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ON THE APPLICATION OF LIE-SERIES TO THE PROBLEMS OF CELESTIAL MECHANICS

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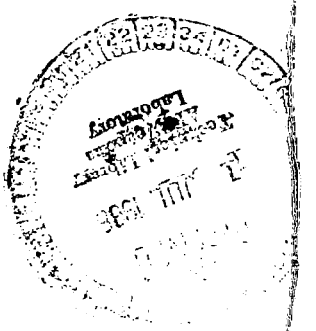

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ABSTRACT

Since Newton, the stumbling block in celestial mechanics has been the three-body problem. Only restricted cases have yielded solutions. This paper describes a device, the "Lie-Series," that first appeared in Lie's work on analytical transformations; Grŏbner has shown that they can be used to solve systems of differential equations by applying differential operators to known functions or to invert systems of analytical functions. The series are applied to Kepler's problem of an undisturbed planet round the sun (two-body problem), to the study of perturbations, and to the process of obtaining the characteristics for any general dynamical problem.

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ON THE APPLICATION OF LIE-SERIES TO THE PROBLEMS OF CELESTIAL MECHANICS*

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DEFINITION AND PROPERTIES OF LIE-SERIES

The main problem of celestial mechanics is the study of the motions in space of n mass points that attract one another according to Newton's law of gravitation. This problem can be solved for $n = 2$, but for $n \geq 3$ no general solutions exist that are valid for any values of the masses, initial conditions, and range of time. Since Newton, many attempts have been made to construct solutions for special cases, for instance, the solar system or the sun, earth, and moon system. The difficulty of solving the n-body problem justifies introducing any new mathematical method that gives deeper insight and facilitates solution.

Recently, a new method has been proposed by Wolfgang Gröbner (Reference 1). He uses the "Lie-Series" in solving a restricted class of differential equations. It is not yet clear that this new method is better than others; it has not been sufficiently used. But it is worthwhile to give a summary of the technique. The Lie-Series were originated by the celebrated Norwegian mathematician Sophus Lie, but they appear only incidentally in his works on analytical transformations. Their usefulness in solving problems like those of celestial mechanics was ascertained by Professor Gröbner.

Suppose that

$$\theta_i(z) = \theta_i(z_1, z_2, \ldots, z_n)$$

are n functions of the n complex variables z_k, and are holomorphic (i.e., can be developed into regular and convergent power series) within a certain neighborhood G around the position

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) .$$

*This paper originates from lectures given at the Goddard Space Flight Center, Laboratory for Theoretical Studies, in February and March, 1967.

Define the operator D as

$$D = \vartheta_1(z) \frac{\partial}{\partial z_1} + \vartheta_2(z) \frac{\partial}{\partial z_2} + \cdots + \vartheta_n(z) \frac{\partial}{\partial z_n} \quad . \tag{1}$$

This operator can be applied to any function $f(z)$ that is holomorphic in G. Applying this operator repeatedly on $f(z)$ gives

$$D^2 f = D(Df), \qquad D^3 f = D\left(D^2 f\right) ,$$

and so on. Define a Lie-Series as

$$L(z,t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu f(z) = \left[1 + tD + \frac{t^2}{2!} D^2 + \cdots\right] f(z) \quad , \tag{2}$$

and write it symbolically

$$L = e^{tD} f(z) \quad .$$

This series is holomorphic in the region G and converges for all values of $|t|$ smaller than a certain positive number T. The proof is left to the mathematicians. We shall assume that there is always a positive radius of convergence, no matter how small.

The fundamental properties of the Lie-Series can be derived from the algebraic behavior of the operator D. It is enough to write down the principal aspects of the algebra. The following rules can easily be demonstrated:

$$\begin{aligned}
D\left[f_1(z) + f_2(z)\right] &= Df_1(z) + Df_2(z) \quad , \\
D\left[c\, f(z)\right] &= c\, Df(z) \quad , \\
Dc &= 0 \quad , \\
D\left[f_1(z) \cdot f_2(z)\right] &= f_1 Df_2(z) + f_2 Df_1(z) \quad , \\
D^n\left[f_1(z) \cdot f_2(z)\right] &= \sum_{\alpha=0}^{n} \binom{n}{\alpha} D^\alpha f_1(z)\, D^{n-\alpha} f_2(z) \quad .
\end{aligned}$$

From these, rules for the Lie-Series can be generated, for instance:

$$e^{tD}\left[c_1 f_1(z) + \cdots + c_n f_n(z)\right] = c_1 e^{tD} f_1(z) + \cdots + c_n e^{tD} f_n(z) \quad .$$

$$e^{tD}\left[f_1(z) \cdot f_2(z) \cdots f_n(z)\right] = \left[e^{tD} f_1(z)\right] \cdot \left[e^{tD} f_2(z)\right] \cdots \left[e^{tD} f_n(z)\right] \quad .$$

These two relations illustrate the following remarkable theorem:

"The Lie-Series of a sum or a product of different functions is equal to the sum or the product of the Lie-Series of the components."

This statement generalizes to the important rule

$$e^{tD}P(f_1, f_2, \ldots, f_n) = P(e^{tD} f_1, e^{tD} f_2, \ldots, e^{tD} f_n) ,$$

i.e. the Lie-Series of a polynomial P is equal to the polynomial of the Lie-Series. This law is also valid for any analytical function of the z_k. If we let z stand for $z_1, z_2, \ldots, z_n$ and $F(z)$ for $F(z_1, z_2, \ldots, z_n)$, we may write

$$e^{tD}F(z) = F(e^{tD} z) \quad \text{or} \quad e^{tD}F[f(z)] = F[e^{tD} f(z)] ; \tag{3}$$

i.e., the symbol e^{tD} and the function symbol F can be interchanged. This "rule of interchanging" is one of the most useful features of Lie-Series. The proof follows from the assumed uniform convergence of the Lie-Series within the region of their validity. Consider the simple example where z is a single complex variable (the operator $D = d/dz$). Then

$$Dz = 1, \; D^n z = 0 \quad \text{for} \quad n = 2, 3, \ldots$$

and

$$e^{tD} z = (1 + tD)z = z + t .$$

The rule of interchanging gives

$$F(e^{tD} z) = e^{tD} F(z) ,$$

or

$$F(z + t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} F^{(\nu)}(z) ,$$

i.e., the well known Taylor series, which may be considered as a special Lie-Series. It is easy to extend these considerations to functions of several variables.

The above simple application of the rule of interchanging shows that Lie-Series can demonstrate certain analytical properties in a very direct manner. This is still true when solving systems of first-order differential equations.

