QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXVIII, NUMBER 3
SEPTEMBER 2010, PAGES 469–485
S 0033-569X(2010)01163-6
Article electronically published on May 21, 2010

ON THE ARTIFICIAL COMPRESSIBILITY METHOD FOR THE NAVIER-STOKES-FOURIER SYSTEM

By

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Abstract. This paper deals with the artificial compressibility approximation method adapted to the incompressible Navier Stokes Fourier system. Two different types of approximations will be analyzed: one for the full Navier Stokes Fourier system (or the so-called Rayleigh-Benard equations) where viscous heating effects are considered and the other for when the dissipative function $\mathbb{S} : \nabla u$ is omitted. The convergence of the approximating sequences is achieved by exploiting the dispersive properties of the wave equation structure of the pressure of the approximating system.

1. Introduction. This paper is concerned with the incompressible Navier-Stokes equations in $\mathbb{R}^3$. As is well known, an incompressible fluid is subject to the constraint

$$\operatorname{div} u = 0. \tag{1.1}$$

The motion of the fluid is described by the following equation representing the conservation of momentum:

$$\partial_t u + (u \cdot \nabla)u - \mu\Delta u = \nabla p, \tag{1.2}$$

where $(x,t) \in \mathbb{R}^3 \times [0,T]$, $u \in \mathbb{R}^3$ denotes the velocity vector field, $p \in \mathbb{R}$ is the pressure of the fluid, and μ is the kinematic viscosity. If we want to consider the fluctuation of the temperature, the equations (1.1) and (1.2) are supplemented by the following equation:

$$\partial_t \theta + u \cdot \nabla\theta = \kappa\Delta\theta + \mathbb{S} : \nabla u, \tag{1.3}$$

where

$$\mathbb{S} = \mu(\nabla u + \nabla u^t), \tag{1.4}$$

$\theta \in \mathbb{R}$ is the temperature and κ is the heat conductivity. In order to simplify the presentation and since it will not affect our analysis, from now on we will take the physical constants to be $\mu = 1$ and $\kappa = 1$. The equations (1.1)-(1.3) together with the initial conditions

$$u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x) \tag{1.5}$$

Received November 19, 2008.
2000 *Mathematics Subject Classification.* Primary 35Q30; Secondary 35B35, 35Q35, 76D03, 76D05.
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are called the Navier-Stokes-Fourier system or often go under the name of Rayleigh-Benard equations (see [20]). The existence of a global in time weak solution for the equations (1.1)-(1.4) is still an open problem even in the class of Leray weak solutions for the Navier-Stokes equation. One major difficulty is due to the dissipative term

$$\mathbb{S} : \nabla u = (\nabla u + \nabla u^t)^2. \tag{1.6}$$

The reason is that from the a priori estimate it is only available that $\mathbb{S} : \nabla u$ is bounded in $L^1((0,T)\times\mathbb{R}^3)$, and so this term is only weakly lower semicontinuous in ∇u. Moreover part of the "kinetic energy" disappears as a positive measure concentrated at a certain point of the domain. Consequently we expect (1.3) not to hold in the weak sense, but rather to be replaced by an inequality

$$\partial_t\theta + u\cdot\nabla\theta + \Delta\theta \geq \mathbb{S} : \nabla u, \tag{1.7}$$

related clearly to the local (kinetic) energy inequality

$$\partial_t\left(\frac{1}{2}|u|^2\right) + \operatorname{div} u\left(\left(\frac{1}{2}|u|^2 + p\right)u\right) - \operatorname{div}(\mathbb{S}u) + \mathbb{S} : \nabla u \leq 0. \tag{1.8}$$

The aim of this paper is to approximate the system given by the equations (1.1), (1.2), (1.7) and the initial conditions (1.5). As is well known, a stumbling block in approximating the incompressible Navier-Stokes equations is the incompressibility constraint, which has high computational costs. In order to overcome these difficulties, Chorin [2, 3], Temam [30, 31] and Oskolkov [22] introduced in the case of a bounded domain the so-called artificial compressibility approximation method. In these papers they overcome the aforementioned difficulty by replacing the constraint (1.1) by the linearized continuity equation around a constant state. So the equations (1.1) and (1.2) are approximated by the system

