

On the Approximation of the Cut and Step Functions by Logistic and Gompertz Functions

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Abstract—We study the uniform approximation of the sigmoid cut function by smooth sigmoid functions such as the logistic and the Gompertz functions. The limiting case of the interval-valued step function is discussed using Hausdorff metric. Various expressions for the error estimates of the corresponding uniform and Hausdorff approximations are obtained. Numerical examples are presented using *CAS MATHEMATICA*.

Keywords—cut function; step function; sigmoid function; logistic function; Gompertz function; squashing function; Hausdorff approximation.

I. INTRODUCTION

In this paper we discuss some computational, modelling and approximation issues related to several classes of sigmoid functions. Sigmoid functions find numerous applications in various fields related to life sciences, chemistry, physics, artificial intelligence, etc. In fields such as signal processing, pattern recognition, machine learning, artificial neural networks, sigmoid functions are also known as “activation” and “squashing” functions. In this work we concentrate on several

practically important classes of sigmoid functions. Two of them are the cut (or ramp) functions and the step functions. Cut functions are continuous but they are not smooth (differentiable) at the two endpoints of the interval where they increase. Step functions can be viewed as limiting case of cut functions; they are not continuous but they are Hausdorff continuous (H-continuous) [4], [43]. In some applications smooth sigmoid functions are preferred, some authors even require smoothness in the definition of sigmoid functions. Two familiar classes of smooth sigmoid functions are the logistic and the Gompertz functions. There are situations when one needs to pass from nonsmooth sigmoid functions (e. g. cut functions) to smooth sigmoid functions, and vice versa. Such a necessity rises the issue of approximating nonsmooth sigmoid functions by smooth sigmoid functions.

One can encounter similar approximation problems when looking for appropriate models for fitting time course measurement data coming e. g. from cellular growth experiments. Depending on the general view of the data one can decide to use

initially a cut function in order to obtain rough initial values for certain parameters, such as the maximum growth rate. Then one can use a more sophisticated model (logistic or Gompertz) to obtain a better fit to the measurement data. The presented results may be used to indicate to what extent and in what sense a model can be improved by another one and how the two models can be compared.

Section 2 contains preliminary definitions and motivations. In Section 3 we study the uniform and Hausdorff approximation of the cut functions by logistic functions. Curiously, the uniform distance between a cut function and the logistic function of best uniform approximation is an absolute constant not depending on the slope of the functions, a result observed in [18]. By contrast, it turns out that the Hausdorff distance (H-distance) depends on the slope and tends to zero when increasing the slope. Showing that the family of logistic functions cannot approximate the cut function arbitrary well, we then consider the limiting case when the cut function tends to the step function (in Hausdorff sense). In this way we obtain an extension of a previous result on the Hausdorff approximation of the step function by logistic functions [4]. In Section 4 we discuss the approximation of the cut function by a family of squashing functions induced by the logistic function. It has been shown in [18] that the latter family approximates uniformly the cut function arbitrary well. We propose a new estimate for the H-distance between the cut function and its best approximating squashing function. Our estimate is then extended to cover the limiting case of the step function. In Section 5 the approximation of the cut function by Gompertz functions is considered using similar techniques as in the previous sections. The application of the logistic and Gompertz functions in life sciences is briefly discussed. Numerical examples are presented throughout the paper using the computer algebra system *MATHEMATICA*.

II. PRELIMINARIES

Sigmoid functions. In this work we consider *sigmoid functions* of a single variable defined on

the real line, that is functions s of the form $s : \mathbb{R} \rightarrow \mathbb{R}$. Sigmoid functions can be defined as bounded monotone non-decreasing functions on \mathbb{R} . One usually makes use of normalized sigmoid functions defined as monotone non-decreasing functions $s(t), t \in \mathbb{R}$, such that $\lim s(t)_{t \rightarrow -\infty} = 0$ and $\lim s(t)_{t \rightarrow \infty} = 1$. In the fields of neural networks and machine learning sigmoid-like functions of many variables are used, familiar under the name *activation functions*. (In some applications the sigmoid functions are normalised so that the lower asymptote is assumed -1 : $\lim s(t)_{t \rightarrow -\infty} = -1$.)

Cut (ramp) functions. Let $\Delta = [\gamma - \delta, \gamma + \delta]$ be an interval on the real line \mathbb{R} with centre $\gamma \in \mathbb{R}$ and radius $\delta \in \mathbb{R}$. A cut function (on Δ) is defined as follows:

Definition 1. The cut function $c_{\gamma,\delta}$ on Δ is defined for $t \in \mathbb{R}$ by

$$c_{\gamma,\delta}(t) = \begin{cases} 0, & \text{if } t < \Delta, \\ \frac{t - \gamma + \delta}{2\delta}, & \text{if } t \in \Delta, \\ 1, & \text{if } \Delta < t. \end{cases} \quad (1)$$

Note that the slope of function $c_{\gamma,\delta}(t)$ on the interval Δ is $1/(2\delta)$ (the slope is constant in the whole interval Δ). Two special cases are of interest for our discussion in the sequel.