Consider the equations

$$\frac{dz_i}{dt} = \theta_i(z) ; \qquad i = 1, 2, \ldots, n , \qquad (z) = (z_1, z_2, \ldots, z_n) .$$

The functions $\theta_i(z)$ are holomorphic in a certain region G, and do not explicitly contain the independent variable t. Such a system is called "autonomous." (Non-autonomous systems can easily be reduced to autonomous ones, so that restriction to autonomous systems is not essential.) To solve the system in Equation (4) let the differential operator D be

$$D = \theta_1(\zeta)\frac{\partial}{\partial\zeta_1} + \theta_2(\zeta)\frac{\partial}{\partial\zeta_2} + \cdots + \theta_n(\zeta)\frac{\partial}{\partial\zeta_n} ,$$

where

$$(\zeta) = (\zeta_1, \zeta_2, \ldots \zeta_n)$$

is a position in the region G that provisionally may be considered as variable but independent of the time. Then

$$z_i = \phi_i(\zeta, t) = e^{tD}\zeta_i \qquad (i = 1, 2, \ldots, n)$$

are n holomorphic functions of time t and of position ζ which, for $t = 0$, take the values

$$(z_i)_{t=0} = \phi_i(\zeta, 0) = \zeta_i .$$

Differentiating z_i with respect to time gives

$$\frac{dz_i}{dt} = D e^{tD}\zeta_i = e^{tD} D\zeta_i ,$$

according to the rule of interchanging. But because

$$D\zeta_i = \theta_i(\zeta) ;$$

therefore,

$$\frac{dz_i}{dt} = e^{tD}\theta_i(\zeta) = \theta_i\left(e^{tD}\zeta\right) = \theta_i(z) ,$$

i.e. the functions

$$z_i = e^{tD}\zeta_i \tag{5}$$

are solutions of Equations 4, with the initial conditions $z_i(0) = \zeta_i$. The ζ_i are to be considered as variables during the application of the operator D, but after having performed all partial

differentiations connected with this application they must be replaced by the constant values ζ_i prescribed by the initial conditions of the problem.

If the system of differential equations is not autonomous, then

$$\frac{dz_i}{dt} = \theta_i(z,t) \quad ,$$

where the functions θ_i are holomorphic in a certain region G of the complex variables $z_1, z_2, \ldots, z_n, t$. Now introduce, only by a change of notation, the new variable $z_0 = t$, which is the solution of the equation $dz_0/dt = 1$, with the initial condition $(z_0)_{t=0} = 0$. This allows the nonautonomous system of n equations to be replaced by the autonomous system of $n+1$ equations

$$\frac{dz_i}{dt} = \theta_i(z_\nu) \quad , \qquad (i = 0, 1, 2, \ldots, n)$$

with $\theta_0(z) = 1$. This system can be solved using the differential operator

$$D = \frac{\partial}{\partial \zeta_0} + \sum_{i=1}^{n} \theta_i(\zeta) \frac{\partial}{\partial \zeta_i}$$

in the same manner as before.

The Lie-Series is applied here to a simple problem. Let λ be the time, or another independent variable replacing the time, and let $z(\lambda)$ be a real or complex function of λ. Then, if α^2 is a positive constant,

$$z'' + \alpha^2 z = 0, \qquad z' = \frac{dz}{d\lambda} \tag{6}$$

is the differential equation of a harmonic oscillation with the period $2\pi/\alpha$. This problem can be solved directly by classical methods; however, we shall here use Lie-Series. The second-order differential equation is replaced by a system of two first-order differential equations:

$$\frac{dz}{d\lambda} = y = \theta_1(z, y) \quad ,$$

$$\frac{dy}{d\lambda} = -\alpha^2 z = \theta_2(z, y) \quad .$$

We let (ζ, η) be a certain position in the (z, y)-space, and use the operator

$$D = \eta \frac{\partial}{\partial \zeta} - \alpha^2 \zeta \frac{\partial}{\partial \eta} \quad .$$

Solving Equation 6, with the initial conditions

$$z(0) = \zeta \ , \qquad y(0) = \left(\frac{dz}{d\lambda}\right)_{\lambda=0} = \eta \ ,$$

for $\lambda = 0$, gives

$$z = e^{\lambda D}\zeta = \left[1 + \lambda D + \frac{\lambda^2}{2} D^2 + \cdots\right] \zeta \quad .$$

It is easy to apply the operator on ζ, as in this case D has a simple linear form. Indeed, we get

$$D\zeta = \eta$$

$$D^2\zeta = D\eta = -\alpha^2\zeta \ ,$$

$$D^3\zeta = -\alpha^2 D\zeta = -\alpha^2\eta \ , \ \ldots$$

and generally

$$D^{2n}\zeta = (-1)^n \alpha^{2n}\zeta \ , \qquad D^{2n+1}\zeta = (-1)^n \alpha^{2n}\eta \quad .$$

Therefore, the solution z is given by

$$z = \zeta + \lambda\eta - \frac{\lambda^2}{2!}\alpha^2\zeta - \frac{\lambda^3}{3!}\alpha^2\eta + \frac{\lambda^4}{4!}\alpha^4\zeta + \cdots$$

$$= \zeta\left(1 - \frac{\lambda^2\alpha^2}{2!} + \frac{\lambda^4\alpha^4}{4!} - \cdots\right) + \eta\left(\lambda - \frac{\lambda^3\alpha^2}{3!} + \frac{\lambda^5\alpha^4}{5!} - \cdots\right)$$

or

$$z = \zeta \cos\alpha\lambda + \frac{\eta}{\alpha}\sin\alpha\lambda \quad , \tag{7}$$

with the initial conditions

$$z(0) = \zeta \ , \qquad \left(\frac{dz}{d\lambda}\right)_0 = \eta \quad . \tag{8}$$

This is the well-known general solution of Equation 6.

Suppose that the general solution of a system of differential equations

$$\frac{dz_i}{dt} = \theta_i(z)$$

$$D = \sum_{i=1}^{n} \theta_i(\zeta) \frac{\partial}{\partial \zeta_i}$$

and

$$z_i = e^{tD}\zeta_i \quad ,$$

the ζ_i being the coordinates of a certain point of the region G, in which the functions $\theta_i(z)$ are holomorphic. Then every analytical function $F(z)$ of the solutions z_k is holomorphic in G, and may be written as a Lie-Series

$$F(z) = e^{tD}F(\zeta) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu \left[F(\zeta)\right] \quad .$$

This means that we can formally write down the development of any function $F(z)$ without having explicitly calculated the solutions z_k. Among these functions F, there are some for which

$$DF(\zeta) = 0 \quad .$$

These are called "characteristics" or "integrals" of the problem. Indeed, if F is a characteristic, then

$$F(z) = e^{tD}F(\zeta) = F(\zeta)$$

i.e., $F(z)$ is constant on every trajectory $z = z(t)$ which solves the problem. The last section of this paper discusses methods of constructing all the characteristics of a given system of differential equations.