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu\Delta u^\varepsilon - (u^\varepsilon\cdot\nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon)u^\varepsilon \\ \varepsilon\partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0, \end{cases} \tag{1.9}$$

where the term

$$-\frac{1}{2}(\operatorname{div} u^\varepsilon)u^\varepsilon$$

is added as a correction in order to avoid an increase of energy along the motion. The paper of Temam [30, 31] and his book [32] discuss the convergence of these approximations on bounded domains by using the classical Sobolev compactness embedding theorems and recover the compactness in time via the classical Lions [19] method of fractional derivatives. In [7] the authors deal with the same approximation in the whole space $\mathbb{R}^3$. In this case they cannot make use of the classical compactness theorems, therefore they exploit the underlying wave equation structure and recover the necessary compactness by means of dispersive type estimates. Here we will use a similar approach adapted to the Navier-Stokes-Fourier equations (1.1)-(1.3). Our approximating system will be the following one:

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \\ \partial_t \theta^\varepsilon + \operatorname{div}(P u^\varepsilon \theta^\varepsilon) = \Delta \theta^\varepsilon + \mathbb{S}^\varepsilon : \nabla u^\varepsilon, \end{cases} \tag{1.10}$$

where $(x,t) \in \mathbb{R}^3 \times [0,T]$, $u^\varepsilon = u^\varepsilon(x,t) \in \mathbb{R}^3$, $\theta^\varepsilon = \theta^\varepsilon(x,t) \in \mathbb{R}$, $p^\varepsilon = p^\varepsilon(x,t) \in \mathbb{R}$, $\mathbb{S}^\varepsilon = \nabla u^\varepsilon + (\nabla u^\varepsilon)^t$ and P is the Leray projector on the space of divergence free vector fields (see Section A.1 in the Appendix). The system (1.10) is endowed with the following initial conditions:

$$u^\varepsilon(x,0) = u_0^\varepsilon(x), \qquad \theta^\varepsilon(x,0) = \theta_0^\varepsilon(x), \qquad p^\varepsilon(x,0) = p_0^\varepsilon(x). \tag{1.11}$$

We want to point out that the Navier-Stokes-Fourier system requires only two initial conditions, one for the velocity u and one for the temperature θ. Hence our approximation will be consistent if the initial datum on the pressure can be eliminated by an "initial layer" phenomenon. Taking into account the fact that in order to get a priori estimates we will need finite initial energy, we require the initial data to satisfy the following conditions:

$$u_0^\varepsilon \in L^2(\mathbb{R}^3), \quad \theta_0^\varepsilon \in L^1(\mathbb{R}^3), \quad \sqrt{\varepsilon} p_0^\varepsilon \in L^2(\mathbb{R}^3). \tag{1.12}$$

We will be able to prove the following theorem.

Theorem 1.1. Let $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ be a sequence of weak solutions in $\mathbb{R}^3$ of the system (1.10), and assume that the initial data satisfy (1.12). Then:

(i) There exists $u \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; \dot{H}^1(\mathbb{R}^3))$ such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2([0,T]; \dot{H}^1(\mathbb{R}^3)).$$

(ii) The gradient component Qu^ε of the vector field u^ε satisfies

$$Qu^\varepsilon \longrightarrow 0 \quad \text{strongly in } L^2([0,T]; L^p(\mathbb{R}^3)), \text{ for any } p \in [4,6).$$

(iii) The divergence free component Pu^ε of the vector field u^ε satisfies

$$Pu^\varepsilon \longrightarrow Pu = u \quad \text{strongly in } L^2([0,T]; L^2_{loc}(\mathbb{R}^3)),$$

and $u = Pu$ will satisfy the following equation in $\mathcal{D}'([0,T] \times \mathbb{R}^3)$:

$$P(\partial_t u - \nabla u + (u \cdot \nabla) u) = 0.$$

(iv) The sequence $\{p^\varepsilon\}$ will converge in the sense of distribution to

$$p = \Delta^{-1} \operatorname{div}((u \cdot \nabla) u) = \Delta^{-1} tr((Du)^2).$$

(v) There exists $\theta \in C([0,T]; L^1(\mathbb{R}^3)) \cap L^1([0,T]; L^q(\mathbb{R}^3))$, $q \in [1,3)$, such that

$$\int_0^T \int_{\mathbb{R}^3} \theta \partial_t \varphi + \theta u \cdot \nabla \varphi + \theta \Delta \varphi dx dt \leq \int_0^T \int_{\mathbb{R}^3} (\mathbb{S} : \nabla u) \varphi dx dt,$$

for any $\varphi \in \mathcal{D}([0,T] \times \mathbb{R}^3)$.