Special case 1. For $\gamma = 0$ we obtain a cut function on the interval $\Delta = [-\delta, \delta]$:

$$c_{0,\delta}(t) = \begin{cases} 0, & \text{if } t < -\delta, \\ \frac{t + \delta}{2\delta}, & \text{if } -\delta \leq t \leq \delta, \\ 1, & \text{if } \delta < t. \end{cases} \quad (2)$$

Special case 2. For $\gamma = \delta$ we obtain the cut function on $\Delta = [0, 2\delta]$:

$$c_{\delta,\delta}(t) = \begin{cases} 0, & \text{if } t < 0, \\ \frac{t}{2\delta}, & \text{if } 0 \leq t \leq 2\delta, \\ 1, & \text{if } 2\delta < t. \end{cases} \quad (3)$$

Step functions. The step function (with “jump” at $\gamma \in \mathbb{R}$) can be defined by

$$h_\gamma(t) = c_{\gamma,0}(t) = \begin{cases} 0, & \text{if } t < \gamma, \\ [0, 1], & \text{if } t = \gamma, \\ 1, & \text{if } t > \gamma, \end{cases} \quad (4)$$

which is an *interval-valued function* (or just *interval function*) [4], [43]. In the literature various point values, such as 0, 1/2 or 1, are prescribed to the step function (4) at the point γ ; we prefer the interval value $[0, 1]$. When the jump is at the origin, that is $\gamma = 0$, then the step function is known as the Heaviside step function; its “interval” formulation is:

$$h_0(t) = c_{0,0}(t) = \begin{cases} 0, & \text{if } t < 0, \\ [0, 1], & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (5)$$

H-distance. The step function can be perceived as a limiting case of the cut function. Namely, for $\delta \rightarrow 0$, the cut function $c_{\delta,\delta}$ tends in “Hausdorff sense” to the step function. Here “Hausdorff sense” means *Hausdorff distance*, briefly *H-distance*. The H-distance $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$ [24], [41]. More precisely,

$$\rho(f, g) = \max\left\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \|A - B\|, \sup_{B \in F(g)} \inf_{A \in F(f)} \|A - B\| \right\}, \quad (6)$$

wherein $\|\cdot\|$ is any norm in \mathbb{R}^2 , e. g. the maximum norm $\|(t, x)\| = \max\{|t|, |x|\}$.

To prove that (3) tends to (5) let h be the H-distance between the step function (5) and the cut function (3) using the maximum norm, that is a square (box) unit ball. By definition (6) h is the side of the smallest unit square, centered at the point $(0, 1)$ touching the graph of the cut function. Hence we have $1 - c_{\delta,\delta}(h) = h$, that is $1 - h/(2\delta) = h$, implying

$$h = \frac{2\delta}{1 + 2\delta} = 2\delta + O(\delta^2).$$

For the sake of simplicity throughout the paper we shall work with some of the special cut functions (2), (3), instead of the more general (arbitrary shifted) cut function (1); these special choices will not lead to any loss of generality concerning the results obtained. Moreover, for all sigmoid functions considered in the sequel we shall define a “basic” sigmoid function such that any member of the corresponding class is obtained by replacing the argument t by $t - \gamma$, that is by shifting the basic function by some $\gamma \in \mathbb{R}$.

Logistic and Gompertz functions: applications to life-sciences. In this work we focus on two familiar smooth sigmoid functions, namely the Gompertz function and the Verhulst logistic function. Both their inventors, B. Gompertz and P.-F. Verhulst, have been motivated by the famous demographic studies of Thomas Malthus.

The Gompertz function was introduced by Benjamin Gompertz [22] for the study of demographic phenomena, more specifically human aging [38], [39], [47]. Gompertz functions find numerous applications in biology, ecology and medicine. A. K. Laird successfully used the Gompertz curve to fit data of growth of tumors [32]; tumors are cellular populations growing in a confined space where the availability of nutrients is limited [1], [2], [15], [19].

A number of experimental scientists apply Gompertz models in bacterial cell growth, more specifically in food control [10], [31], [42], [48], [49], [50]. Gompertz models prove to be useful in animal and agro-sciences as well [8], [21], [27], [48]. The Gompertz model has been applied in modelling aggregation processes [25], [26]; it is a subject of numerous theoretical modelling studies as well [6], [7], [9], [20], [37], [40].