APPLICATION TO KEPLER'S PROBLEM

The preceding section presented symbolically the general solution of a system of first-order differential equations with the aid of Lie-Series. Suppose that the equations are of the form

$$\frac{dz_i}{dt} = \theta_i(z) = \theta_i(z_1, z_2, \ldots, z_n) \quad , \qquad i = 1, 2, \ldots, n \quad ,$$

the z_i being complex or real coordinates, and the θ_i being functions that are holomorphic within a neighborhood G around a certain position $(\zeta) = (\zeta_1, \zeta_2, \ldots, \zeta_n)$. The solutions of this system can be expressed by the Lie-Series

$$z_i = e^{(t-t_0)D} \zeta_i ,$$

where

$$D = \theta_1(\zeta) \frac{\partial}{\partial \zeta_1} + \cdots + \theta_n(\zeta) \frac{\partial}{\partial \zeta_n}$$

is an operator, and $z_i (t_0) = \zeta_i$ are the initial conditions. All operations refer to ζ as a variable, but afterwards ζ must be replaced by its constant value $\zeta = z(t_0)$.

We may use this method to solve Kepler's problem of the undisturbed motion of a planet around the sun. The heliocentric rectangular coordinates of the planet being x_1, x_2, x_3, the differential equations of the problem are

$$\ddot{x}_i + \frac{x_i}{r^3} = 0 , \qquad r = \sqrt{\Sigma x_i^2} , \qquad i = 1, 2, 3, \tag{9}$$

if the units are canonic, i.e., if they are determined so that the gravitational constant $k = 1$, and the sum of the masses $M+m = 1$. These equations may be written as a system of the first-order equations:

$$\begin{aligned} \dot{x}_i &= u_i , \\ \dot{u}_i &= -\frac{x_i}{r^3} . \end{aligned} \tag{10}$$

If the initial conditions for $t = t_0$ are

$$(x_i)_0 = \xi_i , \qquad (u_i)_0 = \eta_i ,$$

the differential operator will be

$$D = \Sigma \eta_i \frac{\partial}{\partial \xi_i} - \Sigma \frac{\xi_i}{\rho^3} \frac{\partial}{\partial \eta_i} , \qquad \rho = \sqrt{\Sigma \xi_i^2} ,$$

and the solutions which correspond to the conditions are

$$x_i = e^{\tau D} \xi_i , \qquad u_i = e^{\tau D} \eta_i , \qquad \tau = t - t_0 .$$

Carrying out the partial differentiations involved in the first of these formulas (the second is not absolutely needed) gives the well-known development of the undisturbed planetary coordinates in a power series of intermediate time $\tau = t - t_0$. Unfortunately, this development is useful only for small intermediate times. For large τ, higher-order terms in the series are needed, the expressions for the coefficients of τ^n become complicated for large n, and the convergence of the series is poor. However, if we introduce independent variable λ, where

$$\frac{d\lambda}{d\tau} = \frac{1}{r} \quad , \tag{11}$$

the operator D becomes linear. The resulting solutions converge for all values of λ and may be written in closed form. Indeed, with pseudo-time λ as independent parameter $(x' = dx/d\lambda)$, Equations 9 change into linear ones, $x_i'' + \alpha^2 x_i = \beta_i = \text{const.}$, $\alpha^2 = 1/a = \text{const.}$, a being the major semiaxis of a conic section; and, instead of Equation 10,

$$x_i' = u_i \quad ,$$

$$u_i' = -\alpha^2 x_i + \beta_i \quad .$$

Another interesting method (Reference 2) of solving this problem makes use of the Levi-Civita transformation of the plane (x_1, x_2)-coordinates,

$$x_1 = \xi_1^2 - \xi_2^2 \quad , \qquad x_2 = 2\xi_1 \xi_2 \quad ,$$

which leads to the equations

$$\xi_i'' + \omega^2 \xi_i = 0 \quad , \qquad \omega^2 = \frac{1}{4a} = \text{const.}$$

that are identical with Equation 6, and have the general solutions given in Equation 7,

$$\xi_i = \zeta_i \cos \omega\lambda + \frac{\eta_i}{\omega} \sin \omega\lambda \quad ,$$

with the initial conditions of Equation 8:

$$\xi_i(0) = \zeta_i \quad , \qquad \xi_i'(0) = \eta_i \quad .$$

Kepler's problem of the undisturbed motion of two bodies in a heliocentric system requires six integrals. The integral constants are the six elements of the orbit

$$a,\ e,\ T;\ i,\ \Omega,\ \omega\ .$$

The first three of these are independent of the choice of coordinate system; the last three are not. The first three are essential, as they determine the shape and the spatial extension of the orbit and its relation to the current of time. The last three, however, are merely directions for imbedding the orbit into an arbitrary given system of reference. Therefore to solve Kepler's problem it is sufficient to reduce the system of differential equations to one of the third order, using only variables that are independent of the system of coordinates. For instance, we may solve equation given on page 201 of Reference 3:

$$\dddot{r} + 3\frac{\dot{r}\ddot{r}}{r} + \frac{\dot{r}}{r^3} = 0$$

for the radius vector r, provided that for $\tau = t - t_0 = 0$ the initial conditions $r(0)$, $\dot{r}(0)$, $\ddot{r}(0)$ are given. When the pseudo-time λ is introduced, the differential equation takes the linear form

$$r''' + \alpha^2 r' = 0\ , \qquad \alpha = \frac{1 - r''}{r} = \frac{1}{a} = \text{const.}\ ,$$

which has the well-known solution

$$r = a + b\cos\alpha\lambda + c\sin\alpha\lambda\ , \tag{12}$$

a, b, c being constants depending on r_0, $\dot{r}_0$, $\ddot{r}_0$. To obtain this solution by using Lie-Series, replace the one third-order differential equation by first-order equations. For instance, by the system

$$r' = \phi(r, \phi, \psi)\ ,$$

$$\phi' = \psi(r, \phi, \psi)\ ,$$

$$\psi' = -\alpha^2\phi\ ,$$

$$\tau' = r\ ,$$

which is accomplished using Equation 11 (which connects λ and time). Then the differential operator

$$D = \phi\frac{\partial}{\partial r_0} + \psi\frac{\partial}{\partial \phi_0} - \alpha^2\phi\frac{\partial}{\partial \psi_0} + r\frac{\partial}{\partial \tau_0}\ ,$$

where r_0, ϕ_0, ψ_0, τ_0 are the values of r, ϕ, ψ, τ for $\lambda=0$, gives the general solutions

$$r = e^{\lambda D} r_0 , \qquad \phi = e^{\lambda D}\phi_0 , \qquad \psi = e^{\lambda D}\psi_0 , \qquad \tau = e^{\lambda D}\tau_0 .$$

It is sufficient here to consider the first and the last of these formulas. Considering the first, we have

$$r = \left(1 + \lambda D + \frac{\lambda^2}{2!} D^2 + \cdots\right) r_0 ,$$

$$Dr_0 = \phi_0 = r_0' ,$$

$$D^2 r_0 = D\phi_0 = \psi_0 = r_0'' ,$$

$$D^3 r_0 = D\psi_0 = -\alpha^2\phi_0 = -\alpha^2 r_0' ,$$

and generally, for $n \geq 1$,

$$D^{2n} r_0 = (-\alpha)^{n-1} r_0'' , \qquad D^{2n-1} r_0 = (-\alpha)^{n-1} r_0' . \tag{13}$$