Let us point out that from (iii) we get that in the limit the equation (1.2) is satisfied in the sense of distribution. Moreover the inequality (1.7) is satisfied in its weak formulation represented by the integral inequality in (v). We want to remark that a similar approach is also followed by Feireisl in [10] in the context of variational solutions for the compressible Navier-Stokes equations.

Sometimes in mathematical models the dissipative function $\mathbb{S} : \nabla u$ representing the irreversible transfer of mechanical energy into heat is neglected, so that the equation (1.3) takes the simpler form

$$\partial_t \theta + u \cdot \nabla \theta = \Delta \theta. \tag{1.13}$$

For the system (1.1), (1.2) and (1.13) it is possible to extend the fundamental theorem of Leray [18]; the following result holds.

THEOREM 1.2. There exists at least one weak solution (u, θ) of the Navier-Stokes-Fourier system (1.1), (1.2) and (1.13), namely the pair (u, θ) which satisfies

$$\int_0^T \int_{\mathbb{R}^d} \left(\mu \nabla u \cdot \nabla \varphi - u_i u_j \partial_i \varphi_j - u \cdot \frac{\partial \varphi}{\partial t} \right) dxdt = \int_{\mathbb{R}^d} u_0 \cdot \varphi dx$$

and

$$\int_0^T \int_{\mathbb{R}^d} \left(\kappa \nabla \theta \cdot \nabla \varphi - \theta u \cdot \nabla \varphi - \theta \cdot \frac{\partial \varphi}{\partial t} \right) dxdt = \int_{\mathbb{R}^d} \theta_0 \cdot \varphi dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d \times [0, T])$ such that $\operatorname{div} \varphi = 0$ and

$$\operatorname{div} u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times [0, T]).$$

Moreover the following energy inequality holds:

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^d} (|u(x,t)|^2 + |\theta(x,t)|^2) dx + \int_0^t \int_{\mathbb{R}^d} (\mu |\nabla u(x,s)|^2 + \kappa |\nabla \theta(x,s)|^2) dxds \\ &\quad \leq \frac{1}{2} \int_{\mathbb{R}^d} (|u_0|^2 + |\theta_0|^2) dx, \qquad \text{for all } t \geq 0. \end{aligned} \tag{1.14}$$

Sometimes one also refers to such solutions as Leray weak solutions, although Leray himself didn't study thermal effects. In this case we can approximate the equations (1.1), (1.2) and (1.13) by the system

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2} (\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon = \kappa \Delta \theta^\varepsilon - \frac{1}{2} (\operatorname{div} u^\varepsilon) \theta^\varepsilon, \end{cases} \tag{1.15}$$

endowed with initial data (1.11). Since in the limit we have to deal with weak solutions that satisfy (1.14) it is reasonable to require the finite energy constraint to be satisfied by the approximating sequences $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$. So we can deduce a natural behaviour to be imposed on the initial data $(u_0^\varepsilon, \theta_0^\varepsilon, p_0^\varepsilon)$, namely

$$\left. \begin{array}{l} u_0^\varepsilon = u^\varepsilon(\cdot, 0) \longrightarrow u_0 = u(\cdot, 0) \text{ strongly in } L^2(\mathbb{R}^3) \\ \theta_0^\varepsilon = u^\varepsilon(\cdot, 0) \longrightarrow \theta_0 = \theta(\cdot, 0) \text{ strongly in } L^2(\mathbb{R}^3) \\ \sqrt{\varepsilon} p_0^\varepsilon = \sqrt{\varepsilon} p^\varepsilon(\cdot, 0) \longrightarrow 0 \text{ strongly in } L^2(\mathbb{R}^3) \end{array} \right\} \tag{1.16}$$

Let us remark that the convergence of $\sqrt{\varepsilon}p_0^\varepsilon$ to 0 is necessary to avoid the presence of concentrations of energy in the limit. Notice, as before, that the term $\frac{1}{2}(\operatorname{div} u^\varepsilon)\theta^\varepsilon$ in the third equation of the system (1.15) is added as a correction term. In this case, since we don't have to deal with the lack of estimates for the term $\mathbb{S}^\varepsilon : \nabla u^\varepsilon$, we are able to recover strong convergence for the sequence θ^ε. In particular, we can prove the following theorem.