The logistic function was introduced by Pierre François Verhulst [44]–[46], who applied it to human population dynamics. Verhulst derived his logistic equation to describe the mechanism of the self-limiting growth of a biological population. The equation was rediscovered in 1911 by A. G. McKendrick [35] for the bacterial growth in

broth and was tested using nonlinear parameter estimation. The logistic function finds applications in an wide range of fields, including biology, ecology, population dynamics, chemistry, demography, economics, geoscience, mathematical psychology, probability, sociology, political science, financial mathematics, statistics, fuzzy set theory, to name a few [12], [13], [11], [14], [18].

Logistic functions are often used in artificial neural networks [5], [16], [17], [23]. Any neural net element computes a linear combination of its input signals, and applies a logistic function to the result; often called “activation” function. Another application of logistic curve is in medicine, where the logistic differential equation is used to model the growth of tumors. This application can be considered an extension of the above-mentioned use in the framework of ecology. In (bio)chemistry the concentration of reactants and products in autocatalytic reactions follow the logistic function.

Other smooth sigmoid functions. The integral (antiderivative) of any smooth, positive, “bump-shaped” or “bell-shaped” function will be sigmoidal. A famous example is the error function, which is the integral (also called the cumulative distribution function) of the Gaussian normal distribution. The logistic function is also used as a base for the derivation of other sigmoid functions, a notable example is the generalized logistic function, also known as Richards curve [37]. Another example is the Dombi-Gera-squashing function introduced and studied in [18] obtained as an antiderivative (indefinite integral) of the difference of two shifted logistic functions.

In what follows we shall be interested in the approximation of the cut function by smooth sigmoid functions, more specifically the Gompertz, the logistic and the Dombi-Gera-squashing function. We shall focus first on the Verhulst logistic function.

III. APPROXIMATION OF THE CUT FUNCTION BY LOGISTIC FUNCTIONS

Definition 2. Define the logistic (Verhulst) function v on \mathbb{R} as [44]–[46]

$$v_{\gamma,k}(t) = \frac{1}{1 + e^{-4k(t-\gamma)}}. \tag{7}$$

Note that the logistic function (7) has an inflection at its “centre” $(\gamma, 1/2)$ and its slope at γ is equal to k .

Proposition 1. [18] The function $v_{\gamma,k}(t)$ defined by (7) with $k = 1/(2\delta)$: i) is the logistic function of best uniform one-sided approximation to function $c_{\gamma,\delta}(t)$ in the interval $[\gamma, \infty)$ (as well as in the interval $(-\infty, \gamma]$); ii) approximates the cut function $c_{\gamma,\delta}(t)$ in uniform metric with an error

$$\rho = \rho(c, v) = \frac{1}{1 + e^2} = 0.11920292\dots \tag{8}$$

Proof. Consider functions (1) and (7) with same centres $\gamma = \delta$, that is functions $c_{\delta,\delta}$ and $v_{\delta,k}$. In addition chose c and v to have same slopes at their coinciding centres, that is assume $k = 1/(2\delta)$, cf. Figure 1. Then, noticing that the largest uniform distance between the cut and logistic functions is achieved at the endpoints of the underlying interval $[0, 2\delta]$, we have:

$$\rho = v_{\delta,k}(0) - c_{\delta,\delta} = \frac{1}{1 + e^{4k\delta}} = \frac{1}{1 + e^2}. \tag{9}$$

This completes the proof of the proposition.

We note that the uniform distance (9) is an absolute constant that does not depend on the width of the underlying interval Δ , resp. on the slope k . The next proposition shows that this is not the case whenever H-distance is used.

Proposition 2. The function $v(t) = v_{0,k}(t)$ with $k = 1/(2\delta)$ is the logistic function of best Hausdorff one-sided approximation to function $c(t) = c_{0,k}(t)$ in the interval $[0, \infty)$ (resp. in the interval $(-\infty, 0]$). The function $v(t)$, approximates function $c(t)$ in H-distance with an error $h = h(c, v)$ that satisfies the relation:

$$\ln \frac{1 - h}{h} = 2 + 4kh. \tag{10}$$

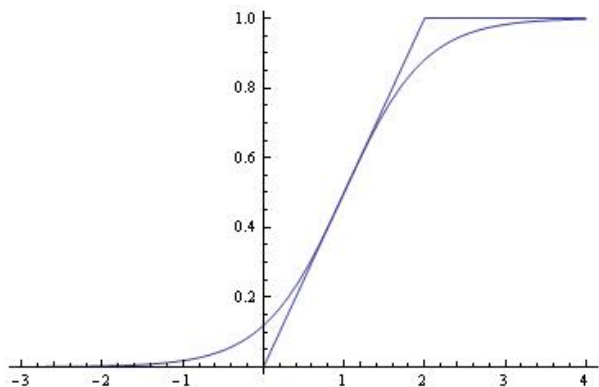


Fig. 1. The cut and logistic functions for $\gamma = \delta = 1$, $k = 1/2$.