Therefore,

$$r = r_0 + \lambda r_0' + \frac{\lambda^2}{2!} r_0'' - \frac{\lambda^3}{3!}\alpha^2 r_0' - \frac{\lambda^4}{4!}\alpha^2 r_0'' + \cdots$$

$$= r_0 + r_0'\left(\lambda - \frac{\lambda^3}{3!}\alpha^2 + \frac{\lambda^5}{5!}\alpha^4 - \cdots\right) + r_0''\left(\frac{\lambda^2}{2!} - \frac{\lambda^4}{4!}\alpha^2 + \frac{\lambda^6}{6!}\alpha^4 - \cdots\right)$$

or

$$r = r_0 + \frac{r_0'}{\alpha}\sin\alpha\lambda + \frac{r_0''}{\alpha^2}(1 - \cos\alpha\lambda) ,$$

which is Equation 12 in another form. If we introduce the c-functions

$$c_0 = \cos\alpha\lambda , \qquad c_1 = \frac{\sin\alpha\lambda}{\alpha\lambda} , \qquad c_2 = \frac{1 - \cos\alpha\lambda}{(\alpha\lambda)^2} , \qquad c_3 = \frac{\alpha\lambda - \sin\alpha\lambda}{(\alpha\lambda)^3} , \ldots ,$$

we may write the solution for r in the elegant form

$$r = r_0 + r_0' c_1\lambda + r_0'' c_2\lambda^2 .$$

In a similar way, we derive the connection between time τ and pseudo-time λ by solving the differential equation

$$\frac{d\tau}{d\lambda} = r$$

by the Lie-Series

$$\tau = e^{\lambda D}\tau_0 \quad .$$

As $\tau_0 = 0$,

$$D\tau_0 = r_0 , \qquad D^2\tau_0 = Dr_0 , \ldots , D^n\tau_0 = D^{n-1}r_0 ,$$

Using Equation 13 for the operation of D^n on r gives

$$\tau = r_0\lambda + r_0' c_2\lambda^2 + r_0'' c_3\lambda^3 ,$$

the "main equation" of Kepler's problem, which is equivalent to Kepler's equation

$$n(t-T) = E - e\sin E \quad .$$

Trying to solve Kepler's equation has fascinated generations of astronomers and mathematicians. Countless methods have been used, especially to overcome the difficulties when orbit eccentricity e approaches 1 (the parabolic case.) Today, interest in this problem has notably decreased because the main equation (the regularized form of Kepler's equation) avoids these difficulties and is applicable to orbits of all eccentricities without singularities.

Solving either Kepler's equation or the main equation may be considered an example of inverting analytical functions. We have equations of the form

$$\Theta = t - T = f(E) \quad \text{(Kepler's equation)} ,$$

$$\tau = t - t_0 = g(\lambda) \quad \text{(Main equation)} ,$$

and the problem is to find the reverse relations

$$E = F(\Theta) , \qquad \lambda = G(\tau) \quad .$$

This inversion is possible under the following general conditions. Given a system of analytical functions

$$y_i = \phi_i(x) = \phi_i(x_1, x_2, \ldots, x_n) ,$$

where the functions ϕ_i are holomorphic in the environment of a position

$$\eta_i = \phi_i(\xi) = \phi_i(\xi_1, \xi_2, \ldots, \xi_n) ,$$

and the Jacobian matrix

$$J = (\phi_{ik}) = \left(\frac{\partial \phi_i}{\partial x_k}\right) = \left(\frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (x_1, x_2, \ldots, x_n)}\right)$$

is different from zero at the position ξ, then the system of functions is reversible; i.e., there exists the solution

$$x_i = \Phi_i(y) = \Phi_i(y_1, y_2, \ldots, y_n) ,$$

with the inverse Jacobian matrix

$$J^{-1} = (\Phi_{ik}) = \left(\frac{\partial x_i}{\partial y_k}\right) .$$

The Lie-Series are helpful in solving this problem. Indeed, the x_i as functions of the y_i must satisfy the system of partial differential equations

$$\frac{\partial x_i}{\partial y_k} = \Phi_{ik}(x)$$

with the initial conditions

$$x_i = \xi_i \quad \text{for} \quad y_i = \eta_i ,$$

and it can easily be shown that it is possible to write the solution of this problem in the form of Lie-Series.

For simplicity and with special attention to the problem of solving Kepler's equation, we shall consider here only the case of one function

$$y = f(x) ; \qquad y = \eta \quad \text{for} \quad x = \xi .$$

We can solve for the inverted function

$$x = \phi(y) \quad ,$$

provided that $\partial y/\partial x \neq 0$ for $x = \xi$. In this case, we can write

$$\frac{dy}{dx} = \theta(x) = f'(x)$$

as a common differential equation, the non-singular inverse is

$$\frac{dx}{dy} = \frac{1}{f'(x)} \quad , \tag{14}$$

provided $f'(\xi) \neq 0$.

Then by Lie-Series, with

$$D = \frac{1}{f'(\xi)} \frac{d}{d\xi} , \qquad x = e^{(y-\eta)D} \xi$$

is the solution of Equation 14 with the prescribed initial conditions.

Apply this rule to Kepler's equation

$$y = x - e \sin x \quad .$$

Then $y' = 1 - e \cos x$ is different from zero for $x = 0$ and $e < 1$, and

$$D = \frac{1}{1 - e \cos \xi} \frac{d}{d\xi} .$$

Applying this operator repeatedly to ξ, we obtain

$$D\xi = \frac{1}{1 - e \cos \xi} ,$$

$$D^2\xi = - \frac{e \sin \xi}{(1 - e \cos \xi)^3} ,$$

$$D^3\xi = - \frac{e \cos \xi}{(1 - e \cos \xi)^4} + \frac{3 e^2 \sin^2 \xi}{(1 - e \cos \xi)^5} , \ \ldots$$

Therefore,

$$x = \xi + (y-\eta)\,D\xi + \frac{1}{2!}(y-\eta)^2 D^2\xi + \frac{1}{3!}(y-\eta)^3 D^3\xi + \cdots$$

and, for $\xi = \eta = 0$,

$$x = \frac{y}{1-e} - \frac{y^3}{3!}\frac{e}{(1-e)^4} + \cdots \quad .$$

This is indeed one of the simplest special series solving Kepler's equation.

