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On the Application of the Eddy Viscosity Concept in the Inertial Sub-range of Turbulence

by

D. K. Lilly
National Center for Atmospheric Research

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NATIONAL CENTER FOR ATMOSPHERIC RESEARCH

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ABSTRACT

It is shown that an eddy diffusion hypothesis suggested by Smagorinsky for use in numerical solutions of turbulent flow problems is consistent with the existence of an inertial sub-range at the smallest resolvable scale of the numerical model. The arbitrary constant, assumed by Smagorinsky to be of order unity, is shown to be a unique function of the constant of the Kolmogoroff energy spectrum function. An alternative hypothesis, involving an explicit turbulent intensity, is introduced as a possible improvement for flows with large space and time variations of turbulent stress.

On the Application of the Eddy Viscosity Concept in the Inertial Sub-range of Turbulence

Despite persistent efforts by fluid dynamicists and mathematicians the problem of obtaining useful analytic solutions of turbulent flow equations remains formidable. By comparison to the slow progress of analytic theory, the development of digital computers continues to proceed at a rapid pace. Thus the "brute force" methods of solution of fluid dynamics problems become more attractive, in spite of the difficulty of obtaining generalized results and the many annoying, purely numerical difficulties. Numerical methods have become virtually indispensible for solutions of non-linear two-dimensional or quasi-two-dimensional problems, such as occur in large-scale atmospheric and oceanic dynamics. Fully turbulent three-dimensional initial-boundary value flow problems are now just beginning to be approachable for computer solutions. It is still totally inconceivable that a computer could resolve both the energy containing and dissipative scales in a high Reynolds number regime, but it is not at all unlikely that the limits of resolution could extend from the largest energy containing scale into the inertial sub-range.

It is the purpose of this paper to describe and rationalize methods of simulating the turbulent energy exchange between the scales of motion explicitly computed and the dissipation scale, through an assumed idealized inertial sub-range. The methods are not entirely new, having been used previously, although somewhat inappropriately in two-dimensional computations, by Smagorinsky (1963, 1965) and

Lilly (1962). It is ~~thus~~ not intended to present a complete theory, but only to partly rationalize some methods or recipes which have already been found moderately successful, but are believed to be most valid in computational models only now becoming accessible.

For purposes of simplification we consider incompressible flow of a fluid of constant density. The continuous Eulerian equations of fluid motion and the continuity equation, in standard tensor subscript notation , are written as follows:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) + \frac{\partial (p/\rho)}{\partial x_i} - \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j}\right) = 0 \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

These, together with appropriate initial and boundary conditions, define a particular fluid motion problem with, in general, a unique solution. Numerical approximations to the equation may be obtained in various different ways, but in several respects the most direct and flexible is the finite difference grid network approach. The variables $\bar{u}_i$, $\bar{p}$ are defined at the centers of cubes of side h , and are considered to represent averages of u_i , p over the volume of the corresponding cube, that is

$$\bar{u}_i(x_1, x_2, x_3) = \frac{1}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} u_i(x_1+y_1, x_2+y_2, x_3+y_3)\, dy_1 dy_2 dy_3 \tag{3}$$

and similarly for p . The derivative expressions in equations (1) and (2) are then replaced by numerical approximations and time integration accomplished by some appropriate numerical method. It is not the purpose here to describe or discuss these numerical approximations except to note that, in principle, they may be made as accurate as desired, within certain limitations imposed by wave number aliasing in the non-linear terms. In any case, methods are available which conserve kinetic energy within the non-linear and pressure gradient terms. These aspects have been discussed in a previous paper (Lilly, 1965). The problem to be attacked here is the treatment of the Reynolds stresses, resulting from substitution of

$$\begin{aligned} u_i &= \bar{u}_i + u_i' \\ p &= \bar{p} + p' \end{aligned} \tag{4}$$

into (1) and (2) and then averaging over the grid cube. When this is done and the usual Reynolds postulates are made, the averaged equations may be written