Proof. Using $\delta = 1/(2k)$ we can write $\delta + h = (1 + 2hk)/(2k)$, resp.:

$$v(-\delta - h) = \frac{1}{1 + e^{2(1+2hk)}}.$$

The H-distance h using square unit ball (with a side h) satisfies the relation $v(-\delta - h) = h$, which implies (10). This completes the proof of the proposition.

Relation (10) shows that the H-distance h depends on the slope k , $h = h(k)$. The next result gives additional information on this dependence.

Proposition 3. For the H-distance $h(k)$ the following holds for $k > 5$:

$$\frac{1}{4k + 1} < h(k) < \frac{\ln(4k + 1)}{4k + 1}. \quad (11)$$

Proof. We need to express h in terms of k , using (10). Let us examine the function

$$f(h) = 2 + 4hk - \ln(1 - h) - \ln \frac{1}{h}.$$

From

$$f'(h) = 4k + \frac{1}{1 - h} + \frac{1}{h} > 0$$

we conclude that function f is strictly monotone increasing. Consider the function

$$g(h) = 2 + h(1 + 4k) - \ln \frac{1}{h}.$$

Then $g(h) - f(h) = h + \ln(1 - h) = O(h^2)$ using the Taylor expansion $\ln(1 - h) = -h + O(h^2)$. Hence $g(h)$ approximates $f(h)$ with $h \rightarrow 0$ as $O(h^2)$. In addition $g'(h) = 1 + 4k + 1/h > 0$, hence function g is monotone increasing. Further, for $k \geq 5$

$$g\left(\frac{1}{1 + 4k}\right) = 3 - \ln(1 + 4k) < 0,$$

$$g\left(\frac{\ln(4k + 1)}{4k + 1}\right) = 2 + \ln \ln(1 + 4k) > 0.$$

This completes the proof of the proposition.

Relation (11) implies that when the slope k of functions c and v tends to infinity, the h-distance $h(c, v)$ between the two functions tends to zero (differently to the uniform distance $\rho(c, v)$ which remains constant).

The following proposition gives more precise upper and lower bounds for $h(k)$. For brevity denote $K = 4k + 1$.

Proposition 4. For the H-distance h the following inequalities hold for $k \geq 5$:

$$\frac{\ln K}{K} - \frac{2 + \ln \ln K}{K \left(1 + \frac{1}{\ln K}\right)} < h(k) < \quad (12)$$

$$\frac{\ln K}{K} + \frac{2 + \ln \ln K}{K \left(\frac{\ln \ln K}{1 - \ln K} - 1\right)}, K = 4k + 1.$$

Proof. Evidently, the second derivative of $g(h) = 2 + h(1 + 4k) - \ln(1/h)$, namely $g''(h) = -\frac{1}{h^2} < 0$, has a constant sign on $[\frac{1}{K}, \frac{\ln K}{K}]$. The straight line, defined by the points $(\frac{1}{K}, g(\frac{1}{K}))$ and $(\frac{\ln K}{K}, g(\frac{\ln K}{K}))$, and the tangent to g at the point $(\frac{\ln K}{K}, g(\frac{\ln K}{K}))$ cross the abscissa at the points

$$\frac{\ln K}{K} + \frac{2 + \ln \ln K}{K \left(\frac{\ln \ln K}{1 - \ln K} - 1\right)}, \quad \frac{\ln K}{K} - \frac{2 + \ln \ln K}{K \left(1 + \frac{1}{\ln K}\right)},$$

respectively. This completes the proof of the Proposition.

Propositions 2, 3 and 4 extend similar results from [4] stating that the Heaviside interval-valued step function is approximated arbitrary well by

logistic functions in Hausdorff metric. The Hausdorff approximation of the Heaviside step function by sigmoid functions is discussed from various computational and modelling aspects in [28], [29], [30].

IV. APPROXIMATION OF THE CUT FUNCTION BY A SQUASHING FUNCTION

The results obtained in Section 3 state that the cut function cannot be approximated arbitrary well by the family of logistic functions. This result justifies the discussion of other families of smooth sigmoid functions having better approximating properties. Such are the squashing functions proposed in [18] further denoted DG-squashing functions.