It is interesting to apply this method to the main equation

$$\tau = r_0\lambda + r_0' c_2\lambda^2 + r_0'' c_3\lambda^3 \quad .$$

If

$$D = \frac{1}{\tau_0'}\frac{\partial}{\partial\lambda_0} = \frac{1}{r_0}\frac{\partial}{\partial\lambda_0}$$

denotes the solving operator, then

$$\lambda = e^{\tau D}\lambda_0 = \left[1 + \tau D + \frac{\tau^2}{2!}D^2 + \cdots\right]\lambda_0 \quad ,$$

and

$$\lambda_0 = 0 \quad ,$$

$$D\lambda_0 = \frac{1}{r_0} \quad ,$$

$$D^2\lambda_0 = D\frac{1}{r_0} = \frac{1}{r_0}\left(-\frac{r_0'}{r_0^2}\right) = -\frac{r_0'}{r_0^3} \quad ,$$

$$D^3\lambda_0 = -D\left(\frac{r_0'}{r_0^3}\right) = -\frac{1}{r_0}\left(\frac{r_0''}{r_0^3} - \frac{3r_0'^2}{r_0^4}\right) \quad , \quad \cdots \quad .$$

Therefore,

$$\lambda = \frac{\tau}{r_0} - \frac{\tau^2}{2!}\frac{r_0'}{r_0^3} - \frac{\tau^3}{3!}\left(\frac{r_0''}{r_0^4} - \frac{3r_0'^2}{r_0^5}\right) + \cdots \quad ,$$

or, introducing the dimensionless quantity

$$z_\lambda = \frac{r_0 \lambda}{\tau}$$

and multiplying λ by r_0/τ gives

$$z = 1 - \frac{\tau}{2!}\frac{r_0'}{r_0^2} - \frac{\tau^2}{3!}\left(\frac{r_0''}{r_0^3} - \frac{3r_0'^2}{r_0^4}\right) + \cdots .$$

When these considerations are extended to systems of several functions of several variables, the process of inversion is complicated as we are dealing with partial differential equations. Without going into details, note that this problem requires, not one operator D but n operators, such that

$$D_j = \sum_{i=1}^{n} \theta_{ij} \frac{\partial}{\partial \xi_i}, \qquad j = 1, 2, \ldots, n ,$$

and these operators must fulfill the conditions of reversibility

$$D_j D_k = D_k D_j ,$$

which correspond to the conditions of compatibility,

$$\frac{\partial^2 x_i}{\partial y_j \, \partial y_k} = \frac{\partial^2 x_i}{\partial y_k \, \partial y_j} ,$$

between second-order partial derivatives.

APPLICATION TO THE PROBLEM OF PERTURBATIONS

Lie-Series are helpful in symbolically representing the solution of any system of first-order differential equations. A theorem of Lagrange, applied to a more complicated problem, demonstrates the usefulness of Lie-Series.

Suppose that three bodies (for instance, the sun and two planets) are moving according to Newton's law of gravitation. Let the masses be m_1, m_3, m_5, and let $\vec{q}_1$, $\vec{q}_3$, $\vec{q}_5$ be the vectors leading from m_3 to m_5, m_5 to m_1, m_1 to m_3 respectively, and

$$\dot{\vec{q}}_1 = \vec{q}_2 , \qquad \dot{\vec{q}}_3 = \vec{q}_4 , \qquad \dot{\vec{q}}_5 = \vec{q}_6$$

be the corresponding velocity vectors. This system of relative motions requires 12 integrals of motion. Between the q's there exist the vector relations

$$\vec{q}_1 + \vec{q}_3 + \vec{q}_5 = 0 , \qquad \vec{q}_2 + \vec{q}_4 + \vec{q}_6 = 0 .$$

Therefore, only four independent vectors need be solved for, e.g., $\vec{q}_1$, $\vec{q}_2$, $\vec{q}_3$, $\vec{q}_4$. Lagrange states that this three-body problem is solved if we know, as functions of time, nine independent functions of the $\vec{q}_i$ that are invariant under coordinate transformations. This solution being achieved, the remaining three integrals can be obtained directly or by simple quadratures. The problem of finding the nine first integrals is called the "reduced problem of three bodies" (see Reference 4, or Chapter XII of Reference 5). If

$$p_{ik} = (\vec{q}_i \cdot \vec{q}_k) = x_i x_k + y_i y_k + z_i z_k ,$$

(the scalar products between any two of the six vectors $\vec{q}$, inclusive of the cases $i = k$), it is sufficient to solve for the nine independent quantities

$$p_{11}, p_{33}, p_{55}; p_{12}, p_{34}, p_{56}; p_{22}, p_{44}, p_{66} \tag{15}$$

as functions of the time. These quantities may be called "fundamental invariants" of the problem, as they are independent of the system of coordinates and of one another, and form a symmetrically builtup system. These nine quantities are solutions of the differential equations

$$\begin{aligned} \frac{1}{2}\dot{p}_{11} &= p_{12} , \\ \dot{p}_{12} &= p_{22} - \mu_1 p_{11} + m_1 p_1 , \\ \frac{1}{2}\dot{p}_{22} &= -\mu_1 p_{12} + m_1 p_2 , \end{aligned} \qquad \text{with} \quad \begin{cases} \mu_1 = \dfrac{1}{r_1^3} = p_{11}^{-3/2} , \\ p_1 = \mu_1 p_{11} + \mu_3 p_{13} + \mu_5 p_{15} , \\ p_2 = \mu_1 p_{21} + \mu_3 p_{23} + \mu_5 p_{25} , \end{cases} \tag{16}$$

and six others are obtained by cyclic permutation of the indices 1, 3, 5,and 2, 4, 6, respectively. The invariants p_1 and p_2 (likewise p_3 and p_4, p_5, and p_6) on the right-hand sides can be expressed as functions of the fundamental invariants by a number of easily constructed relations between the p_{ik} (e.g., $p_{11} + p_{13} + p_{15} = 0$, $p_{32} + p_{34} + p_{36} = 0$, etc.). We could supplement this system of nine differential equations by those for all the other possible p_{ik} invariants. The total number of different invariants is 21 (for N bodies there are $N \cdot (2N+1)$ p's). The system

$$\frac{dp_{ik}}{dt} = \theta_{ik}(p_{11}, p_{12}, \ldots, p_{56}, p_{66}) = \theta_{ik}(p)$$

may be solved by Lie-Series, if we use the operator

$$D = \Sigma \theta_{ik} \frac{\partial}{\partial p_{ik}}$$

the sum containing 21 expressions. The solutions can be written

$$p_{ik} = e^{(t-t_0)D} \pi_{ik} ,$$

where π_{ik} are the values of the p_{ik} for $t = t_0$. In discussing Lagrange's theorem it is enough to write down these developments for nine of the fundamental invariants (Equation 15) or for nine independent functions of these invariants. The procedure may be improved by introducing a pseudo-time s instead of t.

This method of developing the solutions of the reduced three-body problem does not seem to have been attempted. Perhaps it is as intricate and cumbersome as any of the classical methods, but the short and elegant symbolic form of the solution may be more tractable for computer programming, as the right-hand sides of the differential equations are finite algebraic expressions of the invariants themselves.