THEOREM 1.3. Let $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ be a sequence of weak solutions in $\mathbb{R}^3$ of the system (1.15), and assume that the initial data satisfy (1.16). Then:

(i) There exists $u \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; \dot{H}^1(\mathbb{R}^3))$ such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2([0,T]; \dot{H}^1(\mathbb{R}^3)).$$

(ii) The gradient component Qu^ε of the vector field u^ε satisfies

$$Qu^\varepsilon \longrightarrow 0 \quad \text{strongly in } L^2([0,T]; L^p(\mathbb{R}^3)), \text{ for any } p \in [4,6).$$

(iii) The divergence free component Pu^ε of the vector field u^ε satisfies

$$Pu^\varepsilon \longrightarrow Pu = u \quad \text{strongly in } L^2([0,T]; L^2_{loc}(\mathbb{R}^3)).$$

(iv) There exists $\theta \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; \dot{H}^1(\mathbb{R}^3))$ such that

$$\theta^\varepsilon \longrightarrow \theta \quad \text{strongly in } L^2([0,T]; L^2_{loc}(\mathbb{R}^3)),$$
$$\nabla\theta^\varepsilon \rightharpoonup \nabla\theta \quad \text{weakly in } L^2([0,T] \times \mathbb{R}^3).$$

(v) The sequence $\{p^\varepsilon\}$ will converge in the sense of distribution to

$$p = \Delta^{-1} \operatorname{div}((u \cdot \nabla)u) = \Delta^{-1} tr((Du)^2).$$

(vi) $u = Pu$ and θ are weak solutions to the incompressible Navier-Stokes-Fourier system, and the following energy inequality holds for all $t \in [0,T]$:

$$\frac{1}{2}\int_{\mathbb{R}^3} (|u(x,t)|^2 + |\theta(x,t)|^2)dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla u(x,s)|^2 + |\nabla\theta(x,s)|^2)dxds \leq \frac{1}{2}\int_{\mathbb{R}^3} (|u_0(x)|^2 + |\theta_0(x)|^2)dx.$$

This paper is organized as follows. In Section 2 we recover all the a priori estimates for u^ε and p^ε and get the strong convergence for u^ε by analyzing the associated Hodge decomposition. The important idea here is to regard each of the systems (1.10) and (1.15) as a semilinear wave type equation for the pressure function, and the dispersive estimates will then be carried out by using certain classical L^p-type estimates due to Strichartz [13, 15, 29]. In particular our wave equation structure has some similarities with that exploited in various ways in the papers of Desjardin, Grenier, Lions and Masmoudi [5] and Desjardin and Grenier [4]. In Sections 3 and 4 we prove Theorems 1.1 and 1.3 respectively. Finally in the Appendix we recall the mathematical tools and notation that we use throughout the paper.

2. Estimates and convergence for the velocity vector field and the pressure. In this section we wish to recover all the a priori estimates, independent of ε, that are necessary to obtain convergence of the velocity u^ε and of the pressure p^ε. As we will see, we will achieve this goal in two steps. First of all, we derive estimates that are naturally related to the first two equations of the systems (1.10) and (1.15). Then, in order to obtain compactness of the approximating sequences, we will go deeper by exploiting the hyperbolic nature of the system, namely the underlying wave equation structure for the pressure.

2.1. *A priori estimates.*

2.1.1. *Energy estimates.* The next result concerns an energy type estimate for the equations $(1.10)_{1,2}$ and $(1.15)_{1,2}$.

Theorem 2.1. Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.10) (respectively, (1.15)). Assume that the hypotheses (1.12) (respectively, (1.16)) hold; then one has

$$E(t) + \int_0^t \int_{\mathbb{R}^3} |\nabla u^\varepsilon(x,s)|^2 dx ds = E(0), \tag{2.1}$$

where we set

$$E(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|u^\varepsilon(x,t)|^2 + \frac{\varepsilon}{2}|p^\varepsilon(x,t)|^2 \right) dx. \tag{2.2}$$

Proof. We multiply, as usual, the first equation of the system (1.10) (resp. (1.15)) by u^ε and the second by p^ε; then we sum up and integrate by parts in space and time, and hence get (2.1). □