Definition 3. The DG-squashing function s_{Δ} on the interval $\Delta = [\gamma - \delta, \gamma + \delta]$ is defined by

$$s_{\Delta}^{(\beta)}(t) = s_{\gamma,\delta}^{(\beta)}(t) = \frac{1}{2\delta} \ln \left(\frac{1 + e^{\beta(t-\gamma+\delta)}}{1 + e^{\beta(t-\gamma-\delta)}} \right)^{\frac{1}{\beta}}. \tag{13}$$

Note that the squashing function (13) has an inflection at its ‘‘centre’’ γ and its slope at γ is equal to $(2\delta)^{-1}$.

The squashing function (13) with centre $\gamma = \delta$:

$$s_{\delta,\delta}^{(\beta)}(t) = \frac{1}{2\delta} \ln \left(\frac{1 + e^{\beta t}}{1 + e^{\beta(t-2\delta)}} \right)^{\frac{1}{\beta}}, \tag{14}$$

is the function of best uniform approximation to the cut function (3). Indeed, functions $c_{\delta,\delta}$ and $s_{\gamma,\delta}^{(\beta)}$ have same centre $\gamma = \delta$ and equal slopes $1/(2\delta)$ at their coinciding centres. As in the case with the logistic function, one observes that the uniform distance $\rho = \rho(c, s)$ between the cut and squashing function is achieved at the endpoints of the interval Δ , more specifically at the origin. Denoting the width of the interval Δ by $w = 2\delta$ we obtain

$$\rho = s_{\delta,\delta}^{(\beta)}(0) = \frac{1}{w} \ln \left(\frac{2}{1 + e^{\beta(-w)}} \right)^{1/\beta} < \tag{15}$$

$$\frac{\ln 2}{w} \frac{1}{\beta} = \text{const} \frac{1}{\beta}.$$

The estimate (15) has been found by Dombi and Gera [18]. This result shows that any cut

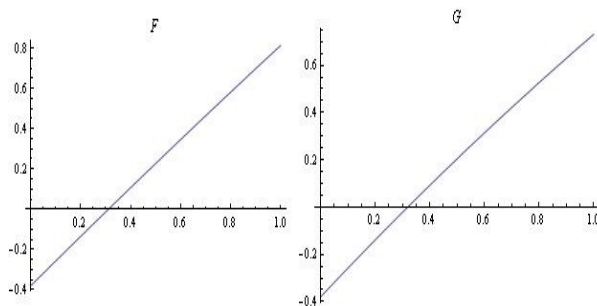


Fig. 2. The functions $F(d)$ and $G(d)$.

function c_{Δ} can be approximated arbitrary well by squashing functions $s_{\Delta}^{(\beta)}$ from the class (13). The approximation becomes better with the increase of the value of the parameter β . Thus β affects the quality of the approximation; as we shall see below the practically interesting values of β are integers greater than 4.

In what follows we aim at an analogous result using Hausdorff distance. Let us fix again the centres of the cut and squashing functions to be $\gamma = \delta$ so that the form of the cut function is $c_{\delta,\delta}$, namely (3), whereas the form of the squashing function is $s_{\delta,\delta}^{(\beta)}$ as given by (14). Both functions $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ have equal slopes $1/w$, $w = 2\delta$, at their centres δ .

Denoting the square-based H-distance between $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ by $d = d(w; \beta)$, $w = 2\delta$, we have the relation

$$s_{\delta,\delta}^{(\beta)}(w + d) = \frac{1}{w} \ln \left(\frac{1 + e^{\beta(w+d)}}{1 + e^{\beta d}} \right)^{\frac{1}{\beta}} = 1 - d$$

or

$$\ln \frac{1 + e^{\beta(w+d)}}{1 + e^{\beta d}} = \beta w(1 - d). \tag{16}$$

The following proposition gives an upper bound for $d = d(w; \beta)$ as implicitly defined by (16):

Proposition 5. For the distance d the following holds for $\beta \geq 5$:

$$d < \ln 2 \frac{\ln(4\beta w + 1)}{4w\beta + 1}. \tag{17}$$

Proof. We examine the function:

$$F(d) = -\beta w(1-d) + \ln(1 + e^{\beta(w+d)}) + \ln \frac{1}{1 + e^{\beta d}}.$$

From $F'(d) > 0$ we conclude that function $F(d)$ is strictly monotone increasing. We define the function

$$G(d) = -\beta w + \ln(1 + e^{\beta w}) +$$

$$d\beta \left(w + \frac{e^{\beta w}}{1 + e^{\beta w}} \right) + \ln \frac{1}{1 + e^{\beta d}}.$$

We examine $G(d) - F(d)$:

$$G(d) - F(d) =$$

$$\ln(1 + e^{\beta w}) + \frac{e^{\beta w} \beta d}{1 + e^{\beta w}} - \ln(1 + e^{\beta(w+d)}).$$

From Taylor expansion

$$\ln(1 + e^{\beta(w+d)}) = \ln(1 + e^{\beta w}) + \frac{e^{\beta w} \beta d}{1 + e^{\beta w}} + O(d^2)$$

we see that function $G(d)$ approximates $F(d)$ with $d \rightarrow 0$ as $O(d^2)$ (cf. Fig. 2).