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) + \frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho} + \frac{1}{3}\overline{u_i'^2}\right) - \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{u}_i}{\partial x_j}\right) = \frac{\partial \tau_{ij}}{\partial x_j} \tag{5}$$

$$\tau_{ij} = -\left(\overline{u_i' u_j'} - \frac{1}{3}\delta_{ij}\overline{u_k'^2}\right) \tag{6}$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{7}$$

where δ_{ij} is the Kronecker delta. The term $\overline{u_i'^2}/3$ has been combined with pressure and removed from the Reynolds stresses so that the latter will vanish in isotropic turbulence. The problem of representing the small-scale turbulence effects by functions of the explicit variables is thus formally similar to the general problem of prediction of turbulent stresses from mean flow functions. The difference is that the "mean" flow is itself variable in space and time and is defined artificially within a computationally convenient volume. The adequacy of the Reynolds postulates is therefore in some doubt. It probably depends on the existence of sufficient scale separation between the energy containing motions and the grid interval.

We now introduce a standard eddy-viscosity hypothesis, relating the eddy stresses to mean flow gradients. The eddy coefficient is assumed to be proportional to the product of the velocity scale (grid-interval) and turbulent velocity magnitude, where the latter is determined in one of two ways, of ascending complexity. The principle justification of this hypothesis arises from the results to follow, which show unambiguous and necessary relationships between the numerical proportionality coefficients required and the value of the coefficient α in the Kolmogoroff universal equilibrium spectral function. The eddy-viscosity hypothesis, for incompressible flow, consists of replacing (6) by the following:

$$\tau_{ij} = K\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \qquad (8)$$

where K is an eddy viscosity coefficient, variable in space and time.

Upon multiplication of (5) by $\bar{u}_i$, with use of (8), we may obtain the mean flow kinetic energy equation, as follows:

$$\frac{\partial}{\partial t}(\bar{u}_i^2/2) + \frac{\partial}{\partial x_j}\left[\bar{u}_j\left(\frac{\bar{u}_i^2}{2} + \frac{\bar{p}}{\rho} + \frac{\overline{u_i'^2}}{3}\right) - \nu\frac{\partial(\bar{u}_i^2/2)}{\partial x_j}\right] + \nu\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^2 = \frac{\partial}{\partial x_j}\left[K\left(\frac{\partial \bar{u}_i^2/2}{\partial x_j} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_i}\right)\right] - K\frac{\partial \bar{u}_j}{\partial x_i}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \quad (9)$$

The first term on the right represents the diffusion of mean energy by turbulence, and the second is the transfer of mean flow (large scale) energy to that of smaller scales. This transfer is always in the direction of decreasing the large scale flow if K is positive. For simplicity in the following derivations we define the positive transfer term S to be the product of K and the squared deformation ~~term~~ tensor of the large scale flow, i.e.

$$S = K \, Def^2 \quad (10)$$

$$Def^2 = \frac{\partial \bar{u}_j}{\partial x_i}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \quad (11)$$

Smagorinsky's method -- analytic

We now introduce a formula for K consistent with dimensional analysis and mixing length theory and similar to that suggested and used by Smagorinsky (1963). It is

$$K = (kh)^2 \langle Def^2 \rangle^{1/2} \quad (12)$$

where k is a constant of order unity, analogous to the von Karman constant of boundary layer turbulence theory. The brackets $\langle\rangle$ denote some form of averaging sufficient to assure stability of velocity covariances. If the turbulence is reasonably homogeneous then the averaging could appropriately be over all space. Otherwise an average over an ensemble of flows with identical initial values of $\bar{u}_i$ would seem optimal, except that substitution of (8) for the Reynolds stresses effectively eliminates the ensemble variance. Smagorinsky considered the mesh box itself to be the appropriate averaging interval, in which case the bracket average becomes equivalent to the bar average.

