On the analytic properties of partial wave scattering amplitudes obtained from the Schrödinger equation

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RESUME

On étudie les propriétés analytiques de l'amplitude de diffusion pour une onde partielle donnée à partir des propriétés de la solution radiale de l'équation de Schrödinger pour une énergie complexe. Les résultats connus dans le cas des potentiels à portée finie sont rétablis par cette méthode. On examine ensuite, en se limitant à l'onde S, les propriétés analytiques pour des potentiels décroîssant exponentiellement. On retrouve ainsi une bande d'analyticité commune à tous ces potentiels. Deux exemples sont traités complètement: la cas d'un potentiel produit d'une exponentielle par une fonction sinusoïdale, le cas d'une famille assez générale de potentiels comprenant en particulier le potentiel de Yukawa et une somme de potentiels de Yukawa. Le comportement à l'infini de l'amplitude de diffusion est obtenu et il est possible d'écrire une relation de dispersion pour cette quantité. Sous des conditions plus restrictives, on peut écrire une relation de dispersion pour une quantité reliée à la fonction d'onde à une distance r de l'origine. La méthode employée pour évaluer l'amplitude de diffusion semble être, en outre, une méthode pratique d'approximation valable.

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I. Introduction.

The analytic properties of the partial wave amplitudes have been already established for finite range interactions vanishing rigorously outside a finite region. On the other hand, interesting results have been obtained by Jost who showed the existence of an analycity strip for interactions decreasing as fast as $e^{-\mu r}$ at infinity [1], and by Bowcock and Walecka [2] for a sum of Yukawa potentials; they showed that for a fixed physical angle the scattering amplitude is analytic in the whole complex k^2 plane, except on known cuts and poles, and except perhaps at infinity, provided the interaction is sufficiently weak to make the Born series convergent; from these results at fixed angle one can obviously get information on partial waves.

The method we propose here consists in studying the behaviour of the radial reduced wave function for complex values of the energy or rather of the momentum k; this has already been done in the past by Kapur and Peierls in their study of nuclear reactions [3] and, more recently, by Jost [9]. The results we have obtained by this method overlap with those obtained by the above-mentioned authors, but, in the case of infinite range potentials we can get some additional information, irrespective of the strength of the interaction, on the behaviour of the scattering amplitude for $|k| \to \infty$, which enables one to write dispersion relations. It is possible to treat the case of oscillating potentials, as was suggested by Jost [4], and to show in this case the existence of positive extrapoles outside the imaginary axis for complex k. We should also mention that in the case of finite range potentials we get some additional information on the scattering amplitude in the lower half complex k plane, i.e., in the second Riemann sheet in k^2 plane, for instance on decaying states.

II. Finite range potentials.

a) S wave.

The Schrödinger equation for the reduced wave function can be written

$$u''+k^2u = V(r)u \,, \qquad V(r) = 0 \qquad \text{for} \quad r > r_o \tag{1}$$

the results we derive are also valid for a non local interaction :

$$u''+k^2u = \int V(r,r')u(r')dr' \tag{1'}$$

where

$$V(r,r') = 0 \qquad \text{for} \quad r \text{ or } r' > r_o$$

and

$$V(r,r') = V^*(r',r).$$

Equations (1) or (1') can be solved for complex k; the solution of equation (1) is unique up to a multiplying factor; the solution of equation (1') is unique outside the interaction region [5]. Outside the interaction region the solution can be written explicitly :

$$u = -f(k)e^{ikr} + g(k)\, e^{-ikr} \tag{2}$$

one has to define the S-matrix as

$$S(k) = \frac{f(k)}{g(k)} = \frac{(u'+iku)e^{-ikr}}{(u'-iku)e^{ikr}} \tag{3}$$

for any r larger than r_o. In the case of real k this quantity is just $e^{2i\,\delta(k)}$. What we have to do is to find when the right hand side of (3) is well defined for complex k and to show that it is an analytic function. The numerator and the denominator of (3) are well defined finite quantities so that all we have to do to study the existence of $S(k)$ is to study the zeros of the denominator .

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Combining equation (1) with its complex conjugate, and making use of the reality of the potential, one can derive the following relation :

$$(u^*u'-u^*u')_r \quad + \quad (k^2-k^{*2}) \int_0^r \left| u(r')\right|^2 dr' = 0 \qquad (4)$$

This relation holds for any value of r. For the non local case it holds only for $r > r_0$. Equation (4) enables us to write the modulus of the denominator of (3) in the following way

$$\left| u'-iku \right|_r^2 = \left| u'+\mathrm{Im}\,k\, u \right|_r^2 + (\mathrm{Re}k)^2 \left| u \right|_r^2$$

$$+ 4\,(\mathrm{Re}k)^2\, \mathrm{Im}\,k \int_0^r \left| u\,(r')\right|^2 dr' \qquad (5)$$

This equation shows that the denominator of S cannot vanish in the upper half complex k plane except perhaps on the imaginary axis: When $g(i\varkappa) = 0$, $\varkappa > 0$ this means that $u = \mathrm{ct} \times e^{-\varkappa r}$ outside the interaction region; the zeros of $g(k)$ in the upper half plane correspond to bound states. The vanishing of right hand side of (5), for $(\mathrm{Re}k) \neq 0$ gives a necessary and sufficient condition for the existance of a pole in the lower half complex k plane; this will be explored in IIc.

The holomorphy in the upper half plane will be proven if we show that $S(k)$ as defined by (3) has a derivative. This can be done in the same way as on the real axis (6) (7) ; the starting equation is obtained by comparing the Schrödinger equations for two neighbouring energies :

$$\left[\frac{\partial u}{\partial r}\frac{\partial u}{\partial k} - u\frac{\partial^2 u}{\partial r \partial k} \right]_r = 2k \int_0^r u^2(r')dr' \qquad (6)$$

Assuming $r > r_o$ one can replace u by its asymptotic form (2) and get after some algebraic manipulations :

$$\frac{dS}{dk} = \frac{1}{2ik} \frac{\left[S - e^{-ikr_o}\right]^2}{u^2(r_o)e^{-ikr_o}} \left[2k \int_o^{r_o} u^2 dr + \frac{u(r_o)u'(r_o)}{k}\right] - 2ir_o S, \tag{7}$$

where $u' \equiv \frac{\partial u}{\partial r}$. From (7), we see that S has a well defined derivative whenever it exists. This equation gives also the residue of the poles, especially on the positive imaginary axis: in this case the asymptotic wave function can be taken as $e^{-\gamma r}$ $(k=i\gamma)$, which fixes the normalization. It is easily seen that near such a pole

$$\frac{dS}{dk} = -i\, S^2 \int_0^\infty u^2\, dr \tag{8}$$

Therefore the residue of the pole is $\dfrac{i}{\int_0^\infty u^2 dr}$.