Corollary 2.2. Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.10) (resp. (1.15)). Let us assume that the hypotheses (1.12) (resp. (1.16)) hold; then it follows that

$$\sqrt{\varepsilon} p^\varepsilon \quad \text{is bounded in } L^\infty([0,T]; L^2(\mathbb{R}^3)), \tag{2.3}$$

$$\varepsilon p_t^\varepsilon \quad \text{is relatively compact in } H^{-1}([0,T] \times \mathbb{R}^3), \tag{2.4}$$

$$\nabla u^\varepsilon \quad \text{is bounded in } L^2([0,T] \times \mathbb{R}^3), \tag{2.5}$$

$$u^\varepsilon \quad \text{is bounded in } L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; L^6(\mathbb{R}^3)), \tag{2.6}$$

$$(u^\varepsilon \cdot \nabla) u^\varepsilon \quad \text{is bounded in } L^2([0,T]; L^1(\mathbb{R}^3)) \cap L^1([0,T]; L^{3/2}(\mathbb{R}^3)), \tag{2.7}$$

$$(\operatorname{div} u^\varepsilon) u^\varepsilon \quad \text{is bounded in } L^2([0,T]; L^1(\mathbb{R}^3)) \cap L^1([0,T]; L^{3/2}(\mathbb{R}^3)). \tag{2.8}$$

Proof. Assertions (2.3), (2.4) and (2.5) follow from (2.1), while (2.6) follows from (2.1) and Sobolev's embedding theorems. Finally (2.7) and (2.8) come from (2.5) and (2.6). □

2.1.2. *Pressure wave equation.* Since from the previous estimates we get only weak convergence of the approximating sequences, here we will recover more delicate estimates for the pressure sequence, which, as we will see later, combined with the estimates of the previous section will ensure strong convergence for the velocity. As in the paper [7] we observe that p^ε satisfies a wave equation. In fact, upon differentiating with respect to time the equation $(1.10)_2$ (resp. $(1.15)_2$) and using $(1.10)_1$ (resp. $(1.15)_1$) we have

$$\varepsilon\partial_{tt}p^\varepsilon - \Delta p^\varepsilon + \Delta \operatorname{div} u^\varepsilon - \operatorname{div}\left((u^\varepsilon\cdot\nabla)u^\varepsilon + \frac{1}{2}(\operatorname{div} u^\varepsilon)u^\varepsilon\right) = 0,$$

which is the so-called acoustic pressure wave. As we know, its fast oscillations may cause a lack of strong convergence. Here we will overcome this difficulty by investigating the dispersive nature of the pressure wave equation. Now, by rescaling the time variable to

$$\tau = \frac{t}{\sqrt{\varepsilon}}$$

and consequently the velocity, temperature and pressure as

$$\tilde{u}(x,\tau) = u^\varepsilon(x,\sqrt{\varepsilon}\tau), \quad \tilde{\theta}(x,\tau) = \theta^\varepsilon(x,\sqrt{\varepsilon}\tau), \quad \tilde{p}(x,\tau) = p^\varepsilon(x,\sqrt{\varepsilon}\tau), \tag{2.9}$$

we get that $\tilde{p}$ satisfies the wave equation

$$\partial_{\tau\tau}\tilde{p} - \Delta\tilde{p} + \Delta \operatorname{div}\tilde{u} - \operatorname{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\right) = 0. \tag{2.10}$$

This structure allows us to use on $\tilde{p}$ the Strichartz estimates (A.4) and (A.6). Now the analysis for $\tilde{p}$ will follow the same line of argument as in [7]. So we split $\tilde{p}$ as $\tilde{p} = \tilde{p}_1 + \tilde{p}_2$ where $\tilde{p}_1$ and $\tilde{p}_2$ solve the following wave equations:

$$\begin{cases} \partial_{\tau\tau}\tilde{p}_1 - \Delta\tilde{p}_1 = -\Delta \operatorname{div}\tilde{u} = F_1 \\ \tilde{p}_1(x,0) = \partial_\tau\tilde{p}_1(x,0) = 0, \end{cases} \tag{2.11}$$

$$\begin{cases} \partial_{\tau\tau}\tilde{p}_2 - \Delta\tilde{p}_2 = \operatorname{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\right) = F_2 \\ \tilde{p}_2(x,0) = \tilde{p}(x,0), \quad \partial_\tau\tilde{p}_2(x,0) = \partial_\tau\tilde{p}(x,0). \end{cases} \tag{2.12}$$

Therefore we are able to prove the following theorem.