The bracket average of (10) may now be written as

$$\langle S \rangle = (kh)^2 \left\langle \langle Def^2 \rangle^{1/2} Def^2 \right\rangle \approx (kh)^2 \langle Def^2 \rangle^{3/2} \tag{13}$$

in which the approximation presumably improves as the spatial averaging area is increased. In the remainder of this section we relate $\langle Def^2 \rangle$ to the energy dissipation rate through the velocity auto-correlation functions and the Kolmogoroff $-5/3$ power spectrum appropriate to the inertial sub-range of isotropic homogeneous turbulence. The relationship obtained appears to justify the use of (8) and (12) with k specified as a unique function of Kolmogoroff's universal constant α, provided that the grid separation h lies within the inertial sub-range of turbulence and that within a bracket averaging region the turbulence is essentially homogeneous. For actual numerical solution by finite difference methods the derivatives in (8) and (12) are replaced by finite difference formulae, which are shown to be equally appropriate provided that the value of k is adjusted slightly.

The bracket average of equation (11) may be expanded into the following component terms :

$$\left\langle De\, f^2 \right\rangle = \left\langle 2\left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 + 2\left(\frac{\partial \bar{u}_2}{\partial x_2}\right)^2 + 2\left(\frac{\partial \bar{u}_3}{\partial x_3}\right)^2 + \left(\frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1}\right)^2 + \left(\frac{\partial \bar{u}_1}{\partial x_3} + \frac{\partial \bar{u}_3}{\partial x_1}\right)^2 + \left(\frac{\partial \bar{u}_2}{\partial x_3} + \frac{\partial \bar{u}_3}{\partial x_1}\right)^2 \right\rangle \tag{14}$$

where use has been made of the averaged continuity equation, (7). For isotropic turbulence the first three terms of (14) must be equal, as must the last three. The fourth term may be expanded into:

$$\left\langle \left(\frac{\partial \bar{u}_2}{\partial x_1} + \frac{\partial \bar{u}_1}{\partial x_2}\right)^2 \right\rangle = \left\langle \left(\frac{\partial \bar{u}_2}{\partial x_1}\right)^2 + \left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 + 2\,\frac{\partial \bar{u}_2}{\partial x_1}\frac{\partial \bar{u}_1}{\partial x_2} \right\rangle \tag{15}$$

Again, for isotropic turbulence, the first and second terms on the right have the same value. For evaluation of the third term we introduce another assumption on the bracket average, that is

$$\left\langle \frac{\partial \bar{u}_2}{\partial x_1}\,\frac{\partial \bar{u}_1}{\partial x_2} \right\rangle = \left\langle \frac{\partial \bar{u}_1}{\partial x_1}\,\frac{\partial \bar{u}_2}{\partial x_2} \right\rangle = -\frac{1}{2}\left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 \right\rangle \tag{16}$$

The first equality is strictly true only if the average extends over all space, the second is a consequence of continuity and isotropy. Under these assumptions (14) may be written

$$\left\langle De\, f^2 \right\rangle = 3\left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 \right\rangle + 6\left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle \tag{17}$$

To evaluate these terms, we first expand them into their original definite integral forms, i.e.

$$\left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 \right\rangle = \frac{1}{h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\langle \frac{\partial u_1(\mathbf{x}+\mathbf{y})}{\partial x_1} \quad \frac{\partial u_1(\mathbf{x}+\mathbf{z})}{\partial x_1} \right\rangle d\mathbf{y}\, d\mathbf{z} \tag{18}$$

$$\left\langle \left(\frac{\partial \bar{u}_2}{\partial x_1}\right)^2 \right\rangle = \frac{1}{h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\langle \frac{\partial u_2(\mathbf{x}+\mathbf{y})}{\partial x_1} \quad \frac{\partial u_2(\mathbf{x}+\mathbf{z})}{\partial x_1} \right\rangle d\mathbf{y}\, d\mathbf{z} \tag{19}$$