Equation (7) makes apparent also the essential singularity at infinity e^{-ikr_o}, coming from the last term of the right hand side, but it has to be proven that the first term does not contribute. This is easier to do by looking at an alternative expression of $S(k)$; use of the Schrödinger equation leads to the following equalities

$$(u'+iku)_r\, e^{-ikr} = u'(o) + \int_o^r u(r')e^{-ikr'}\, V(r')dr'$$

$$(u'-iku)_r\, e^{ikr} = u'(o) + \int_o^r u(r')e^{ikr'}\, V(r')dr' \tag{9}$$

$S(k)$ is just the ratio of the two right hand sides. For sufficiently large $|k|$ u can be replaced by $\mathrm{Sin}(kr)$ and it can be shown that if $V(r_1) \neq 0$ $(r_1 < r_o)$

$$\left| S(k) e^{2ikr_1} \right| \longrightarrow \infty \quad \text{for} \quad |k| \to \infty, \quad \mathrm{Im}\, k > 0,$$

if $r_2 > r_o$

$S(k) e^{2ikr_2} \longrightarrow 0.$ the precise nature of the essential singularity depends on the way $V(r)$ goes to zero for $r \to r_o$.

The symmetries of the function $S(k)$ come from the symmetries of the solution of the Schrödinger equation :

$$u(k,r) = C(k) u(-k,r)$$

$$u(k^*,r) = C'(k) u^*(k,r) \quad ;$$

Inserting in equation (3) we get

$$S(-k) = [S(k)]^{-1}$$

$$S(k^*) = S^*(-k) \tag{10}$$

This shows that the singularities in the lower half plane are necessarily isolated poles corresponding to zeros in the upper half plane. The complete structure in the complex k plane is given by Figure 1.

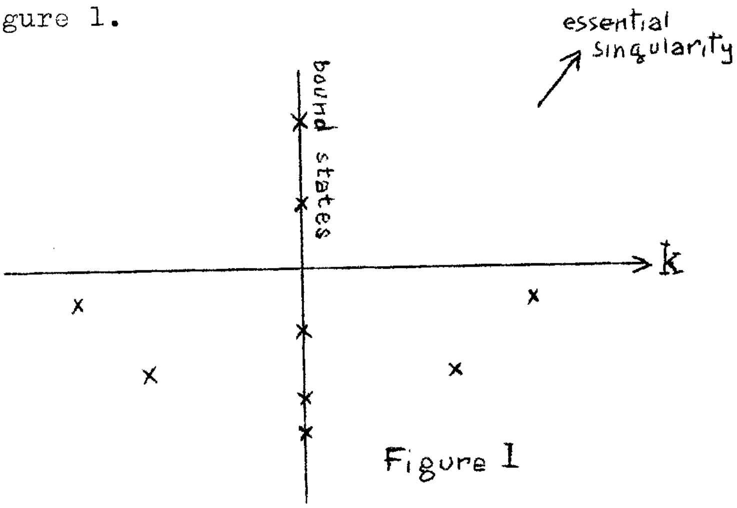

Figure 1

b) Extension to $l \neq 0$.

One has to define solutions of the free Schrödinger equation :

$$O_l = kr\, h_l^{(1)}(kr) \sim e^{\,(kr-\frac{l\pi}{2})} \text{ at infinity}$$
$$I_l = kr\, h_l^{(2)}(kr) \sim e^{-i(kr-\frac{l\pi}{2})} \text{ at infinity} \tag{11}$$

Notice that for k complex $O_l^* \neq I_l$. The S matrix will be defined by

$$S_l(k) = \left(\frac{I_l\, u' - I_l'\, u}{O_l\, u' - O_l'\, u}\right)_{r \geqslant r_o} \tag{12}$$

The singularities of at finite distance are given by

$$O_l\, u' - O_l'\, u = 0 \quad \text{for} \quad r \geqslant r_o.$$

Equation (4) still holds :

$$\left[\frac{u'}{u} - \left(\frac{u'}{u}\right)^*\right]_r + \frac{k^2-k^{*2}}{|u_r|^2} \int_0^r |u(r')|^2\, dr' = 0$$

so that a necessary condition for the existence of a pole is

$$\left[\frac{O'}{O} - \left(\frac{O'}{O}\right)^*\right]_{r_o} + \frac{k^2-k^{*2}}{u_r^2} \int_0^r |u(r')|^2\, dr' = 0 \tag{13}$$

Now we can apply the trick used to get equation (4) to the free Schrödinger equation, and <u>assuming</u> $\mathcal{I}m\, k > 0$, we obtain

$$\left[\frac{O'}{O} - \left(\frac{O'}{O}\right)^*\right]_{r_o} = \frac{k^2-k^{*2}}{|O|^2_{r_o}} \int_{r_o}^{\infty} |O|^2\, dr ,$$

because $O(\infty) = 0$.

Therefore condition (13) becomes :

$$(k^2-k^{*2}) \left[\frac{\int_0^{r_o} |u|^2 dr}{|u|^2_{r_o}} + \frac{\int_{r_o}^{\infty} |O|^2 dr}{|O|^2_{r_o}} \right] = 0. \qquad (14)$$

So that there cannot be any singularity in the upper half plane except for $\mathcal{R}ek = 0$. It seems unnecessary to go into the details as was done in IIa).

c) Lower half plane. Poles near the real axis.

In what follows r_o is fixed and chosen as small as possible, outside the interaction region. Equation (5) shows that a pole given by

$$f(\mathcal{R}ek, \mathcal{I}mk) \equiv \left| u' - iku \right|^2_{r_o} = 0 \qquad (15)$$

cannot be infinitely near the real axis because $u(k, r_o)$ and $u'(k, r_o)$ are smooth functions of k. This qualitative statement can be put in a more quantitative way, when the pole is very close to the real axis, which corresponds to a metastable state (e.g., an α particle in a nuclear potential); assume that such a pole k exists. Since the function $f(\mathcal{R}ek, \mathcal{I}mk)$ is everywhere $\geqslant 0$ it will have a minimum on the real axis, for variable $\mathcal{R}ek$ in the neighbourhood of k_p (see Figure 2)