Theorem 2.3. Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.10) (resp. (1.15)). Assume that the hypotheses (1.12) (resp. (1.16)) hold. Then we have the following estimate:

$$\begin{aligned} \varepsilon^{3/8}\|p^\varepsilon\|_{L^4_t W^{-2,4}_x} + \varepsilon^{7/8}\|\partial_t p^\varepsilon\|_{L^4_t W^{-3,4}_x} &\lesssim \sqrt{\varepsilon}\|p^\varepsilon_0\|_{L^2_x} + \|\operatorname{div} u^\varepsilon_0\|_{H^{-1}_x} \\ &+ \sqrt{T}\|\operatorname{div} u^\varepsilon\|_{L^2_t L^2_x} + \|\,(u^\varepsilon\cdot\nabla)\,u^\varepsilon + \frac{1}{2}(\operatorname{div} u^\varepsilon)u^\varepsilon\|_{L^1_t L^{3/2}_x}. \end{aligned} \tag{2.13}$$

Proof. Since $\tilde{p}_1$ and $\tilde{p}_2$ are solutions of the wave equations (2.11) and (2.12), we can apply the Strichartz estimates (A.5) and (A.6), with $(x,\tau) \in \mathbb{R}^3 \times (0, T/\sqrt{\varepsilon})$. By applying the Strichartz estimate (A.6) to $w = \Delta^{-1}\tilde{p}_1$ we get

$$\|\tilde{p}_1\|_{L^4_\tau W^{-2,4}_x} + \|\partial_\tau\tilde{p}_1\|_{L^4_\tau W^{-3,4}_x} \lesssim \frac{\sqrt{T}}{\varepsilon^{1/4}}\|\operatorname{div}\tilde{u}\|_{L^2_\tau L^2_x}. \tag{2.14}$$

In the same way, by using the estimate (A.5) on $w = \Delta^{-1/2}\tilde{p}_2$ we obtain

$$\|\tilde{p}_2\|_{L^4_\tau W^{-1,4}_x} + \|\partial_\tau \tilde{p}_2\|_{L^4_\tau W^{-2,4}_x} \lesssim \|\tilde{p}(x,0)\|_{H^{-1/2}_x} + \|\partial_\tau \tilde{p}(x,0)\|_{H^{-3/2}_x} + \|\,(\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\|_{L^1_\tau L^{3/2}_x}. \tag{2.15}$$

Now by using (2.14) and (2.15) it follows that $\tilde{p}$ satisfies

$$\begin{aligned}\|\tilde{p}\|_{L^4_\tau W^{-2,4}_x} + \|\partial_\tau \tilde{p}\|_{L^4_\tau W^{-3,4}_x} &\le \|\tilde{p}_1\|_{L^4_\tau W^{-2,4}_x} + \|\tilde{p}_2\|_{L^4_\tau W^{-1,4}_x} \qquad (2.16)\\ &\quad + \|\partial_\tau \tilde{p}_1\|_{L^4_\tau W^{-3,4}_x} + \|\partial_\tau \tilde{p}_2\|_{L^4_\tau W^{-2,4}_x}\\ &\lesssim \|\tilde{p}(x,0)\|_{H^{-1/2}_x} + \|\partial_\tau \tilde{p}(x,0)\|_{H^{-3/2}_x}\\ &\quad + \|(\tilde{u}\cdot\nabla)\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\|_{L^1_\tau L^{3/2}_x}\\ &\quad + \frac{\sqrt{T}}{\varepsilon^{1/4}}\|\operatorname{div}\tilde{u}\|_{L^2_\tau L^2_x}. \qquad (2.17)\end{aligned}$$

Finally, since

$$\|\tilde{p}\|_{L^r((0,T/\sqrt{\varepsilon});L^q(\mathbb{R}^3))} = \varepsilon^{-1/2r}\|p^\varepsilon\|_{L^r([0,T];L^q(\mathbb{R}^3))}$$

we end up with (2.13). □

2.2. *Strong convergence.* In this section we will prove the strong convergence of the velocity vector field. This convergence will be obtained by studying separately the behaviour of the gradient part Qu^ε and of the incompressible component Pu^ε of the velocity vector field. But, before going on, we will just state some easy consequences of the estimates of the previous section.