In addition $G(0) < 0$ and $G\left(\ln 2 \frac{\ln(4\beta w + 1)}{4w\beta + 1}\right) > 0$ for $\beta \geq 5$. This completes the proof of the proposition.

Some computational examples using relation (16) and (17) for various β and w are presented in Table 1.

w	β	$d(w; \beta)$ from(16)	$d(w; \beta)$ from(17)
1	30	0.016040	0.027472
5	10	0.012639	0.018288
6	100	0.001068	0.002247
14	5	0.009564	0.013908
50	100	0.000137	0.000343
500	1000	1.38×10^{-6}	5.02×10^{-6}
1000	5000	1.3×10^{-7}	5.8×10^{-7}

TABLE I
BOUNDS FOR $d(w; \beta)$ COMPUTED BY (16) AND (17),
RESPECTIVELY

The numerical results are plotted in Fig. 3 (for the case $\beta = 5, w = 3; d = 0.0398921$) and Fig. 4 (for the case $\beta = 10, w = 4; d = 0.0154697$).

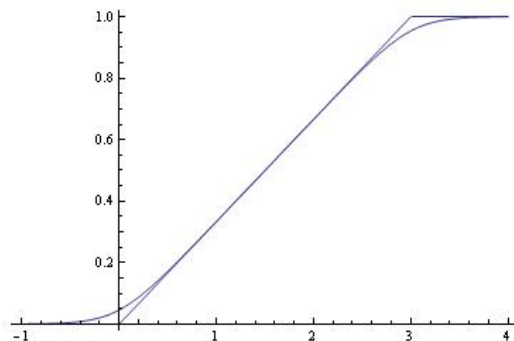


Fig. 3. Functions $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ for $\beta = 5, w = 3; d \leq 0.4$.

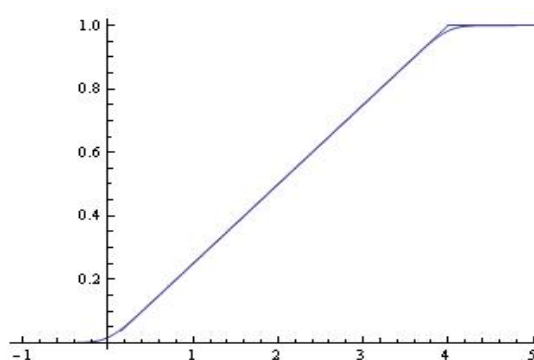


Fig. 4. Functions $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ for $\beta = 10, w = 4; d \leq 0.016$.

V. APPROXIMATION OF THE STEP FUNCTION BY THE GOMPERTZ FUNCTION

In this section we study the Hausdorff approximation of the step function by the Gompertz function and obtain precise upper and lower bounds for the Hausdorff distance. Numerical examples, illustrating our results are given.

Definition 4. The Gompertz function $\sigma_{\alpha,\beta}(t)$ is defined for $\alpha, \beta > 0$ by [22]:

$$\sigma_{\alpha,\beta}(t) = e^{-\alpha e^{-\beta t}}. \tag{18}$$

Special case 3. For $\alpha^* = \ln 2 = 0.69314718\dots$ we obtain the special Gompertz function:

$$\sigma_{\alpha^*,\beta}(t) = e^{-\alpha^* e^{-\beta t}}, \tag{19}$$

such that $\sigma_{\alpha^*,\beta}(0) = 1/2$.

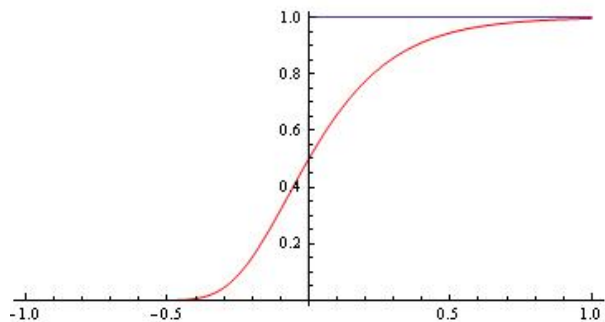


Fig. 5. The Gompertz function with $\alpha = \ln 2$ and $\beta = 5$; H-distance $d = 0.212765$.

We study the Hausdorff approximation of the Heaviside step function $c_0 = h_0(t)$ by Gompertz functions of the form (18) and find an expression for the error of the best approximation.

