm \{(\rho I + \frac{\rho EI}{kG})^2 p^4 - 4EI(\frac{\rho^2 I}{kG}\, p^4 + \rho A p^2)\}^{\frac{1}{2}}}{2EI}\right]^{\frac{1}{2}} .$$

From above expression $m_3 = -m_1$ and $m_4 = -m_2$. Therefore from equation (5.3.7) we obtain,

$$(\bar{w}_B)_c = A_1 e^{-m_1 x} + A_2 e^{-m_2 x} + A_3 e^{m_1 x} + A_4 e^{m_2 x} . \tag{5.3.8}$$

Now to get a particular solution, we use a method called Variation of Parameters from Ince [14]. In the variation of parameter method, to get a particular solution we substitute the functions of x in place of constants in the complementary solution. That is,

$$\text{Particular Solution} = v_1(x) e^{-m_1 x} + v_2(x) e^{-m_2 x} + v_3(x) e^{m_1 x} + v_4(x) e^{m_2 x} . \tag{5.3.9}$$

To find $v_1(x)$, $v_2(x)$, $v_3(x)$ and $v_4(x)$, we have four simultaneous equations because v and $\bar{P}$ has a single relationship only, p. 122 Ince [14].

$$\left.\begin{aligned}
v_1' u_1 + v_2' u_2 + v_3' u_3 + v_4' u_4 &= 0 \\
v_1' u_1' + v_2' u_2' + v_3' u_3' + v_4' u_4' &= 0 \\
v_1' u_1'' + v_2' u_2'' + v_3' u_3'' + v_4' u_4'' &= 0 \\
v_1' u_1''' + v_2' u_2''' + v_3' u_3''' + v_4' u_4''' &= \bar{P}
\end{aligned}\right\} \tag{5.3.10}$$

where $u_1 = e^{-m_1 x}$; $u_2 = e^{-m_2 x}$; $u_3 = e^{m_1 x}$; $u_4 = e^{m_2 x}$ and dash (')

denotes the derivative with respect to x.

The four simultaneous equations in (5.3.10) can be solved by using the matrix method.

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ u_1' & u_2' & u_3' & u_4' \\ u_1'' & u_2'' & u_3'' & u_4'' \\ u_1''' & u_2''' & u_3''' & u_4''' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \overline{P} \end{bmatrix} \qquad (5.3.11)$$

From the above matrix equation (5.3.11) we obtain,

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ u_1' & u_2' & u_3' & u_4' \\ u_1'' & u_2'' & u_3'' & u_4'' \\ u_1''' & u_2''' & u_3''' & u_4''' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \overline{P} \end{bmatrix} \qquad (5.3.12)$$

By substituting $u_1 = e^{-m_1 x}$; $u_2 = e^{-m_2 x}$; $u_3 = e^{m_1 x}$; $u_4 = e^{m_2 x}$ and simplifying we get values for v_1', v_2', v_3' and v_4'.

$$\left.\begin{aligned} v_1' &= \frac{e^{m_1 x}\overline{P}}{2m_1(m_1^2 - m_2^2)} \\ v_2' &= \frac{e^{m_2 x}\overline{P}}{2m_2(m_1^2 - m_2^2)} \\ v_3' &= -\frac{e^{-m_1 x}\overline{P}}{2m_1(m_1^2 - m_2^2)} \\ v_4' &= -\frac{e^{-m_2 x}\overline{P}}{2m_2(m_1^2 - m_2^2)} \end{aligned}\right\} \qquad (5.3.13)$$

Integrating equation (5.3.13) with respect to x results into,

$$\left.\begin{aligned} v_1(x) &= \int \frac{e^{m_1 x}\overline{P}(x,p)}{2m_1(m_1^2 - m_2^2)}\,dx \\ v_2(x) &= \int \frac{e^{m_2 x}\overline{P}(x,p)}{2m_2(m_1^2 - m_2^2)}\,dx \\ v_3(x) &= -\int \frac{e^{-m_1 x}\overline{P}(x,p)}{2m_1(m_1^2 - m_2^2)}\,dx \\ v_4(x) &= -\int \frac{e^{-m_2 x}\overline{P}(x,p)}{2m_2(m_1^2 - m_2^2)}\,dx \end{aligned}\right\} \qquad (5.3.14)$$

We can find v_1, v_2, v_3 and v_4 which depend on $\overline{P}(x,p)$.

The above method is helpful when load applied is complicated and ready made particular solution is not available.

Once v_1, v_2, v_3 and v_4 are known, the particular solution is also known. The form is

$$v_1(x)e^{-m_1x} + v_2(x)e^{-m_2x} + v_3(x)e^{m_1x} + v_4(x)e^{m_2x} .$$

Therefore, a general solution is

$$\bar{w}_B = \{A_1 + v_1(x)\}e^{-m_1x} + \{A_2 + v_2(x)\}e^{-m_2x} + \{A_3 + v_3(x)\}e^{m_1x} + \{A_4 + v_4(x)\}e^{m_2x} \qquad (5.3.15)$$

Now if we apply boundary conditions (5.3.5) to equation (5.3.15) we get four simultaneous equations in quantities A_1, A_2, A_3 and A_4 which can be solved by using the matrix method as shown before.