Proposition 2.4. *Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.10) (resp. (1.15)). Assume that the hypotheses (1.12) (resp. (1.16)) hold. Then, as $\varepsilon \downarrow 0$, one has*

$$\varepsilon p^\varepsilon \longrightarrow 0 \qquad \text{strongly in } L^\infty([0,T];L^2(\mathbb{R}^3)) \cap L^4([0,T];W^{-2,4}(\mathbb{R}^3)), \tag{2.18}$$

$$\operatorname{div} u^\varepsilon \longrightarrow 0 \qquad \text{strongly in } W^{-1,\infty}([0,T];L^2(\mathbb{R}^3)) \cap L^4([0,T];W^{-3,4}(\mathbb{R}^3)). \tag{2.19}$$

Proof. Statements (2.18) and (2.19) follow from the estimates (2.3) and (2.13) and the second equation of the system (1.10) (resp. (1.15)). □

2.2.1. *Strong convergence of Qu^ε.* Now, we wish to show that the gradient part of the velocity field Qu^ε goes strongly to 0 as $\varepsilon \downarrow 0$. As we will see in the next proposition, this is a consequence of the estimate (2.13) and Lemma A.1.

Proposition 2.5. *Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.2). Assume that the hypotheses (1.16) hold. Then as $\varepsilon \downarrow 0$,*

$$Qu^\varepsilon \longrightarrow 0 \quad \text{strongly in } L^2([0,T];L^p(\mathbb{R}^3)) \text{ for any } p \in [4,6). \tag{2.20}$$

Proof. In order to prove Proposition 2.5 we split Qu^ε as follows:

$$\|Qu^\varepsilon\|_{L^2_t L^p_x} \le \|Qu^\varepsilon - Qu^\varepsilon * j_\alpha\|_{L^2_t L^p_x} + \|Qu^\varepsilon * j_\alpha\|_{L^2_t L^p_x} = J_1 + J_2,$$

where j_α is the smoothing kernel defined in Lemma A.1. Now we estimate separately J_1 and J_2. For J_1, by using (A.2) we get

$$J_1 \leq \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} \left(\int_0^T \|\nabla Q u^\varepsilon(t)\|_{L_x^2}^2 dt \right)^{1/2} \leq \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|\nabla u^\varepsilon\|_{L_t^2 L_x^2}. \tag{2.21}$$

Hence from the identity $Qu^\varepsilon = -\varepsilon^{1/8} \nabla \Delta^{-1} \varepsilon^{7/8} \partial_t p$ and the inequality (A.3) we get that J_2 satisfies the estimate

$$J_2 \leq \varepsilon^{1/8} \alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} T^{1/4} \|\varepsilon^{7/8} \partial_t p\|_{L_t^4 W_x^{-3,4}}. \tag{2.22}$$

Therefore, adding (2.21) and (2.22) and using (2.5) and (2.13), we conclude that for any $p \in [4, 6)$,

$$\|Qu^\varepsilon\|_{L_t^2 L_x^p} \leq C \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} + C_T \varepsilon^{1/8} \alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)}. \tag{2.23}$$

Finally we choose α in terms of ε so that the two terms on the right hand side of the previous inequality have the same order, namely

$$\alpha = \varepsilon^{1/18}. \tag{2.24}$$

Therefore we obtain

$$\|Qu^\varepsilon\|_{L_t^2 L_x^p} \leq C_T \varepsilon^{\frac{6-p}{36p}} \quad \text{for any } p \in [4, 6).$$

□

2.2.2. *Strong convergence of Pu^ε.* To prove the strong convergence of the sequence Pu^ε we will use Theorem A.2. So we will need to prove the equicontinuity in time of Pu^ε.

Lemma 2.6. Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.10) (resp. (1.15)). Assume that the hypotheses (1.12) (resp. (1.16)) hold. Then for all $h \in (0, 1)$, we have

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|_{L^2([0,T]\times\mathbb{R}^3)} \leq C_T h^{1/5}. \tag{2.25}$$

Proof. Let us set $z^\varepsilon = u^\varepsilon(t+h) - u^\varepsilon(t)$; then we have

$$\begin{aligned} \|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|_{L^2([0,T]\times\mathbb{R}^3)}^2 &= \int_0^T \int_{\mathbb{R}^3} dt dx (Pz^\varepsilon) \cdot (Pz^\varepsilon - Pz^\varepsilon * j_\alpha) \\ &\quad + \int_0^T \int_{\mathbb{R}^3} dt dx (Pz^\varepsilon) \cdot (Pz^\varepsilon * j_\alpha) = I_1 + I_2. \end{aligned} \tag{2.26}$$