where $\mathbf{y}$ and $\mathbf{z}$ are three-dimensional dummy vector variables and $d\mathbf{y} \equiv dy_1 dy_2 dy_3$, etc. Integration over y_1 and z_1 may be performed immediately and the result written in terms of covariance functions as follows:

$$\left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 \right\rangle = \frac{1}{h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big\{ R_{11}\left[\mathbf{x} + \mathbf{i}\frac{h}{2} + \mathbf{j}\frac{y_2+z_2}{2} + \mathbf{k}\frac{y_3+z_3}{2} \,;\, \mathbf{j}(y_2-z_2) + \mathbf{k}(y_3-z_3)\right] \tag{20}$$

$$- R_{11}\left[\mathbf{x} + \mathbf{j}\frac{y_2+z_2}{2} + \mathbf{k}\frac{y_3+z_3}{2} \,;\, \mathbf{i}h + \mathbf{j}(y_2-z_2) + \mathbf{k}(y_3-z_3)\right] - R_{11}\left[\mathbf{x} + \mathbf{j}\frac{y_2+z_2}{2} + \mathbf{k}\frac{y_3+z_3}{2} \,;\, -\mathbf{i}h + \mathbf{j}(y_2-z_2) + \mathbf{k}(y_3-z_3)\right]$$

$$+ R_{11}\left[\mathbf{x} - \mathbf{i}\frac{h}{2} + \mathbf{j}\frac{y_2+z_2}{2} + \mathbf{k}\frac{y_3+z_3}{2} \,;\, \mathbf{j}(y_2-z_2) + \mathbf{k}(y_3-z_3)\right] \Big\}\, dy_2\, dy_3\, dz_2\, dz_3$$

$$\left\langle \left(\frac{\partial \bar{u}_2}{\partial x_1}\right)^2 \right\rangle = \text{identical except } R_{22} \text{ v. } R_{11} \tag{21}$$

where $$R_{mn}(\mathbf{x}\,;\,\mathbf{y}) = \left\langle u_m\left(\mathbf{x} + \frac{\mathbf{y}}{2}\right) u_n\left(\mathbf{x} - \frac{\mathbf{y}}{2}\right)\right\rangle \tag{22}$$

In this notation for the covariance function the first argument represents the location of the function and the second the separation of its components. Within a mesh box we assume that the covariance is spatially constant, so that the first and last terms of the integrals are equal. The second and third terms are also equal since $\mathbf{y}$ and $\mathbf{z}$ can be interchanged. Equations (21) and (22) then can be simplified into

$$\left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 \right\rangle = \frac{2}{h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big\{ R_{11}\left[\mathbf{j}(y_2-z_2) + \mathbf{k}(y_3-z_3)\right] - R_{11}\left[\mathbf{i}h + \mathbf{j}(y_2-z_2) + \mathbf{k}(y_3-z_3)\right]\Big\}\, dy_2\, dy_3\, dz_2\, dz_3 \tag{23}$$

$$\left\langle \left(\frac{\partial \bar{u}_2}{\partial x_1}\right)^2 \right\rangle = \text{identical except } R_{22} \text{ v. } R_{11} \tag{24}$$

in which only the second argument of the covariance function is retained. By use of equations (3.4.5), (3.4.6), and (3.4.16) of Batchelor (1956) the covariance functions may be written in terms of the longitudinal velocity correlation function $f(r)$ and the energy spectrum function $E(\kappa)$ as follows:

$$R_{nn}(\mathbf{r}) = \frac{\langle u_n^2 \rangle}{2r} \frac{\partial}{\partial r}\left[\left(r^2 - r_n^2\right) f(r)\right] = \frac{1}{r}\frac{\partial}{\partial r}\int_0^\infty \frac{r^2 - r_n^2}{\kappa^2 r^2} E(\kappa)\left(\frac{\sin \kappa r}{\kappa r} - \cos \kappa r\right) d\kappa \tag{25}$$

where κ is the scalar wave number. We now assume that, when κr is of order unity, $E(\kappa)$ is given by the Kolmogoroff function