Thus $\bar{w}_B$ can be found in the Laplace transform domain. Depending on functions in $\bar{w}_B$, either available Laplace transform tables or contour integrations can be used to find w_B.

This shows that it is rather complex to deal with the equation when boundary conditions involve odd derivatives of deflection since the equation does not simplify as it does in the case of even derivatives of the deflection.

The same method of solution by matrices is also applicable to the homogeneous differential equation.

The above method is useful especially in numerical problems, since matrices can be solved by using a computer.

CHAPTER VI

Discussion of Results.

It is mentioned in the introduction that a single equation can be used in dealing with the analysis of bending moment and stresses.

It is shown in Chapter V, Part 1 and Part 2a that equation (4.31) in terms of 'w_B' is sufficient to analyze the problem of both Miklowitz [2] and Dangler [3]. Also a similar equation is used by Eringen [4] .

In addition, depending on the problem, the external load can be considered either in the equation itself or in the boundary condition. Dangler [3] has used the external load in the equation itself where as in the Chapter V, Part 2a, it is accounted for in the boundary condition.

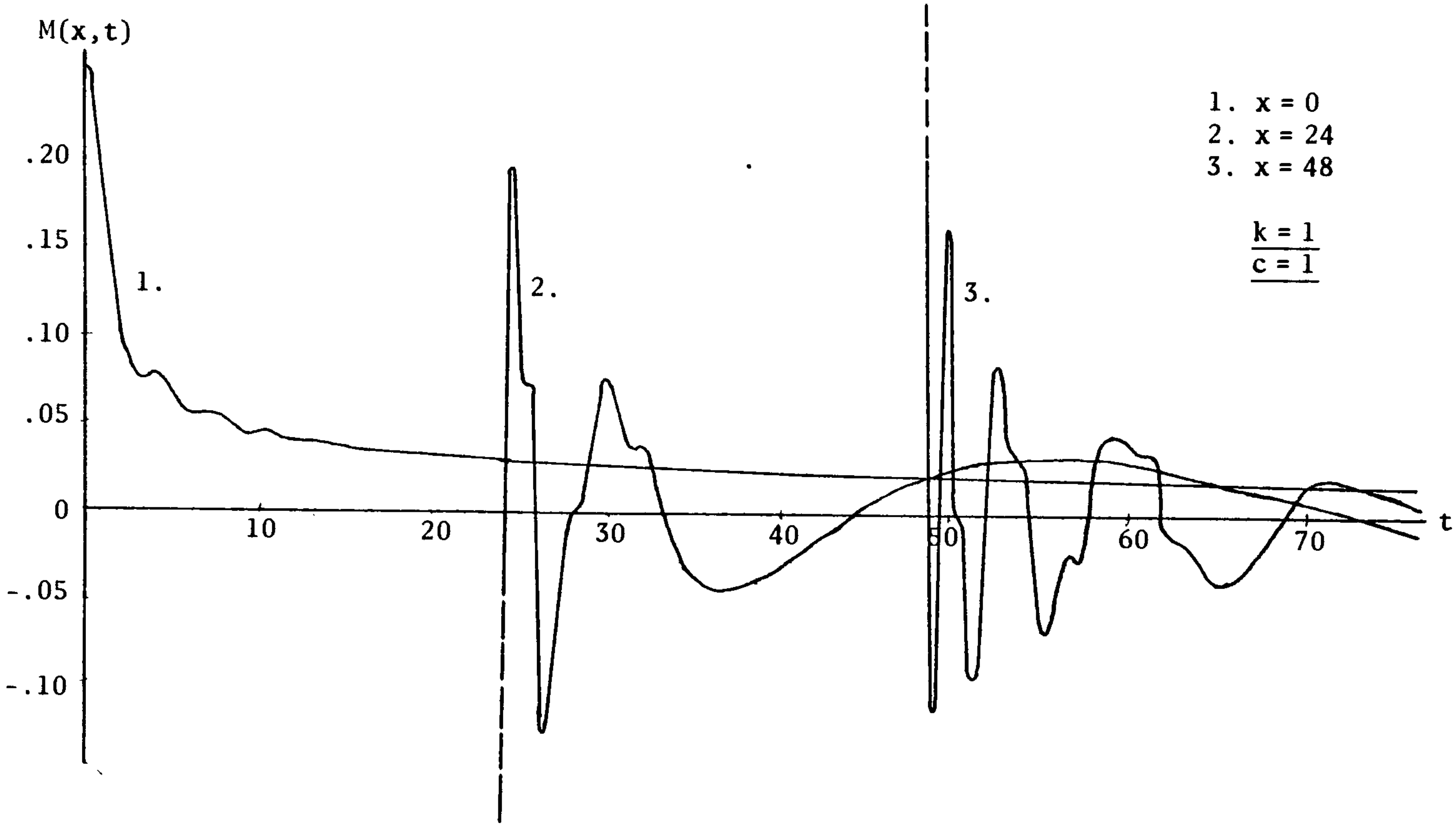
Thus either of the following equations can be used in the analysis depending on the basic problem and the case of the beam considered in the analysis, such as the simply supported beam, infinite beam, fixed beam, cantilever beam, etc.