Another interesting, possible use for Lie-Series is in the case of slightly perturbed planetary orbits. Then, as in classical celestial mechanics, we may divide the motion of a planet into two parts: an intermediate orbit, (for instance, a Keplerian ellipse easily calculable with simple subroutines) and a perturbation term proportional to the small perturbing mass. The differential operator that solves the problem may be written as the sum of two parts,

$$D = D_1 + D_2 .$$

These two operators must fulfill some necessary conditions:

1. D_1 must be constituted so that the solutions

$$L_1 = e^{(t-t_0)D_1} f(t)$$

are simple and well known functions. This is always the case if D_1 represents the linear part of D, for instance if D_1 is the operator of a Kepler motion osculating the true orbit, and if a linearizing pseudo-time is used as independent variable.

2. D_2 must be small compared with D_1; i.e., the coefficients θ_k of D_2 must be noticeably smaller than those of D_1, at least within a certain range of time. If these conditions are satisfied,

the problem can be solved by the Lie-Series

$$f(t) = e^{(t-t_0)(D_1+D_2)} f(t_0) .$$

When $D = D_1 + D_2$ is applied to f, the f is considered as variable, and set equal to $f(t_0)$ only after all differentiations have been performed. However, this exponential expression developed into a power series of $\tau = t - t_0$ gives

$$f = \left[1 + \tau(D_1 + D_2) + \frac{\tau^2}{2}(D_1 + D_2)^2 + \cdots\right] f(t_0) ;$$

and

$$(D_1 + D_2)^2 \quad \text{expands to} \quad D_1^2 + D_1D_2 + D_2D_1 + D_2^2 , \text{ etc.} ,$$

because, generally, $D_1D_2 \neq D_2D_1$, i.e., the operators D_1 and D_2 do not commute, except under very special conditions. Let us digress and consider briefly the commutation relations for the D's.

It can be shown that in the reduced three-body problem D_1 and D_2 do not commute. Write the solutions of the three differential equations for p_{11}, p_{12}, p_{22}, using the differential operator $D = D_1 + D_2$ with

$$D_1 = 2p_{12}\frac{\partial}{\partial p_{11}} + \left(p_{22} - p_{11}^{-1/2}\right)\frac{\partial}{\partial p_{12}} - 2p_{12}p_{11}^{-3/2}\frac{\partial}{\partial p_{22}} ,$$

$$D_2 = m_1\left[p_1\frac{\partial}{\partial p_{12}} + 2p_2\frac{\partial}{\partial p_{22}}\right] ,$$

supposing that p_1, p_2 are functions of p_{11}, p_{12}, p_{22} and known functions of the time (i.e., supposing that the motion of the perturbing mass m_1 is known). Then, for instance,

$$D_1p_{11} = 2p_{12} , \qquad D_1(D_2p_{11}) = D_1(0) = 0 ,$$

$$D_2p_{11} = 0 , \qquad D_2(D_1p_{11}) = 2D_2p_{12} = 2m_1p_1 ;$$

therefore $D_1D_2 \neq D_2D_1$.

It is easy to derive the commutation conditions for any two operators D_1 and D_2. Suppose there are two operators

$$D_1 = \theta_{11}\frac{\partial}{\partial z_1} + \theta_{12}\frac{\partial}{\partial z_2} + \cdots , \qquad D_2 = \theta_{21}\frac{\partial}{\partial z_1} + \theta_{22}\frac{\partial}{\partial z_2} + \cdots .$$

Then

$$D_1D_2 = \theta_{11}\frac{\partial}{\partial z_1}\left(\theta_{21}\frac{\partial}{\partial z_1} + \theta_{22}\frac{\partial}{\partial z_{22}} + \cdots\right) + \theta_{12}\frac{\partial}{\partial z_2}\left(\theta_{21}\frac{\partial}{\partial z_1} + \theta_{22}\frac{\partial}{\partial z_2} + \cdots\right) + \cdots$$

$$= \theta_{11}\left[\frac{\partial\theta_{21}}{\partial z_1}\frac{\partial}{\partial z_1} + \theta_{21}\frac{\partial^2}{\partial z_1^2} + \cdots\right] + \cdots$$

The product of the two operators consists of two terms, one containing first-order partials and the other containing only second-order partials. A simple calculation shows that, for the operator

$$(D_1, D_2) = D_1D_2 - D_2D_1 ,$$

the sum of the second-order terms is zero. Therefore (D_1, D_2) is a linear operator like D_1 and D_2. This operator has the symmetric form

$$(D_1, D_2) = \sum_{i=1}^{n} \Theta_i \frac{\partial}{\partial z_i} ,$$

where

$$\Theta_i = \sum_{k=1}^{n}\left(\theta_{1k}\frac{\partial\theta_{2i}}{\partial z_k} - \theta_{2k}\frac{\partial\theta_{1i}}{\partial z_k}\right) .$$

D_1 and D_2 commute only if (D_1,D_2), the "Jacobi bracket", is zero. The Jacobi brackets behave like the bracket expressions of Lagrange and Poisson, which are well known in celestial mechanics. The Jacobi brackets obey the rule

$$\left((D_1, D_2), D_3\right) + \left((D_2, D_3), D_1\right) + \left((D_3, D_1), D_2\right) = 0 ,$$

which is also valid for the brackets of Lagrange and Poisson.

Let us return to the problem of perturbed motions. Let $z = (z_1, \ldots, z_n)$ be an n-dimensional vector describing the position of a system of bodies as a function of the time, and let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be the position at $t = 0$. Then

$$z(t) = e^{t(D_1+D_2)}\,\zeta = f(t, \zeta)$$

will be the solution of the system of differential equations belonging to this problem. Developing the exponential, we may write

$$z(t) = \left[\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu z + \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} D_1^{\nu-1} D_2 z + \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} D_1^{\nu-2} D_2 D z \right.$$

$$\left. + \cdots + \sum_{\nu=\alpha}^{\infty} \frac{t^\nu}{\nu!} D_1^{\nu-\alpha} D_2 D^{\alpha-1} z + \cdots \right]_{z=\zeta} . \qquad (17)$$

On the right side z must be replaced by ζ when all operations are performed. The first sum,

$$f_0(t, z) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu z = e^{tD_1} z ,$$

contains all terms that depend on D_1 only and symbolize the intermediary orbit. The second term contains D_2 only once and at the end. The third sum has D_2 at the last position but one, and in certain terms at the last one too, and so on. This series is not in a convenient form, as the perturbations of different order are not separated. On the contrary, there are parts of the perturbations of the first order in all sums except the first, and there are parts of the second-order perturbations in all sums but the first and the second. The perturbations of different orders can be obtained separately only by our totally changing the sequence of the terms; this is possible if the series converges absolutely.