$$E(\kappa) = \alpha \langle \epsilon \rangle^{2/3} \kappa^{-5/3} \tag{26}$$

where $\langle \epsilon \rangle$ is the average dissipation rate and α is a dimensionless constant. With this choice of the spectrum function the integrand becomes infinite at $\kappa = 0$. To avoid this difficulty, which is purely mathematical since $E(0) \to 0$ in reality, we subtract the correlation function with argument $\mathbf{r} = 0$ from both sides of (25). Since $R_{nn}(0) = \langle u_n^2 \rangle = (2/3)\int_0^\infty E(\kappa)\, d\kappa$, the singularity at the origin is then cancelled out when (26) is substituted. Equation (25) may then be integrated to yield:

$$R_{nn}(0) - R_{nn}(\mathbf{r}) = \frac{18}{55}\Gamma\left(\tfrac{1}{3}\right) \alpha \langle \epsilon \rangle^{2/3} r^{2/3}\left(1 - \frac{r_n^2}{4r^2}\right) \tag{27}$$

Substitution of (27) into (23) and (24) then leads to the expressions

$$\left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 \right\rangle = \frac{36}{55}\Gamma\left(\tfrac{1}{3}\right) \frac{\alpha \langle \epsilon \rangle^{2/3}}{h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \left[1 - \frac{h^2/4}{h^2 + (y_2 - z_2)^2 + (y_3 - z_3)^2}\right]\left[h^2 + (y_2 - z_2)^2 + (y_3 - z_3)^2\right]^{1/3} - \left[(y_2 - z_2)^2 + (y_3 - z_3)^2\right]^{1/3} \right\} dy_2\, dy_3\, dz_2\, dz_3 \tag{28}$$

$$\left\langle \left(\frac{\partial \bar{u}_2}{\partial x_1}\right)^2 \right\rangle = \frac{36}{55}\Gamma\left(\tfrac{1}{3}\right) \frac{\alpha \langle \epsilon \rangle^{2/3}}{h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \left[1 - \frac{(y_2 - z_2)^2/4}{h^2 + (y_2 - z_2)^2 + (y_3 - z_3)^2}\right]\left[h^2 + (y_2 - z_2)^2 + (y_3 - z_3)^2\right]^{1/3} - \left[1 - \frac{(y_2 - z_2)^2/4}{(y_2 - z_2)^2 + (y_3 - z_3)^2}\right]\left[(y_2 - z_2)^2 + (y_3 - z_3)^2\right]^{1/3} \right\} dy_2\, dy_3\, dz_2\, dz_3 \tag{29}$$

The integrals in (28) and (29) have been evaluated numerically to third decimal accuracy. Pond, Stewart, and Burling (1963) obtained a measured value of the constant $\alpha = 1.41$, with perhaps a 10% error to be expected. By use of this value we evaluate the above relations and (17) as follows:

$$\left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1} \right)^2 \right\rangle \approx .63 \langle \epsilon \rangle^{2/3} h^{-4/3} \tag{30}$$

$$\left\langle \left(\frac{\partial \bar{u}_2}{\partial x_1} \right)^2 \right\rangle \approx 1.26 \langle \epsilon \rangle^{2/3} h^{-4/3} \tag{31}$$

$$\langle Def^2 \rangle \approx 9.46 \langle \epsilon \rangle^{2/3} h^{-4/3} \tag{32}$$

The fact that $\langle (\partial \bar{u}_2/\partial x_1)^2 \rangle = 2\langle (\partial \bar{u}_1/\partial x_1)^2 \rangle$ can also be shown by an extension of Taylor's (1935) proof for the unaveraged quantities. This represents a check on the correctness of the rather tedious calculations involved. The defining characteristic of the inertial sub-range is that the kinetic energy transfer across wave numbers is equal to the dissipation rate. We therefore assume $\langle \epsilon \rangle = \langle S \rangle$ in (13). Upon substitution of (32) into the right side, the dimensional quantities cancel out and we are left with a requirement on the constant k, that is

$$k = \frac{\langle \epsilon \rangle^{1/2}}{h \langle Def^2 \rangle^{3/4}} \approx .185 \tag{33}$$

Since k is proportional to $\alpha^{-3/4}$, we may expect (33) to be in error by something under 10%.