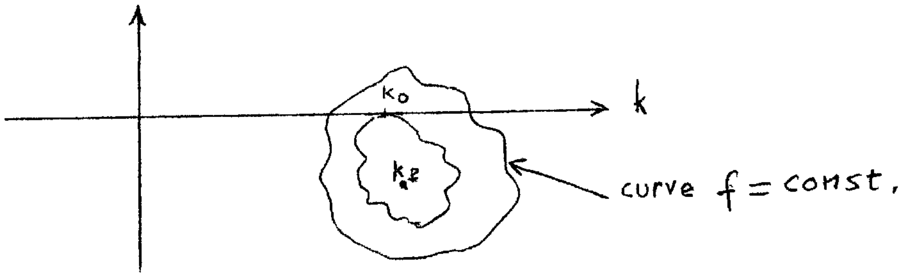

Figure 2

From the definition of k_o it follows that

$$\frac{\partial f}{\partial \,\mathrm{Re}k} = 2\,\mathrm{Re}\left[\overline{u}'+i\,ku)\frac{\partial}{\partial k}(u'-i\,ku)\right] = 0 \qquad (16)$$

Let us try to expand $u'-iku$ in the neighbourhood of k_o:

$$\frac{\partial}{\partial k}(u'-iku)\Big|_{k=k_o} = \frac{i}{u'+iku}\,\mathrm{Im}\left[\overline{u}'+iku)\frac{\partial}{\partial k}(u'-iku)\right], \text{ using (16).}$$

and, using equation (6) $\left(u' \equiv \frac{\partial u}{\partial r}\right)$:

$$\frac{\partial}{\partial k}(u'-iku)\Big|_{k=k_o} = \frac{-i}{u'_{k_o}+ik_o u_{k_o}}\left[2k_o^{\,2}\int_o^{r_o} u_{k_o}^2(r)\,dr+u'_{k_o}u_{k_o}\right] \qquad (17)$$

so that in the neighbourhood of $k = k_o$

$$(u'-iku) = (u'_{k_o}-ik_o u_{k_o})\left[1-i(k-k_o)\frac{2k_o^2\int_o^{r_o} u_{k_o}^2(r)\,dr+u'_{k_o}u_{k_o}}{u'^{\,2}_{k_o}+k_o^{\,2}u_{k_o}^{\,2}}\right] \qquad (18)$$

The assumption that k_p is very near the real axis means that

$$k_p-k_o \simeq -i\,\frac{u'^{\,2}_{k_o}+k_o^{\,2}\,u'_{k_o}\,u_{k_o}}{2k_o^{\,2}\int_o^{r_o} u_{k_o}^{\,2}(r)\,dr+u'_{k_o}u_{k_o}} \qquad (19)$$

the expansion (18) is valid provided the modulus of (19) is very small, i.e.,

$$2k_o^{\,2}\int_o^{r_o} u^2(r)\,dr \gg (u'^{\,2}_{k_o}+k_o^{\,2}u_{k_o}^{\,2})\,r_o$$

This condition is just the condition for the existence of the decaying state [8], which says that the external wave function must be as small as possible as compared to the internal wave function. We recognize on (19) the expression for the life-time of the decaying state (up to trivial factors).

Conversely, setting $\int_0^{r_0} u^2(r)dr = ct$, one might ask if a pole corresponds to a minimum of

$$\frac{(u'^2 + k^2 u^2)}{\int_0^{r_0} u^2(r)dr}$$

for real k. If the value of the minimum happens to be very small, the zero of the expansion (18) lies very near k_0 so that the use of the Taylor expansion is a posteriori justified and it appears very likely that a pole given by (19) exists.

III. Potentials decreasing faster than an exponential.

In what follows we consider local potentials such that there exists a μ giving

$$\lim_{r \to \infty} V(r)e^{+\mu r} = 0 ,$$

and we restrict ourselves to S waves.

a) The analyticity strip.

One can try to define the S matrix for complex k by generalizing (3)

$$S(k) \overset{?}{=} \frac{\lim_{r \to \infty} [(u'+iku)e^{-ikr}]}{\lim_{r \to \infty} [(u'-iku)e^{ikr}]} \qquad (20)$$

The problem is to find when these limits exist and when the denominator is different from zero. According to equation (9) these limits, if they exist, are respectively

$$N(k) = u'(0) + \int_0^\infty u(r)\, V(r) e^{-ikr} dr$$
$$D(k) = u'(0) + \int_0^\infty u(r)\, V(r) e^{+ikr} dr \,, \tag{21}$$

and

$$S(k) = \frac{N(k)}{D(k)}$$

Now the arguments given to show that the Schrödinger equation admits two independent solutions

$$f(r,k)e^{-ikr} \quad , \quad g(r,k)e^{+ikr} \tag{22}$$

such that

$$f(\infty,k) = g(\infty,k) = 1$$

provided $V(r)$ decreases at least as fast as $\frac{1}{r^2}$ are still valid[9] for complex k. For $\mathrm{Im}\, k > 0$ the solution which blows up at infinity is the first one. $u(r)$ is a combination of the two. So we are certain that $u(r)e^{ikr}$ tends to a finite limit (which may be zero) at infinity, for $\mathrm{Im}\, k > 0$. This means that the quantity $D(k)$ exists. On the other hand, $N(k)$ may be written

$$N(k) = u'(0) + \int_0^\infty u(r)e^{ikr} \left[V(r)e^{-2ikr} \right] dr$$

So $N(k)$ is a well defined quantity for

$$2\, \mathrm{Im}\, k < \mu \tag{23}$$

We still have to study the zeros of $D(k)$ By looking at the two independent solutions (22), one is led to the following alternative : for $\mathrm{Im}\,k > 0$, either u behaves like e^{ikr} for $r \to \infty$ or u behaves like e^{-ikr}. So to prove that $D(k) = \lim_{r\to\infty} (u'-iku)e^{ikr} \neq 0$ it is sufficient to prove that $\lim_{r\to\infty} |u'-iku| \neq 0$. Now equation (5), established inside the range of the potential is still valid in the present case. It shows that

$$\lim_{r\to\infty} \left|u'-iku\right|^2 > 4(\mathrm{Re}k)^2 \mathrm{Im}\,k \int_0^\infty |u(r')|^2 dr' \qquad (24)$$

the right hand side is certainly a positive quantity for $\mathrm{Im}\,k > 0$ $\mathrm{Re}k \neq 0$. This means that $D(k)$ is a well defined quantity in the upper half plane, certainly different from zero outside the imaginary axis. When $\lim_{r\to\infty} (u'-iku)e^{ikr} = 0$ this means that $u(r)$ is $e^{ikr} = e^{-\varkappa r}$ $(k=i\varkappa, \varkappa > 0)$ so that the zeros of $D(k)$ on the positive imaginary axis correspond to <u>true bound states</u>.

Therefore, we have shown that it is possible to define $S(k)$ in the strip $0 \leqslant \mathrm{Im}\,k < \frac{\mu}{2}$, with possible poles, corresponding to true bound states, on the imaginary axis.