For simplicity, consider only the first-order perturbations. Select from each sum the terms that contain D_2 only once; i.e., from the sum beginning with $\nu = \alpha$ take the partial sum

$$\sum_{\nu=\alpha}^{\infty} \frac{t^\nu}{\nu!} D_1^{\nu-\alpha} D_2 D_1^{\alpha-1} z .$$

Then the perturbation terms of the first order are completely collected in the double sum

$$f_1(t, z) = \sum_{\alpha=1}^{\infty} \sum_{\nu=\alpha}^{\infty} \frac{t^\nu}{\nu!} D_1^{\nu-\alpha} D_2 D_1^{\alpha-1} z = \sum_{\alpha=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{t^{\nu+\alpha+1}}{(\nu+\alpha+1)!} D_1^\alpha D_2 D_1^\nu z . \qquad (18)$$

Each term of this expression contains D_2 only once, but in every possible position among the factors D_1. Similar considerations lead to the expressions $f_n(t, z)$ for the perturbations of higher order.

Gröbner proposes another method which gives the solutions

$$z(t) = \left[f_0(t, z) + f_1(t, z) + \cdots\right]_{z=\zeta}$$

in the form of an integral equation of the Volterra type. We can rewrite the complete partial sum of Equation 17:

$$\sum_{\nu=\alpha}^{\infty} \frac{t^{\nu}}{\nu!} D_1^{\nu-\alpha} D_2 D^{\alpha-1} z = \sum_{\nu=0}^{\infty} \frac{t^{\nu+\alpha}}{(\nu+\alpha)!} D_1^{\nu} D_2 D^{\alpha-1} z ,$$

by making use of the identity

$$\frac{t^{\nu+\alpha}}{(\nu+\alpha)!} = \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} \frac{\tau^{\nu}}{\nu!} d\tau ,$$

which can easily be proved by repeated integration by parts, in the form

$$\int_0^t \sum_{\nu=0}^{\infty} \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} \frac{\tau^{\nu}}{\nu!} D_1^{\nu} D_2 D^{\alpha-1} z \, d\tau .$$

This is the partial sum with index α. Summing up all these expressions and changing the index α into $\alpha + 1$ gives the total expression for the solution $z(t)$,

$$z(t) = f(t,z)_{z=\zeta} = f_0(t,z)_{z=\zeta} + \int_0^t \sum_{\alpha=0}^{\infty} \frac{(t-\tau)^{\alpha}}{\alpha!} \left(\sum_{\nu=0}^{\infty} \frac{\tau^{\nu}}{\nu!} D_1^{\nu} D_2 D^{\alpha} z \right)_{z=\zeta} d\tau .$$

Now we may write in the inner sum,

$$\sum_{\nu=0}^{\infty} \frac{\tau^{\nu}}{\nu!} D_1^{\nu} = e^{\tau D_1} ,$$

and, according to the rule of interchanging,

$$e^{\tau D_1} \left(D_2 D^{\alpha}\right) z_{(z=\zeta)} = D_2 D^{\alpha} \left(e^{\tau D_1} z\right)_{z=\zeta} = D_2 D^{\alpha} z_{(z=f_0)} .$$

Therefore,

$$\sum_{\alpha=0}^{\infty} \frac{(t-\tau)^{\alpha}}{\alpha!} D_2 D^{\alpha} z = D_2 \sum_{\alpha=0}^{\infty} \frac{(t-\tau)^{\alpha}}{\alpha!} D^{\alpha} z = D_2 f(t-\tau, z) \quad ,$$

and finally

$$z(t) = f_0(t, z)_{z=\zeta} + \int_0^t D_2 f(t-\tau, z)_{z=f_0} \, d\tau \quad .$$

This integral equation can be solved by iteration; this process will rapidly converge if D_2 is small enough. The first steps of the iteration will be

$$z_0 = f_0(t,z)_{z=\zeta} = e^{tD_1} z_{(z=\zeta)} \quad ,$$

$$z_1 = f_0 + \int_0^t D_2 f_0(t-\tau, z)_{z=z_0} \, d\tau \quad ,$$

and one can easily prove that the integral on the right side of the second equation is identical with the double sum in Equation 18 derived for the first-order perturbations.

APPLICATION TO THE CONSTRUCTION OF THE CHARACTERISTICS OF DYNAMICAL PROCESSES

Three previous sections applied Lie-Series to some problems of celestial mechanics: the two-body problem, the reduced three-body problem, and the special problem of computing general perturbations.

Is this new method of solving dynamical problems an improvement? That depends on our point of view. The development of solutions in terms of power series is scarcely changed by the innovation. But the Lie-Series give short, pregnant solutions for any dynamical process—an obvious advantage. The Lie operators have a universal form that encompasses nearly all the types of linear operators used in former theories. The very interesting properties of the Lie-Series, especially the important rule of interchanging (Equation 3), allow the expressions for the solutions of dynamical problems to be reduced to simple and most lucid forms. This is helpful, especially in developing series for computer programming.

This section discusses a method to obtain the integrals of motion of a dynamical process. Consider a concrete case: the reduced problem of two bodies. This involves three integrals—in

other words, three independent constant functions, which may or may not contain the time explicitly. In the reduced problems, the integrals are invariants with respect to coordinate transformations.

If x_1, y_1, and z_1 are the heliocentric coordinates of a planet, and $x_2 = \dot{x}_1$, $y_2 = \dot{y}_1$, $z_2 = \dot{z}_1$ the coordinates of the velocity, then the reduced problem of two bodies will be solved when three fundamental invariants are found (as functions of time):

$$p_{11} = x_1^2 + y_1^2 + z_1^2 = r^2 ,$$

$$p_{12} = x_1 x_2 + y_1 y_2 + z_1 z_2 = rV \cos \phi ,$$

$$p_{22} = x_2^2 + y_2^2 + z_2^2 = V^2 ,$$

where r is the distance from sun to planet, V is the magnitude of the relative velocity, and ϕ is the angle between the vectors of position and velocity. These invariants are the solutions of Equations 16 (with $\mu_1 = p_{11}^{-3/2}$, and $m_1 = 0$):

$$\dot{p}_{11} = 2p_{12} ,$$

$$\dot{p}_{12} = p_{22} - p_{11}^{-1/2} ,$$

$$\dot{p}_{22} = -2p_{12} p_{11}^{-3/2} .$$

The integrals of this problem are

$$\frac{1}{a} = 2p_{11}^{-1/2} - p_{22} \text{ (the integral of energy)} ,$$

$$a(1-e^2) = p_{11} p_{22} - p_{12}^2 \text{ (the integral of areas)} ,$$

$$T = t - a^{3/2}(E - e \sin E) = f(t, a, e) \text{ (Kepler's equation)}.$$

These integrals provide three arbitrary constants a, e, and T. The first two integrals describe the geometrical form of the heliocentric orbit of the planet; the last integral describes the planet's motion in this orbit. The first two integrals show that certain functions of the variables p_{ik} are true time-independent constants, denoted as "characteristics" of the problem. In the reduced two-body problem the characteristics are finite algebraic functions of the variables. The third integral behaves quite differently. This constant expression contains the time explicitly and involves the time-dependent function E, usually denoted as the "eccentric anomaly". This is a very simple case, showing that among the three integrals of the problem there are two characteristic functions.