Finite difference approximations

Equations (5), (7), (8), and (12) could be integrated by any of several numerical methods. If, for example, the velocity components were expressed as a sum of Fourier components then the above evaluation of k would be appropriate, provided that the bracket average is taken over all computation space and the turbulence is essentially homogeneous. If the equations are approximated by finite difference formulae and the bracket average taken over a smaller volume in inhomogeneous turbulence, then k must be evaluated from the particular finite difference approximations used. If, as in Smagorinsky's (1965) computations, we replace all first derivatives by differences across one grid interval, then the squared deformation tensor in (11) and subsequently is replaced by

$$\mathcal{D}ef_h^2 = \Delta_i \bar{u}_j \left(\Delta_j \bar{u}_i + \Delta_i \bar{u}_j \right) \tag{34}$$

where $$\Delta_i \varphi = \left[\varphi(x_i + \tfrac{h}{2}) - \varphi(x_i - \tfrac{h}{2}) \right] / h \tag{35}$$

for any variable $\varphi(x_i)$. Equations (12) - (19) remain essentially unchanged except for the replacement of differences for derivatives, but the reduction in integral order in (20) and (21) cannot be accomplished. The expression analogous to (20) is

$$\left\langle (\Delta_1 \bar{u}_1)^2 \right\rangle = \frac{1}{h^8} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big\{ R_{11}\left(\mathbf{x} + \mathring{a}\tfrac{h}{2} + \tfrac{\mathbf{y}+\mathbf{z}}{2} ; \mathbf{y}-\mathbf{z}\right) - R_{11}\left(\mathbf{x} + \tfrac{\mathbf{y}+\mathbf{z}}{2} ; -\mathring{a}h + \mathbf{y} - \mathbf{z}\right)$$
$$- R_{11}\left(\mathbf{x} + \tfrac{\mathbf{y}+\mathbf{z}}{2} ; \mathring{a}h + \mathbf{y} - \mathbf{z}\right) + R_{11}\left(\mathbf{x} - \mathring{a}\tfrac{h}{2} + \tfrac{\mathbf{y}+\mathbf{z}}{2} ; \mathbf{y}-\mathbf{z}\right) \Big\} d\mathbf{y}\, d\mathbf{z} \tag{36}$$

while that similar to (23) is the following

$$\left\langle (\Delta_1 \bar{u}_1)^2 \right\rangle = \frac{2}{h^8} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[R_{11}(\mathbf{y}-\mathbf{z}) - R_{11}(\mathbf{y} + \mathring{a}h - \mathbf{z}) \right] d\mathbf{y}\, d\mathbf{z} \tag{37}$$

After substitution of (27) into (37) and into a similar relation for $\langle(\Delta_1 \bar{u}_2)^2\rangle$, the final integral expressions are

$$\langle(\Delta_1 \bar{u}_1)^2\rangle = \frac{36}{55}\Gamma\left(\tfrac{1}{3}\right)\frac{\alpha\langle\epsilon\rangle^{2/3}}{h^8}\int_{-\frac{h}{2}}^{\frac{h}{2}}\cdots\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\left(1-\frac{1}{4}\left|\frac{y_1+h-z_1}{y+\mathring{a}h-z}\right|^2\right)|y+\mathring{a}h-z|^{2/3}-\left(1-\frac{1}{4}\left|\frac{y_1-z_1}{y-z}\right|^2\right)|y-z|^{2/3}\right]dy\,dz \quad (38)$$

$$\langle(\Delta_1 \bar{u}_2)^2\rangle = \frac{36}{55}\Gamma\left(\tfrac{1}{3}\right)\frac{\alpha\langle\epsilon\rangle^{2/3}}{h^8}\int_{-\frac{h}{2}}^{\frac{h}{2}}\cdots\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\left(1-\frac{1}{4}\left|\frac{y_2-z_2}{y+\mathring{a}h-z}\right|^2\right)|y+\mathring{a}h-z|^{2/3}-\left(1-\frac{1}{4}\left|\frac{y_2-z_2}{y-z}\right|^2\right)|y-z|^{2/3}\right]dy\,dz \quad (39)$$