Now it is not too difficult to convince ourselves that the functions

$$N_{r_0}(k) = u'(0) + \int_0^{r_0} u(r)V(r)e^{-ikr}dr$$

$$D_{r_0}(k) = u'(0) + \int_0^{r_0} u(r)V(r)e^{+ikr}dr$$

converge uniformly to $N(k)$ and $D(k)$ in the region

$$0 \leq \mathrm{Im}\, k \leq \frac{\mu}{2} - \epsilon , \qquad |\mathrm{Re}\, k| < A ,$$

with ϵ arbitrarily small and A arbitrarily large. In this region $\lim u(r)e^{ikr} = C(k)$ (for instance with a normalization $u'(0) = ct$) and $|C(k)|$ has a maximum value M; then, for r and r' large enough

$$\left|N_r(k) - N_{r'}(k)\right| < \left|e^{-\epsilon r} - e^{-\epsilon r'}\right| \times \text{const.}$$

and

$$\left|D_r(k) - D_{r'}(k)\right| < \left|e^{-\mu r} - e^{-\mu r'}\right| \times \text{const.}$$

Excluding now a small region around the zeros of $D(k)$ on the imaginary axis, one can also show that $S_{r_o}(k) = \dfrac{N_{r_o}(k)}{D_{r_o}(k)}$ converges uniformly to $S(k)$.

It happens that $S_{r_o}(k)$ is the S matrix for the finite range potential :

$$\begin{cases} V(r) & \text{for } r < r_o \\ 0 & \text{for } r > r_o \end{cases}$$

for which analytic properties are known. From known theorems (15) it follows that $S(k)$ is holomorphic in the strip $0 \leq \mathrm{Im}\, k < \frac{\mu}{2}$ except for possible singularities on the line $\mathrm{Im}\, k = \frac{\mu}{2}$ and at infinity, and poles due to bound states on the imaginary axis.

b) Analytic properties in the whole upper half complex plane.

When the potential is precisely known, it is possible to get more than an analyticity strip for the S matrix, in spite of the breakdown of equation (20) for $\mathrm{Im}\, k \geqslant \frac{\mu}{2}$. Assume that the two independent solutions (22) are known. Then one can write the solution of the Schrödinger equation as

$$u(r) = f(r,k)e^{-ikr} - g(r,k)\,\frac{f(0,k)}{g(0,k)}\,e^{ikr} ,$$

provided (25)

$$g(0,k) \neq 0 \quad ; \quad \text{if} \quad g(0,k) = 0$$

$$u(r) = g(r,k)e^{ikr} \tag{25'}$$

It is natural to define the S matrix for complex k as

$$S(k) = \frac{f(0,k)}{g(0,k)} \tag{26}$$

since this definition holds for real k.

One can immediately show that the zeros of $g(0,k)$ correspond to bound states and lie on the imaginary axis : these zeros, according to (25'), are such that $\lim_{r\to\infty} (u'-iku) = 0$, which as already shown in the preceding section can only occur for $\mathrm{Re}\,k = 0$; the asymptotic form is then

$$u(r) \simeq e^{-\varkappa r} \qquad \varkappa > 0 .$$

By inserting in the Schrödinger equation, one obtains for $f(r,k)$ and $g(r,k)$ the following equations

$$\left.\begin{aligned} f'' - ikf' &= V(r)f, \quad \text{with} \quad \lim_{r\to\infty} f(r,k) = 1 \\ g'' + ikg' &= V(r)g, \quad \text{with} \quad \lim_{r\to\infty} g(r,k) = 1 \end{aligned}\right\} \tag{27}$$

Assuming provisionally, existence and unicity of these functions, we readily get the following symmetries :

$$f(r,-k) = g(r,k)$$

$$f^*(r,k^*) = g(r,k) ,$$

from which follows

$$\mathcal{S}(k) = \mathcal{S}^{-1}(-k) , \qquad \mathcal{S}(-k^*) = \mathcal{S}^*(k) .$$

Following the classical procedure of neglecting first the second order derivatives, we get the behaviour of the k functions for $r \to \infty$:

$$f(r,k) = 1 + \frac{1}{2ik} \int_r^\infty V(r')dr' + \ldots.$$

$$g(r,k) = 1 - \frac{1}{2ik} \int_r^\infty V(r')dr' + \ldots$$

This is all we can say without more detailed information on the shape of the potentials. We shall now thoroughly treat examples, some of which are quite general. Our study will be limited to upper half k plane or first Riemann sheet of the energy plane.

c) Examples

1. Exponential potential, Sum of exponential potentials oscillating potential.

The first case admits an exact solution [10) 11)] and, on the other hand, has been treated by Jost [9)] using an expansion of the functions $f(r,k)$ $g(r,k)$ of the type :

$$1 + \sum_{1}^{\infty} C_n \, e^{-n\mu r}$$

In this way one shows that $\frac{f(0,k)}{g(0,k)} = S(k)$ is holomorphic in the upper complex plane except for poles at $2ik + n\mu = 0$, and that $\lim_{|k| \to \infty} S(k) = 1$ in any direction, provided $\mathrm{Re}\,k \neq 0$. Jost has also applied his treatment to a sum of exponential potentials

$$V = Ae^{-pr} + B\,e^{-qr}$$

then

$$f = \sum_{mn} C_{mn}(k)\; e^{-(mp+nq)r} \; ,$$

$$g = \sum_{mn} D_{mn}(k)\; e^{-(mp+nq)r} \; .$$

The coefficients C_{mn} and D_{mn} are obtained by recursion formulas.

One can notice that this treatment can be extended without trouble to the case

$$A = B = \frac{V_o}{2} \; ,$$

$$p = q^* = \mu + i\alpha \; ,$$

i.e., (30)

$$V(r) = V_o \operatorname{Cos}(\alpha r)\, e^{-\mu r} \; .$$

Then the possible redundant poles (besides the bound states) are given by

$$k = \frac{(n-m)\alpha}{2} + i\,\frac{(m+n)\mu}{2} \; ,$$

some of which lie outside the imaginary axis. It is important to evaluate the residues of these poles to see if they really exist, since from general arguments [12] one would not expect them. When $2ik + Mp + Nq = 0$ all the $C_{M+m\ N+n}$'s $(m \geqslant 0,\ n \geqslant 0)$ are singular. In the neighbourhood of such a singularity, the recursion formula reads :

$$C_{M+m,N+n} \quad [mp + nq - 2ik]\ [mp+nq]$$

$$= \frac{V_o}{2} \left[C_{M+m-1,N+n} + C_{M+m,\ N+n-1} \right]$$

The $C_{M+m,N+n}$ are determined in terms of C_{MN} which is singular and $C_{M-1,N+1+\lambda}$ $C_{M+1+\lambda,N-1}$ $(\lambda \geqslant 0)$ which are not singular. So, to get the singular part of $C_{M+m,N+n}$ one can take the $C_{M-1,N+1+\lambda}$'s and the $C_{M+1+\lambda,N-1}$ equal to zero. Then by comparison with the recursion formula for the D it can be seen that