Generally we can prove that among the n integrals of an n-order problem there are exactly $n-1$ independent characteristics, i.e., $n-1$ independent functions of the variables that are constant along each special solution of the problem and do not contain time explicitly.

The proof that an n-order problem has $n-1$ characteristics at most, is easy. Let

$$D = \theta_1(z)\frac{\partial}{\partial z_1} + \cdots + \theta_n(z)\frac{\partial}{\partial z_n}$$

be the Lie-operator of the problem, and let $F(z)$ be a characteristic function; then $DF(z) = 0$. If $F_1(z), \ldots, F_r(z)$ are r different characteristics, each analytic function $G(F_1, F_2, \ldots, F_r)$ is a characteristic also, as

$$DG = \frac{\partial G}{\partial F_1} DF_1 + \cdots + \frac{\partial G}{\partial F_r} DF_r = 0 \quad .$$

As each characteristic $F_i(z)$ is a function of the n variables $z_1, z_2, \ldots, z_n$ of the problem, it is clear that no more than n independent characteristics exist. Indeed, if there were $n+1$ functions F_K of the n variables z_i, there would exist a relation $G(F_1, F_2, \ldots, F_{n+1}) = 0$. Actually, there are no more than $n-1$ independent characteristics. Assume that there are n characteristic functions

$$w_1 = F_1(z) \ , \ \ldots \ , \ w_n = F_n(z) \quad .$$

If these functions are independent, the Jacobian matrix

$$J = \left(\frac{\partial w_i}{\partial z_k}\right) \quad .$$

must be regular. Therefore, it must be possible to invert the equations and obtain

$$z_1 = \phi_1(w) \ , \ \ldots \ , \ z_n = \phi_n(w) \quad .$$

But the functions $\phi_i(w_1, \ldots, w_n)$ are characteristics, as shown above, and therefore

$$Dz_i = \theta_i(z) = 0 \quad ,$$

i.e., the problem restricts itself to the trivial case

$$\dot{z}_i = \theta_i(z) = 0 \ , \qquad i = 1, 2, \ldots n \quad ,$$

which can be excluded.

Finally, we must prove that $n-1$ independent characteristics really exist. Let us assume that the solutions of a system of n differential equations

$$\dot{z}_i = \varphi_i(z)$$

are

$$z_i = \left(e^{tD} z_i\right)_{z=\zeta} = \Phi_i(t, \zeta) \quad .$$

Then, eliminating t from any two of these equations gives a characteristic. The inverse function of any analytic function

$$y = f(x)$$

in the environment of a position $\eta = f(\xi)$, where $f'(x) \neq 0$, may be expressed by the Lie-Series

$$x = \left(e^{(y-\eta)D} x\right)_{x=\xi}$$

if

$$D = \frac{1}{f'(x)} \frac{d}{dx}$$

is an operator that is regular in the neighborhood of $x = \xi$. If we apply this rule to any two solutions of our problem

$$z_i = \Phi_i(t, \zeta) \quad .$$

$$z_k = \Phi_k(t, \zeta) \quad .$$

we obtain the reversal expressions

$$t = \left(e^{(z_i - \zeta_i)D_i} t\right)_{t=\tau} \quad . \qquad t = \left(e^{(z_k - \zeta_k)D_k} t\right)_{t=\tau} \quad . \tag{19}$$

with the operators

$$D_i = \frac{1}{\dot{\Phi}_i} \frac{d}{dt} \quad . \qquad D_k = \frac{1}{\dot{\Phi}_k} \frac{d}{dt} \quad .$$

which are regular in any region around $t = \tau$ where the derivatives of Φ_i and Φ_k do not vanish. The difference between these two expressions for t gives an equation

$$\psi\left(z_i, z_k; \zeta\right) = 0$$

with constant values ζ and independent of time. This formula represents a characteristic of the problem, as it is valid along each solution. Combining thus the first solution with the other $n-1$ solutions gives $n-1$ independent characteristics

$$\psi_k(z_1, z_k; \zeta) = 0 , \qquad k = 2, 3, \ldots, n . \tag{20}$$

Besides the $n-1$ characteristics in Equation 20, there exists one independent relation (Equation 19) that gives t as a function of the particular z_i, ζ_i; e.g.,

$$t = \left(e^{(z_1-\zeta_1)D_1} t\right)_{t=\tau} .$$

This equation, $t = f(z_1, \zeta_1, \ldots, \zeta_n)$, plays the same role in the general problem as Kepler's equation does in the two-body problem. In this case, z_1 takes a favored position but can obviously be replaced by any other of the z_n.

The above device for constructing the characteristics of a dynamical problem does not quite satisfy our expectations, as each of the cases of Equation 20 contains only two of the n variables z_i and more or less of the constants ζ_i. (Pertinent to this question there is an unpublished remark made in 1950 by K. Stumpff.)

Let $\zeta_1 \ldots \zeta_n = \zeta$ be the position of a vector $z_1 \ldots z_n = z(t)$ at $t = t_0$. Then, $\tau = t - t_0$,

$$\zeta_i = z_i - \dot{z}_i\tau + \frac{\ddot{z}_i}{2}\tau^2 - \frac{\dddot{z}_i}{6}\tau^3 + \cdots ,$$

where z_i and its derivatives are taken at time t_0. If $\dot{z}_i$ is different from zero,

$$\frac{z_i - \zeta_i}{\dot{z}_i} = \tau - \frac{\ddot{z}_i}{\dot{z}_i}\frac{\tau^2}{2} + \frac{\dddot{z}_i}{\dot{z}_i}\frac{\tau^3}{6} - \cdots = \phi_i(\tau, z_1, \ldots, z_n) ,$$

as the derivatives $z_i^{(n)}$ are functions of the z themselves, and may be derived from the differential equations $\dot{z}_i = \theta_i(z)$. Reversing these equations gives, as before, n equations

$$\tau = f_i(z, \zeta_i) .$$

Suppose that one of the ζ_i (for instance ζ_n) is zero. Then

$$\tau = f_n(z)$$

is a function of the variables z only. Substituting f_n for τ the equations for $f_1, f_2, \ldots, f_{n-1}$ gives $n-1$ equations

$$f_i(z, \zeta_i) = f_n(z) ,$$

from which

$$\zeta_i = \zeta_i(z) = \text{const.}, \qquad i = 1, 2, \ldots, n-1 ,$$

by inversion, provided that the position ζ is so chosen that in a certain surrounding region G the $\dot{z}_1$ are not zero. These equations, which can easily be expressed by Lie-Series, are the characteristics in a convenient form, and they describe the geometrical feature of the trajectories. These $n-1$ constants are supplemented by the main equation $\tau = f_n(z)$, which expresses the connection between the variables z and the time.

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