Evaluation of the integrals leads to the results

$$\langle(\Delta_1 \bar{u}_1)^2\rangle \approx .50\,\langle\epsilon\rangle^{2/3}\,h^{-4/3} \quad (40)$$

$$\langle(\Delta_1 \bar{u}_2)^2\rangle \approx .93\,\langle\epsilon\rangle^{2/3}\,h^{-4/3} \quad (41)$$

$$\langle D_e f_h^2\rangle \approx 7.08\,\langle\epsilon\rangle^{2/3}\,h^{-4/3} \quad (42)$$

$$k \approx .230 \quad (43)$$

Here there is no reason to expect an even ratio between (40) and (41).

As might be expected, the truncation errors of finite differencing lead to some decrease of magnitude of the derivatives, which is compensated for by an increase in the viscosity coefficient. It is interesting to note that in recent integrations of the general circulation equations Smagorinsky used, on empirical grounds, a value of $k = 0.28$. The agreement with (43) may be fortuitous and perhaps is related to L. F. Richardson's (1926) discovery that large-scale quasi-two-dimensional atmospheric motions seem to conform to the same scale relationships as ~~were later postulated and verified for~~ three-dimensional isotropic turbulence.

Turbulent energy method

The use of Smagorinsky's method rests on an implicit assumption that an equilibrium inertial sub-range is developed instantly as soon as velocity gradients appear. In highly inhomogeneous flows, it may be important to consider the lag between changes of the mean flow gradients and the turbulent response. It appears to be possible to at least partly take account of these transients by defining and predicting changes in a small scale turbulent energy parameter which is only indirectly related to the mean flow gradients. In place of (12) we assume an eddy viscosity proportional to the turbulent intensity,

$$K = k_v h \langle E_T \rangle^{1/2} \tag{44}$$

where k_v is now another constant, to be determined below, and

$$E_T = \frac{1}{2} \overline{u_i'^2} = \frac{1}{2} \left(\overline{u_i^2} - \bar{u}_i^2 \right) \tag{45}$$

Formal replacement of the terms of the bracket average of (45) by correlation functions leads to the relations

$$\langle E_T \rangle = \frac{1}{2h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[R_{ii}(0) - R_{ii}(y-z) \right] dy\, dz = \frac{9}{20} \Gamma\left(\tfrac{1}{3}\right) \frac{\alpha \langle \epsilon \rangle^{2/3}}{h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} |y-z|^{2/3} dy\, dz \tag{46}$$

in which we have again assumed the Kolmogoroff spectrum function. The integral, which already appeared in (38), has the numerical value 0.761 $h^{20/3}$, so that (46) becomes

$$\langle E_T \rangle \approx 1.29 \langle \epsilon \rangle^{2/3} h^{2/3} \tag{47}$$

A time dependent differential equation for E_T may be written (see e.g. Hinze, 1959, equation 1-99) in the tensor notation, as

$$\frac{\partial E_T}{\partial t} + \frac{\partial}{\partial x_j}(u_j E_T) + \frac{\partial}{\partial x_j}\left[\overline{u_j'\left(\frac{u_i'^2}{2} + \frac{p'}{\rho}\right)}\right] - \frac{\partial}{\partial x_j}\left(\nu \frac{\partial E_T}{\partial x_j}\right) = \tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{48}$$