$$\frac{\text{singular part of } C_{M+m,N+n}}{C_{MN}} = D_{mn}$$

So $f(r,k)$, near $2ik + Mp+Nq = 0$ behaves like $C_{MN}g(r,k)$. In the special case $2ik + Mp \equiv 2ik + M(\mu+i\alpha) = 0$,

$$C_{MO} \cong \frac{V_o^M}{2^M (Mp+2ik)\ [(M-1)!]^2 M\mu^{2M-1}} .$$

Therefore the residues of

$$S(k) = \frac{f(0,k)}{g(0,k)}$$

at the poles

$$k = \frac{M(i\mu \pm \alpha)}{2}$$

are certainly different from zero because $g(0,k)$ is, from general arguments different from zero, so that there is no indetermination in the ratio defining $S(k)$. We have also shown by direct calculation that the first poles corresponding to $MN \neq 0$ exist. One gets the picture given in Figure 3.

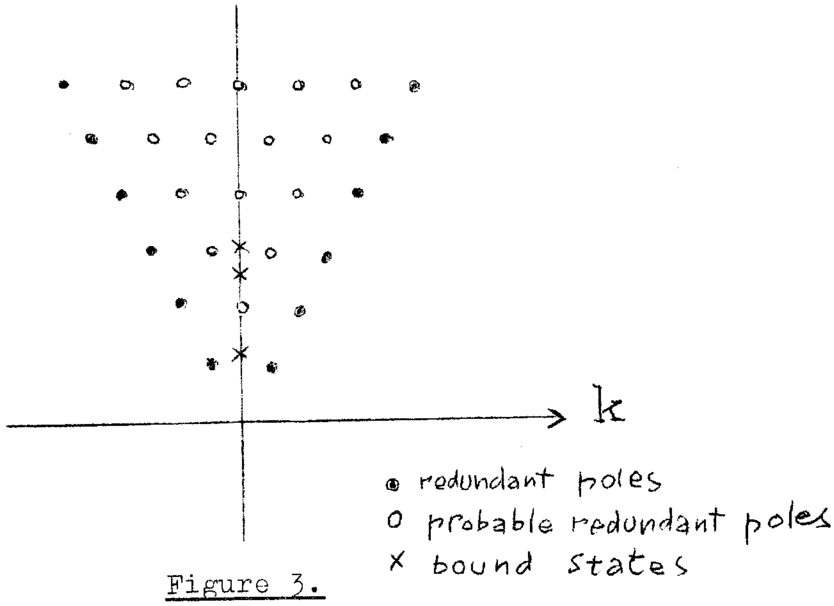

Figure 3.

The problem is now to understand why the arguments against the existence of such poles, based on conservation of probability, fail in the present case. These arguments require that the asymptotic behaviour of the wave function be :

$$e^{-ikr} - S(k)e^{ikr} \quad .$$

Here the wave function is

$$[1+C_{10}\ e^{-(\mu+i\alpha)r} + C_{01}\ e^{-(\mu-i\alpha)r} + \ldots]\ e^{-ikr} - [1+\ldots]\ S(k)e^{ikr}$$

To identify this expression with the preceding one, it is necessary that $e^{-\mu r}\ e^{-ikr}$ be negligible as compared to e^{ikr}. This means that k must be in the analyticity strip $0 < \mathrm{Im}\, k < \frac{\mu}{2}$.

Outside this strip these arguments are no longer valid.

2) Yukawa potential and generalizations.

It is quite clear that example one could be generalized to any finite sum of real exponential potentials, $\sum_i A_{p_i} e^{-p_i r}$, then the expected redundant poles are given by $\sum n_i p_i + 2ik = 0$, on the imaginary axis. So, above the pole given by the minimum p_i, $k = \frac{i p_{min}}{2}$, the poles will become more and more dense as the number of p_i's increases. This enables us to guess what will happen for a potential

$$V(r) = V_0 e^{-\mu r} \int_0^\infty C(\alpha) e^{-\alpha r} d\alpha, \text{ with } |C(\alpha)| < M \tag{31}$$

One expects a cut on the imaginary axis starting at $k = i\frac{\mu}{2}$.
This family of potentials contains as a special case, the Yukawa potential for $C(\alpha) = ct.$ and sums of Yukawa potentials when $C(\alpha)$ is made of a sum of step functions. It is equivalent to the family

$$\frac{e^{-\mu r}}{r} \int_0^\infty C'(\alpha) e^{-\alpha r} d\alpha ,$$

where $C'(\alpha)$ may now be a distribution, with the restriction that $\int_0^\infty C'(\alpha) d\alpha$ is finite (this excludes potentials behaving as $\frac{1}{r^{1+\epsilon}}$ at the origin).

Here the natural "ansatz" for the functions f and g is:

$$\begin{aligned} f(r,k) &= 1 + e^{-\mu r} \int_0^\infty \rho_k(\alpha) e^{-\alpha r} d\alpha \\ g(r,k) &= 1 + e^{-\mu r} \int_0^\infty \sigma_k(\alpha) e^{-\alpha r} d\alpha . \end{aligned} \tag{32}$$

Inserting in equations (27) with potential (31), we get, making use of the known properties of products of Laplace transforms, and equating the coefficients of $e^{-\alpha r}$ in both sides of the equation thus obtained;

$$\rho_k(\alpha)\,[\alpha+\mu]\,[\alpha+\mu+2ik] = V_0\left[C(\alpha) + \int_0^{\alpha-\mu} C(\alpha-\mu-\beta)\,\rho_k(\beta)\,d\beta\right]$$

$$\sigma_k(\alpha)\,[\alpha+\mu]\,[\alpha+\mu-2ik] = V_0\left[C(\alpha) + \int_0^{\alpha-\mu} C(\alpha-\mu-\beta)\,\sigma_k(\beta)\,d\beta\right] \qquad (33)$$

where one should keep in mind that

$$\rho(\lambda) = \sigma(\lambda) = C(\lambda) = 0 \text{ for } \lambda < 0^{*)} .$$

These equations are not true integral equations. Assume $0 < \alpha < \mu$ then the integrals do not contribute and

$$\rho(\alpha) = V_0\, C(\alpha)\,[\alpha+\mu]^{-1}\,[\alpha+\mu+2ik]^{-1}$$

$$\sigma(\alpha) = V_0\, C(\alpha)\,[\alpha+\mu]^{-1}\,[\alpha+\mu-2ik]^{-1} \qquad (0 \leqslant \alpha \leqslant \mu)$$

The functions ( ) and ( ) can be determined by recursion. Assume $\rho(\alpha)$ and $\sigma(\alpha)$ are known for $0 \leqslant \alpha \leqslant n\mu$. Then the right hand sides of equations (33) are known for $0 \leqslant \alpha \leqslant (n+1)\mu$. It is clear that $\sigma_k(\alpha)$ is everywhere a well defined quantity when k is in the upper half plane. $\rho_k(\alpha)$ is well defined when $\alpha+\mu+2ik$ cannot vanish, i.e., outside a cut along the imaginary axis starting at $k=i\frac{\mu}{2}$. The problem is now to study the convergence of the integrals

$$1 + e^{-\mu r}\int_0^{\infty}\rho_k(\alpha)e^{-\alpha r}d\alpha, \quad 1+e^{-\mu r}\int_0^{\infty}\sigma_k(\alpha)e^{-\alpha r}d\alpha .$$

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*) One can check that if the functi ns defined by (32) and (33) exist they satisfy the Schrödinger equation with the right potential.