The H-distance $d = d(\alpha^*, \beta)$ between the Heaviside step function $h_0(t)$ and the Gompertz function (19) satisfies the relation

$$\sigma_{\alpha^*, \beta}(d) = e^{-\alpha^* e^{-\beta d}} = 1 - d,$$

or

$$\ln(1 - d) + \alpha^* e^{-\beta d} = 0. \tag{20}$$

The following theorem gives upper and lower bounds for $d(\alpha^*, \beta)$. For brevity we denote $\alpha = \alpha^*$ in Theorem 1 and its proof.

Theorem 1. The Hausdorff distance $d = d(\alpha, \beta)$ between the step function h_0 and the Gompertz function (19) can be expressed in terms of the parameter β for any real $\beta \geq 2$ as follows:

$$\frac{2\alpha - 1}{1 + \alpha\beta} < d < \frac{\ln(1 + \alpha\beta)}{1 + \alpha\beta}. \tag{21}$$

Proof. We need to express d in terms of α and β , using (20). Let us examine the function $F(d) = \ln(1 - d) + \alpha e^{-\beta d}$. From

$$F'(d) = -\frac{1}{1 - d} - \alpha\beta e^{-\beta d} < 0$$

we conclude that the function F is strictly monotone decreasing. Consider function $G(d) = \alpha - (1 + \alpha\beta)d$. From Taylor expansion

$$\alpha - (1 + \alpha\beta)d - \ln(1 - d) - \alpha e^{-\beta d} = O(d^2)$$

we obtain $G(d) - F(d) = \alpha - (1 + \alpha\beta)d - \ln(1 - d) - \alpha e^{-\beta d} = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \rightarrow 0$ as $O(d^2)$. In addition $G'(d) = -(1 + \alpha\beta) < 0$. Further, for $\beta \geq 2$,

$$G\left(\frac{2\alpha - 1}{1 + \alpha\beta}\right) = 1 - \alpha > 0,$$

$$G\left(\frac{\ln(1 + \alpha\beta)}{1 + \alpha\beta}\right) = \alpha - \ln(1 + \alpha\beta) < 0.$$

This completes the proof of the theorem.

Some computational examples using relation (20) are presented in Table 2.

β	$d(\alpha^*, \beta)$
2	0.310825
5	0.212765
10	0.147136
50	0.0514763
100	0.0309364
500	0.00873829
1000	0.00494117

TABLE 2
BOUNDS FOR $d(\alpha^*, \beta)$ COMPUTED BY (20) FOR VARIOUS β .

The calculation of the value of the H-distance between the Gompertz sigmoid function and the Heaviside step function is given in Appendix 1.

The numerical results are plotted in Fig. 5 (for the case $\alpha^* = \ln 2$, $\beta = 5$, H-distance $d = 0.212765$) and Fig. 6 (for the case $\alpha^* = \ln 2$, $\beta = 20$, H-distance $d = 0.0962215$).

Remark 1. For some comparisons of the Gompertz and logistic equation from both practical and theoretical perspective, see [6], [8], [40]. As can be seen from Figure 6 the graph of the Gompertz function is “skewed”, it is not symmetric with respect to the inflection point. In biology, the Gompertz function is commonly used to model growth process where the period of increasing growth is shorter than the period in which growth decreases [8], [33].

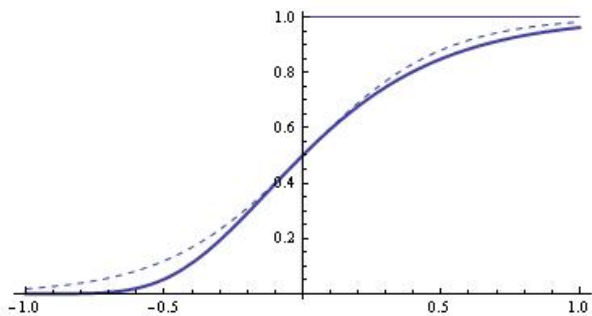


Fig. 6. The logistic (dotted line) and the Gompertz function (dense line) with same point and same rate (at that point).

Remark 2. For $k > 0, \beta > 0$ consider the differential equation

$$y' = ke^{-\beta t}y, \quad \frac{k}{\beta} = \alpha. \quad (22)$$

We have

$$\frac{dy}{dt} = ke^{-\beta t}y; \quad \frac{dy}{y} = ke^{-\beta t} dt$$

$$\ln y = -\frac{k}{\beta}e^{-\beta t} = -\alpha e^{-\beta t}; \quad y = e^{-\alpha e^{-\beta t}}.$$

We see that the solution of differential equation (22) is the Gompertz function $\sigma_{\alpha,\beta}(t)$ (18) [6]. As shown in [28], equation (22) can be interpreted as $y' = ksy$, wherein $s = s(t)$ is the nutrient substrate used for the growth of the population; one see that s is a decay exponential function in the Gompertz model (a similar interpretation can be found in [21]), [40]). For other interpretations see [6]), [8], [20].