This equation is not practical for numerical solution because of the unknown triple correlation terms and the requirement for high resolution in the viscous terms. We substitute for it an equation in which the triple-correlation terms are replaced by an eddy diffusion, and the averaged dissipation is computed from (47), that is

$$\frac{\partial E_T}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j E_T) - \frac{\partial}{\partial x_j}\left(K_e \frac{\partial E_T}{\partial x_j}\right) = K\, Def^2 - .679\,\frac{E_T \langle E_T\rangle^{1/2}}{h} \tag{49}$$

The first term on the right, the transfer from the explicit flow, is taken directly from (10). An equation similar to this has been derived in a somewhat different context by Prandtl (1945) and used by Emmons (   ) and others. The diffusion coefficient for turbulent energy, K_e , apparently cannot be determined from the isotropic turbulence assumptions. It is not likely to be of any great significance, except as a device for assuring smoothness of the E_T field. The assumption $K_e = K$ is probably satisfactory. The coefficient k_v appearing in (44) can be unambiguously determined by requiring that (12) hold in steady-state homogeneous turbulence. In those conditions the bracket average of (49) becomes

$$K\,\langle Def^2\rangle = .679\,\langle E_T\rangle^{3/2}/h \tag{50}$$

By elimination between (32), (44), (47), and (50) we find that

$$k_v \approx \frac{.679\,(1.29)}{9.46} = .093 \tag{51}$$

If Def^2 is replaced by Def_h^2 in (49) and (50) the corresponding finite difference result is

$$k_v \approx \frac{.679\,(1.29)}{7.08} = .124 \tag{52}$$

Discussion

The above results bear some resemblance to certain previous theoretical models, in particular those of Kovasznay (1948) and Heisenberg (1948), for the energy transfer function. It can be shown, from (25), that the integral for almost any reasonable energy spectrum function is dominated by the region of the spectrum near $\kappa r = 1$. Thus, typically,

$$R_{nn}(0) - R_{nn}(\mathbf{r}) \propto r^{-1} E(r^{-1}) \tag{53}$$

The result of the averaging integrations, by dimensional analysis, must therefore yield

$$\langle Def^2 \rangle \propto h^{-3} E(h^{-1}) \tag{54}$$

or, for Smagorinsky's method,

$$\langle S \rangle \propto h^2 \langle Def^2 \rangle^{1/2} \propto h^{-5/2} \left[E(h^{-1})\right]^{1/2} \tag{55}$$

This is essentially identical to Kovasznay's hypothesis for the energy transfer across the wave number $\kappa = h^{-1}$. On the other hand Heisenberg's transfer function can be written in the form

$$\langle S \rangle = K(h) \int_0^{h^{-1}} 2\kappa^2 E(\kappa)\, d\kappa \tag{56}$$

where the integral is the contribution to mean square vorticity (or deformation) from all wave numbers lower than h^{-1} and $K(h)$ is an eddy viscosity obtained from an integral expression involving only scales smaller than h^{-1}, i.e.

$$K(h) = \gamma \int_{h^{-1}}^{\infty} \kappa^{-3/2} \left[E(\kappa) \right]^{1/2} d\kappa \tag{57}$$

where γ is a constant. The expressions developed in the turbulent energy method are closely related to these, identical when the Kolmogoroff spectrum function is valid for $\kappa \geqslant h^{-1}$.

Kovasznay's and Heisenberg's formulae have been criticized on various grounds, both theoretical and practical. Kraichnan ( ) suggested that, if applied to time integration of a complete energy spectrum, Heisenberg's equations would be unstable. This objection cannot hold for the application considered here, that is for the transfer of kinetic energy to scales smaller than those computed by direct integration of the large scale dynamic equations. It is easily seen, from equation (49), that oscillations of E_T are always damped. The dissipation term, being of a higher degree in E_T than either the transfer or advection terms, always tends to force K;

computed from (44), to approach the value given by Smagorinsky's method, from (12). This can be shown more rigorously by a perturbation expansion and has been verified by some unpublished integrations of thermal convection problems.

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