This is possible because equations (33) are very suitable to get upper bounds of $|\rho_k(\alpha)|$ and $|\sigma_k(\alpha)|$. The following results are demonstrated in appendix I :

- $g(r,k)$ is a holomorphic function in the whole upper complex plane and $\lim_{|k|\to\infty} (g(r,k) - 1) |k|^{1-\eta} = 0$ (η arbitrarily small). - $f(r,k)$ is a holomorphic function such that in $\lim_{|k|\to\infty} (f(r,k) - 1) |k|^{1-\eta} = 0$ in the union of the domains :

$$0 \leqslant \mathrm{Im}\, k \leqslant \frac{\mu}{2} - \epsilon , \quad 0 \leqslant \mathrm{Arg}k \leqslant \frac{\pi}{2} - \epsilon , \quad 0 \leqslant \pi - \mathrm{Arg}k \leqslant \frac{\pi}{2} - \epsilon$$

(ϵ arbitrarily small)

Moreover $\lim_{\mathrm{Im}k \to +\infty} f(r,k) = 1$ provided

$$\mathrm{Re}k > C \, [\mathrm{Im}k]^{-n}$$

where n is an arbitrary positive number, so that the limit of $f(r,k)$ as $|k| \to \infty$ is unity even if one asymptotically approaches the imaginary axis at infinity.

Concerning the existence of the limiting function

$$\lim_{\substack{\mathrm{Re}k \to 0 \\ \mathrm{Re}k > 0}} f(r,k)$$

we have only considered two particular cases, but it is clear that much more general cases could be solved :

- if the weight function in the integral representation of the potential is such that

$$\left| C(\alpha) - C(\beta) \right| < B \left| \alpha - \beta \right| \tag{34}$$

in addition to $|C| < M$, where B is independent on α and β, the weight function $\rho_k(\alpha)$ has a pole at $\alpha+\mu+2ik = 0$, but the limit of $f(r,k)$ exists everywhere along the cut, except at $2ik+\mu = 0$ where there is a logarithmic singularity. This case does not contain the Yukawa potential because then $C(\alpha)$ has a discontinuity at $\alpha = 0$.

- for the Yukawa potential $C(\alpha) = 1$ for $\alpha > 0$. The weight function $\rho_k(\alpha)$ has a pole at $\alpha+\mu+2ik = 0$ and a logarithmic singularity at $\alpha+2ik = 0$. For $\mu+2ik \neq 0$ the limit of $f(r,k)$ exists along the cut, but for $\mu+2ik = 0$ the limit of $f(r,k)$ has a logarithmic singularity. This singularity is harmless when one writes dispersion relations.

These properties are sufficient to give a dispersion relation for the quantity

$$A(E) = \frac{1}{2i}\left[\frac{f(0,k)}{g(0,k)} - 1\right] \qquad (E = k^2)$$

if one remembers that the zeros of $g(0,k)$ lie on the positive k imaginary axis (or E real, negative) and correspond to bound states :

$$\operatorname{Re} A(E_0) = \text{bound states} + \frac{P}{\pi}\int_0^{\infty} \frac{\operatorname{Im} A(E+i\epsilon)\,dE}{E-E_0}$$

$$- \frac{1}{\pi}\int_{\left(\frac{\mu}{2}\right)^2}^{\infty} \frac{\operatorname{Im} A(-E+i\epsilon)\,dE}{E+E_0}$$

This relation is not very useful from a practical point of view, because one has a very large unphysical range. If we assume that the N-P potential belongs to the above considered family, we see that the pole corresponding to the deuteron lies at $E = -0.105\,\mu^2$, while the cut begins at $-0.25\,\mu^2$.

The quantity $\mathcal{S}(r,k) = \frac{f(r,k)}{g(r,k)}$ has the same analytic properties as $\mathcal{S}(k)$. However, there is no general argument to guarantee that $g(r,k)$ has no zeros outside the imaginary axis. We have shown that a sufficient condition to avoid zeros in the upper half plane is that $|V_0| \mathcal{M} < \frac{2}{3}$ (appendix II). This is certainly far from being necessary. Under this condition one can write a dispersion relation for

$$A(r,E) = \frac{1}{2i} \left(\mathcal{S}(r,k) - 1 \right) ,$$

which can be considered as a dispersion relation for the wave function. This problem is being studied for finite range potentials by Bosco and Fubini [13]. For real k or $E > 0$, $|S(r,k)| = 1$, so that there is an obvious connection between the real and imaginary partsof $A(r,E)$. This would permit to transform the dispersion relation for $A(r,E)$ into a Low integral equation, but the presence of the unphysical range makes it rather useless.

A by-product of equations (33) is a new practical method of computation of the scattering matrix and of the bound states. Some work has already been done in that direction by Koppe [14]. In appendix III, we show that one can very quickly get an accurate value of the Yukawa well depth parameter.

IV Concluding Remarks.

In the case of finite potentials, our method is straightforward but does not bring anything really new. Its use is rather to guide us in the case of infinite range potentials. In the case of infinite range potentials, we restricted ourselves to S wave; we hope to be able to extend this method to higher waves soon. The main results we get are, first, the anomalous poles in the upper half plane outside the imaginary axis for the oscillating potential, second, the complete analytic behaviour for Yukawa potential and generalization, including the behaviour at infinity, third, analytic properties of a quantity related to the wave function at a point r fixed, giving rise to generalized dispersion relations at least if the potential is not too strong. Except for the last result, all the other results are derived without any assumption on the strength of the potential. This is due to the fact that we have separated the spurious poles or cuts arising in the complex plane, which appear in the numerator of the S matrix, from the poles due to bound states which appear as zeros of the denominator .

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NOTE:

While this paper was mimiographed, we received a preprint by R.E. Peierls, in which some of the results presented here are derived under more restrictive assumptions. The behaviour at infinity is not studied.