Equation 1,

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \rho A \frac{\partial^2 w_B}{\partial t^2} = 0$$

Equation 2,

$$EI \frac{\partial^4 w_B}{\partial x^4} - (\rho I + \frac{\rho EI}{kG}) \frac{\partial^4 w_B}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG} \frac{\partial^4 w_B}{\partial t^4} + \rho A \frac{\partial^2 w_B}{\partial t^2} = P$$



(Figure taken from Dangler [3].)

where P means an external load.

Both equations are simple Timoshenko equations without any internal damping.

Dangler [3] has shown that an impulse load applied to an infinite beam without considering an internal damping gives an increasing amplitude in the propagation of bending moment $M(x,t)$. The same case is approached differently in Chapter V, Part 2a and the same conclusions as those reached by Dangler [3] are obtained.

It is seen from the equation (5.2.21), which corresponds to equation (20) in [3], that there is an increasing amplitude in the propagation of $M(x,t)$ when t is equal to x. Also the same phenomenon is seen in the figure taken from [3] on page 53.

The above mentioned problem is solved in Chapter V, Part 2b by considering a modified equation (4.32) in the analysis. This equation is derived by including internal rotatory damping in the theory.

The result of the inclusion of an internal rotatory damping in the Timoshenko equation is seen in the equation (5.2.47).

$$\lim_{t \to x} M(x,t) = \frac{A}{2(Z-Y)} \left[\frac{Z}{2} e^{-\frac{Z}{2}x} \left\{1 + x\frac{Z}{4}\right\} - \frac{Y}{2} e^{-\frac{Y}{2}x} \left\{1 + x\frac{Y}{4}\right\} \right]$$

In the above equation all the terms are known to be convergent. They are $e^{-\frac{Z}{2}x}$, $e^{-\frac{Y}{2}x}$, $x \cdot e^{-\frac{Z}{2}x}$ and $x \cdot e^{-\frac{Y}{2}x}$.

Because the limit of $M(x,t)$ as $t \to x$ is convergent, it is obvious that an increasing amplitude at $t = x$ is eliminated.

Also, it is seen that only rotatory damping is found necessary to assure the convergence of the solution. Eringer [4] used both linear and rotatory damping to get the results in the case of a random loading.

In Chapter V, Part 3, the use of variation of parameter method is explained to get the solution of a non-homogeneous differential equation. In general, this method is applicable to any differential equation.

CHAPTER VII

Suggestions for Further Continuation of the Work

In the case of an impulse load and a random load application to a beam, Euler's theory is found inadequate as the results obtained for bending stresses is divergent which is not acceptable in reality. The Timoshenko beam theory, which takes into account corrections for shear and rotatory inertia, is not always adequate since it does not include internal damping.

It is possible to use linear and rotatory damping concepts as developed and used by Eringen [4]. When both linear and rotatory damping are used together, the equation becomes too complex to analyze. However, it is possible to use either only linear or only rotatory damping in the equation. The inclusion of only one type of damping leads to somewhat easier analysis.

When solving the problem associated with an impulse load, an equation with only linear damping was considered before using an equation with only rotatory damping. The equation with linear damping became very complex, and a complicated contour integration would have to be performed in order to get the inverse Laplace transform of the result. When only rotatory damping was used, the equation became somewhat easier to analyze.

As it is seen from Chapter V, Part 2b, only rotatory damping is adequate for the particular case when $c = 1$. Now the analysis can be carried out in a similar fashion for arbitrary c to find out, if indeed only rotatory damping is sufficient for obtaining satisfactory results.

It is possible to analyze the problem with random load along the same line as Eringen [4] to find out if only rotatory damping is adequate in that case.

It is possible that the results might be closer to reality when both linear and rotatory damping are used but to make a decision about the use of linear damping, the following two questions should be answered.

(i) How much improvement in results is achieved by using linear damping with rotatory damping over using only rotatory damping.

(ii) Is the improvement significant enough to justify more complicated analysis.

To answer the above questions, one can perform an analysis along the same line as Eringen [4]. The problem should be solved two times, once with only rotatory damping and then with both rotatory and linear damping. The results obtained can be compared with each other and it can be found out from the convergence of the equation for bending moment or bending stress that if there is any significant improvement.

From the results of the above analysis it will be possible to decide if the rotatory damping plays a major role in the analysis and if linear damping is significant at all.

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