VI. CONCLUSION

In this paper we discuss several computational, modelling and approximation issues related to two familiar classes of sigmoid functions—the logistic (Verhulst) and the Gompertz functions. Both classes find numerous applications in various fields of life sciences, ecology, medicine, artificial neural networks, fuzzy set theory, etc.

bigskip

We study the uniform and Hausdorff approximation of the cut functions by logistic functions. We

demonstrate that the best uniform approximation between a cut function and the respective logistic function is an absolute constant not depending on the (largest) slope k . On the other side we show that the Hausdorff distance (H-distance) depends on the slope k and tends to zero with $k \rightarrow \infty$. We also discuss the limiting case when the cut function tends to the Heaviside step function in Hausdorff sense, thereby extending a related previous result [4].

The approximation of the cut function by a family of squashing functions induced by the logistic function is also discussed. We propose a new estimate for the H-distance between a cut function and its best approximating squashing function. Our estimate extends a known result stating that the cut function can be approximated arbitrary well by squashing functions [18]. Our estimate is also extended to cover the limiting case of the Heaviside step function.

Finally we study the approximation of the cut and step functions by the family of Gompertz functions. New estimates for the H-distance between a cut function and its best approximating Gompertz function are obtained.

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APPENDIX

The Module “Computation of the distance d and visualization of the cut function c_Δ and squashing function $s_\Delta^{(\beta)}$ ” in *CAS MATHEMATICA*.

```
Print["Calculation of the value of the distance d (see Eq. (16)) and graphical visualization
of the generalized cut function [x]_{0,r} and the interval squashing function S_{0,r}^k"];

k = Input[" k"]; (*5 *)
Print["The parameter - k = ", k];

r = Input[" r"]; (*5 *)
Print["The parameter - r = ", r];

Print["The following nonlinear equation is used to determination of the distance d: "];

m = Log[(1 + Exp[k * (r - d)]) / (1 + Exp[d * k])] - k * r * (1 - d);

Print[m, " = 0"];
Print["The unique positive root of the equation is the searched value of d:"];
FindRoot[m == 0, {d, 0}];
Print[TableForm[%]];

Print["Graphical visualization of the generalized cut function [x]_{0,r}:"];
pw = Piecewise[{{0, x <= 0}, {x/r, 0 < x < r}, {1, r <= x}}];
g1 = Plot[pw, {x, -1, 7}];

Print["Graphical visualization of the the interval squashing function S_{0,r}^k:"];
g2 = Plot[1/r + Log[(1 + Exp[k * t]) / (1 + Exp[k * (t - r)])]^(1/k), {t, -1, 7}];

Print["Comparing of both graphical visualizations:"];
Show[g1, g2]
```

Fig. 7. Module in programming environment *MATHEMATICA*.

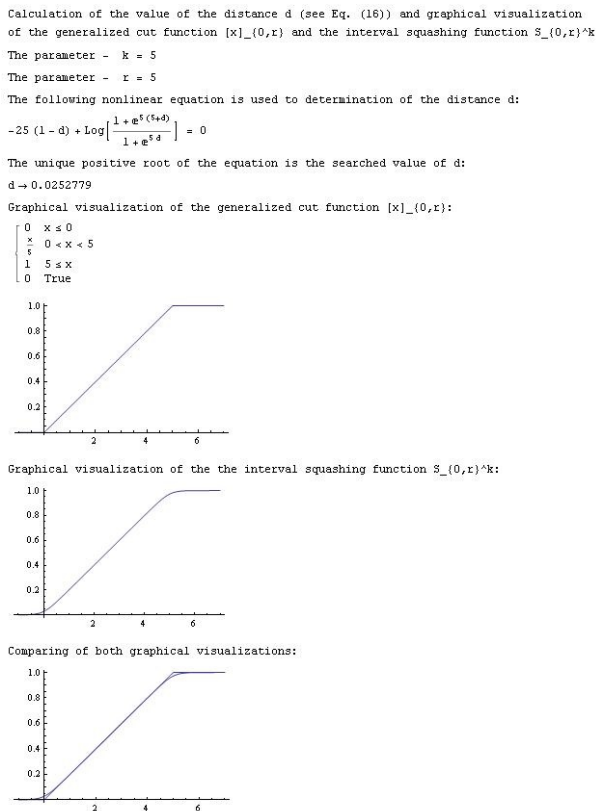


Fig. 8. The test provided on our control example.