APPENDIX I.

In the upper half plane $|\alpha+\mu-2ik| > |\alpha+\mu|$
From the inequality

$$\left|\sigma_k(\alpha)\right| < \frac{|V_0|\ M}{(\alpha+\mu)^2}\ \left[1+\int_0^{\alpha-\mu}\left|\sigma_k(\beta)\right|\ d\beta\right] \qquad (I,1)$$

one sees that an upper bound of $\sigma_k(\alpha)$ is $\Sigma(\alpha)$ given by

$$\Sigma(\alpha) = \frac{|V_0|\ M}{(\alpha+\mu)^2}\ \left[1+\int_0^{\alpha-\mu}\Sigma(\beta)\ d\beta\right] \qquad (I,2)$$

where $\Sigma(\alpha) = 0$ for $\alpha < 0$. From (I,2) one can easily get

$$1+\int_0^{\infty}\Sigma(\alpha)\ d\alpha < \frac{1+\int_0^{\alpha_1}\Sigma(\alpha)d\alpha}{1-\frac{|V_0|\ M}{\alpha_1+\mu}} \qquad (I,3)$$

provided $\alpha_1+\mu > M\,|V_0|$, and also with (I,1) :

$$\int_A^{\infty}\left|\sigma_k(\alpha)\right|\ d\alpha < \frac{|V_0|\ M}{A+\mu}\ \left(1+\int_0^{\infty}\Sigma(\alpha)d\alpha\right)$$

This establishes the <u>uniform</u> and <u>absolute</u> convergence of the integral defining $g(r,k)$ in the whole upper half plane. Since one can see by recursion that $1+e^{-\mu r}\int_0^A e^{-\alpha r}\sigma_k(\alpha)d\alpha$ is a holomorphic function, the same is true for $g(r,k)$ [15].

Since $\lim_{|k| \to \infty} [1 + \int_0^A |\sigma_k(\alpha)| \, d\alpha] = 1$, the same is true for $g(r,k)$ more precisely using equations (33) and (I,3) :

$$\int_0^\infty |\sigma_k(\alpha)| \, d\alpha < \frac{C_1}{|k|} \int_0^A \frac{d\alpha}{\alpha+\mu} + \frac{C_2}{A+\mu}$$

taking $A = |k|$, we see that this goes to zero as fast as $\frac{\log |k|}{|k|}$.

With some precautions, one can extend this treatment to $\rho_k(\alpha)$: - if

$$0 \leqslant \text{Im}k \leqslant \frac{\mu}{2} - \epsilon \quad |\alpha+\mu-2ik| \quad |\alpha+\mu| > (\alpha+\epsilon)^2.$$

In this domain the existence and analyticity of $f(r,k)$ are established in the same way as for $g(r,k)$.

- if

$$\epsilon \leqslant |\text{Arg } i\, k| \leqslant \frac{\pi}{2}$$

$$|(\alpha+\mu)\ (\alpha+\mu+2ik)| \geqslant |\alpha+\mu|^2 \ |\cos \epsilon|$$

So we have only to replace in inequality (I,1) $|V_0|$ by $\left|\frac{V_0}{\cos\epsilon}\right|$ and the holomorphy of $f(r,k)$ is established. In the same way, one can also, using

$$|(\alpha+\mu)\ (\alpha+\mu+2ik)| > 2\ |\alpha+\mu| \ |k| \ |\cos \epsilon| ,$$

show that

$$\int_0^\infty |\rho_k(\alpha)| \, d\alpha \longrightarrow 0$$

as fast as $\frac{\log|k|}{|k|}$ when $|k| \rightarrow \infty$.

The above conditions define a domain which does not contain the cut, and does not give the behaviour of $f(r,k)$ for $\mathrm{Im}k \rightarrow +\infty$, $\mathrm{Re}k$ finite. One has to be more careful :

$$\left|\rho_k(\alpha)\right| < \frac{2\,|V_o|\,M}{(\alpha+\mu)\,[\,|\alpha+\mu-2\,\mathrm{Im}k| + 2\,|\mathrm{Re}k|\,]}\left[1+\int_0^\alpha |\rho_k(\gamma)|\,d\gamma\right]$$

From this one can show that

$$\left[1-\psi(\alpha_1,k)\right] \leqslant \frac{1+\int_0^\infty |\rho_k(\alpha)|\,d\alpha}{1+\int_0^\infty |\rho_k(\alpha)|\,d\alpha}$$

with $\alpha_1 < 2\,\mathrm{Im}k - \mu$ and

$$\psi(\alpha_1 k) = |V_o|\,M\left[\frac{1}{\mathrm{Im}k+\mathrm{Re}k}\log\frac{\mathrm{Re}k}{\mathrm{Im}k}\;\frac{2(\mathrm{Re}k+\mathrm{Im}k)-\alpha_1-\mu}{\alpha_1+\mu} + \frac{1}{\mathrm{Im}k-\mathrm{Re}k}\log\frac{\mathrm{Im}k}{\mathrm{Re}k}\right]$$

For fixed α_1, provided $\mathrm{Re}k > C\,[\mathrm{Im}k]^{-n}$ (n arbitrary), one can make $|\psi(\alpha_1,k)|$ arbitrarily small by taking $\mathrm{Im}k$ large enough. Since

$$\lim_{\mathrm{Im}k\rightarrow+\infty}\int_0^{\alpha_1}\rho_k(\alpha)\,d\alpha = 0$$

the only possibility is

$$\lim_{\mathrm{Im}k\rightarrow+\infty}\int_0^\infty |\rho_k(\alpha)|\,d\alpha = 0.$$

Existence of a limit of $f(r,k)$ along the cut

- Assume first $\left| C(\alpha) - C(\beta) \right| < B \left| \alpha - \beta \right|$ (34)

take first

$$\alpha_o = 2\, I\, mk - \mu > 0$$

$$\rho_k(\alpha) \left[\alpha + \mu \right] \left[\alpha - \alpha_o + 2i \ \mathrm{Re}\, k \right] = V_o \left[C(\alpha) + \int_o^{\alpha - \mu} C(\alpha - \mu - \beta)\, \rho_k(\alpha)\, d\alpha \right]$$

$\lim \rho_k(\alpha)$ exists for $\alpha < \alpha_o$. Near $\alpha = \alpha_o$

$$\rho_k(\alpha) \simeq \phi(\alpha_o) \left[\alpha - \alpha_o + 2i \ \mathrm{Re}\, k \right]^{-1}$$

with

$$\phi(\alpha) = \frac{V_o \left[C(\alpha) + \int_o^{\alpha - \mu} C(\alpha - \mu - \beta)\, \rho_k(\beta)\, d\beta \right]}{\alpha + \mu}$$

If $C(\alpha)$ satisfies condition (34), which contains $C(0) = 0$, $\lim \rho_k(\alpha)$ is well defined for $\alpha > \alpha_o$ because the limit of the integral is well defined. We have now to investigate the convergence for large α :

for $\alpha > \alpha_o + \mu$

$$\left| \rho_k(\alpha) \right| < \frac{1}{\left| \alpha - \alpha_o \right|^2} V_o \left[\begin{array}{l} M + M \int_o^{\alpha_o - \Delta} \left| \rho_k(\beta) \right| d\beta + M \int_{\alpha_o + \Delta}^{\alpha - \mu} \left| \rho_k(\beta) \right| d\beta \\ + M \int_{\alpha_o - \Delta}^{\alpha_o + \mu} {}_k(\ )\, d\beta \quad + 2\Delta . B . \max\limits_{\alpha_o - \Delta}^{\alpha_o + \Delta} \left| \phi(\beta) \right| \end{array} \right]$$

$$= \frac{1}{\left| \alpha - \alpha_o \right|^2} \left[C_1 + C_2 \int_{\alpha_o + \Delta}^{\alpha - \Delta} \left| \rho_k(\beta) \right| d\beta \right]$$

Then one can easily establish the convergence of

$$\int_{\alpha_0+\Delta}^{\infty} |\rho_k(\alpha)| \, d\alpha$$

as was done outside the cut. If $\alpha_0 = 0$, i.e. $2\mathrm{Im}k = \mu$ we get a logarithmic singularity in

$$\int_0^{\infty} |\rho_k(\alpha)| \, d\alpha$$

when $\mathrm{Re}k \to 0$.

- We now treat the Yukawa case $(C(\alpha) = 1$ for $0)$. We have the same logarithmic singularity at $\alpha_0 = 0$ $2\mathrm{Im}k = \mu$ The trouble is now that in addition to the pole $\alpha = \alpha_0$, we have a new singularity when the singularity of $\rho(\beta)$ happens to be at the upper limit of the integral defining $\rho(\alpha)$ for $\alpha > \alpha_0$:

$$\rho_k(\alpha) \, [\alpha+\mu] \, [\alpha-\alpha_0+2i\,\mathrm{Re}k] = V_0 \left[1+ \int_0^{\alpha-\mu} \rho_k(\beta) \, d\beta\right]$$

This happens for $\alpha = \alpha_0 + \mu$. Fortunately this new singularity is a logarithmic singularity, so that the integral

$$\int_0^{\alpha-\mu} \rho_k(\beta) \, d\beta$$

is still well defined for $\alpha > \alpha_0 + 2\mu$. Then, using inequality

$$|\rho_k(\alpha)| < \frac{V_0}{|\alpha-\alpha_0|^2} \left[1 + \left| \int_0^{\alpha_0+2\mu} \rho_k(\beta) \, d\beta \right| + \int_{\alpha_0+2\mu}^{\alpha-\mu} |\rho_k(\beta)| \, d\beta \right]$$

one can prove the existence of $\lim_{\mathrm{Re}k \to 0} f(r,k)$.

APPENDIX II.

Sufficient condition for the absence of zeros of g in the upper half plane

A sufficient condition for this is

$$\int_0^\infty \Sigma(\alpha)\, d\alpha < 1.$$

Then

$$1 + e^{-\mu r} \int_0^\infty \sigma_k(\alpha)\, e^{-\alpha r}\, d\alpha$$

can never vanish. From (I,2) one can show that :

$$1 + \int_0^\infty \Sigma(\alpha)\, d\alpha < \frac{1 + \int_0^{n\mu} \Sigma(\alpha)\, d\alpha}{1 - \frac{|V_o|\, M}{(n+1)\mu}}$$

$\int_0^{n\mu} \Sigma(\alpha)\, d\alpha$ can be computed exactly. With $n = 0$ we get

$$|V_o|\, M < \frac{\mu}{2} ;$$

$n = 1$ gives

$$|V_o|\, M < \frac{2\mu}{3} .$$

This could be still slightly improved, but not much because $M\,|V_o| = \mu$ gives

$$\int_0^\infty \Sigma(\alpha)\, d\alpha > 1.$$

APPENDIX III.

Numerical use of equations (33)

We look on equations (33) at the bound state at zero energy. This gives

$$1 + \int_0^{\infty} \sigma_0(\alpha)\, d\alpha = 0$$

with

$$\sigma_0(\alpha) \left[\alpha+\mu\right]^2 = - V_0 \left[1 + \int_0^{\alpha-\mu} \sigma_0(\gamma)\, d\gamma\right]$$

If we take as a first approximation

$$1 + \int_0^{\infty} \sigma_0(\alpha)\, d\alpha \simeq 1 + \int_0^{\mu} \sigma_0(\alpha)\, d\alpha$$

we get

$$1 - \frac{|V_0|}{2\mu} = 0, \qquad \frac{\mu}{|V_0|} = 0.5$$

The second approximation

$$1 + \int_0^{2\mu} \sigma_0(\alpha)\, d\alpha = 0$$

gives

$$1 - \frac{V_0}{2\mu} - \frac{V_0}{6\mu} + \left|\frac{V_0}{\mu}\right|^2 \left[\frac{1}{3} - \log\frac{4}{3}\right] = 0$$

and

$$\frac{\mu}{|V_0|} = 0.595$$

The exact result [16)] is 0.5953.

R E F E R E N C E S

1) Private communication from H. Lehmann.

2) J. Bowcock and D. Walecka, Preprint CERN 6636/Th.26, to be published in Nuclear Physics.

3) P. Kapur and R.E. Peierls, Proc. Roy. Soc. A 166, 277 (1938).

4) R. Jost, private communication to D. Walecka.

5) A. Martin, Nuovo Cimento X 7, 607 (1957).

6) E.P. Wigner, Phys. Rev. 98, 145 (1955).

7) A. Martin, Comptes Rendus, 243, 22 (1956).

8) E. Fermi, Nuclear Physics, The University of Chicago Press (1950) p. 59.

9) R. Jost, H.P.A. 19, 256 (1946).

10) H.A. Bethe and R. Bacher, Rev. Mod. Phys. 8, 111 (1936).

11) S.T. Ma, Phys. Rev. 69, 668 (1946).

12) E. Corinaldesi, Nuclear Physics, 2, 420 (1956).

13) S. Fubini, private communications.

14) A. Koppe, Zeit. für Naturforschung, 6A, 229 (1951).

15) C. Caratheodory, Theory of functions of a compley variable, 2nd English edition, Chelsea Publishing Company (1958), p. 180 - 197.

16) J. Blatt and V. Weisskopf, Theoretical Nuclear Physics, John Wiley and Sons (1952), p. 56.