By using (A.2) we can estimate I_1 in the following way:

$$I_1 \lesssim \alpha T^{1/2} \|u^\varepsilon\|_{L_t^\infty L_x^2} \|\nabla u^\varepsilon\|_{L_{t,x}^2}. \tag{2.27}$$

Let us reformulate Pz^ε in integral form by using the equation $(1.10)_1$ (resp. $(1.15)_1$); hence

$$I_2 \leq \left| \int_0^T dt \int_{\mathbb{R}^3} dx \int_t^{t+h} ds (\Delta u^\varepsilon - (u^\varepsilon \cdot \nabla) u^\varepsilon - \frac{1}{2} u^\varepsilon (\operatorname{div} u^\varepsilon)(s,x)) \cdot (Pz^\varepsilon * j_\alpha)(t,x) \right|. \tag{2.28}$$

Then, upon integrating by parts and using (A.3) with $s = 0$, $p = \infty$ and $q = 2$, we deduce that

$$I_2 \leq h\|\nabla u^\varepsilon\|^2_{L^2_{t,x}} + C\alpha^{-3/2}T^{1/2}h\|u^\varepsilon\|_{L^\infty_t L^2_x}\| (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon)u^\varepsilon\|_{L^2_t L^1_x}. \tag{2.29}$$

Adding I_1 and I_2 and taking into account (2.5), (2.6), (2.7) and (2.8), we have

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|^2_{L^2([0,T]\times\mathbb{R}^3)} \leq C(\alpha T^{1/2} + h\alpha^{-3/2}T^{1/2} + h), \tag{2.30}$$

and by choosing $\alpha = h^{2/5}$, we end up with (2.25). □

THEOREM 2.7. Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.10) (resp. (1.15)). Assume that the hypotheses (1.12) (resp. (1.16)) hold. Then as $\varepsilon \downarrow 0$,

$$Pu^\varepsilon \longrightarrow Pu \qquad \text{strongly in } L^2(0,T; L^2_{loc}(\mathbb{R}^3)). \tag{2.31}$$

Proof. By using Lemma 2.6, Theorem A.2 and Proposition 2.5 we get (2.31). □

3. The full Navier-Stokes-Fourier system. In this section we are going to prove Theorem 1.1. In the previous section we have collected a priori estimates on the velocity field u^ε and on the pressure p^ε. Here we will start by collecting some estimates on θ^ε.

3.1. *Estimates on the temperature* θ^ε. From the a priori estimates we know that $\mathbb{S}^\varepsilon : \nabla u^\varepsilon \in L^1_{t,x}$, so obviously, from heat equation considerations we cannot expect better integrability for θ^ε than $\theta^\varepsilon \in L^\infty_t L^1_x$. If we look at the equation $(1.10)_3$ we see that it fits in with the theory of renormalized solutions of parabolic equations (see Section A.3 in the Appendix). By using Theorem A.3 we are able to prove the following result.

THEOREM 3.1. Let $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ be a weak solution of the system (1.10), and assume that the hypotheses (1.12) hold; then

$$\theta^\varepsilon \in C([0,T] : L^1(\mathbb{R}^3)) \cap L^1([0,T]; L^q(\mathbb{R}^3)), \quad q \in [1,3], \tag{3.1}$$

$$\nabla\theta^\varepsilon \in L^r(\mathbb{R}^3 \times (0,T)), \quad r \in \left[1, \frac{4}{3}\right]. \tag{3.2}$$

Proof. The proof follows from applying Theorem A.3 to the equation $(1.10)_3$ and the variable $f = \theta^\varepsilon$. By the estimates (2.6) we get that θ^ε fulfills, uniformly in ε, the requirements of Theorem A.3, and so we end up with (3.1) and (3.2). □

3.2. *Proof of Theorem* 1.1.

(i) This follows from the estimate (2.6).

(ii) This is a consequence of Proposition 2.5.

(iii) By taking into account the decomposition $u^\varepsilon = Pu^\varepsilon + Qu^\varepsilon$, from Theorem 2.7 and Proposition 2.5 we have that

$$Pu^\varepsilon \longrightarrow u \qquad \text{strongly in } L^2([0,T]; L^2_{loc}(\mathbb{R}^3)),$$

and so we can pass to the limit in the equation $(1.10)_1$.

(iv) Let us apply the Leray projector Q to the equation $(1.10)_1$; then it follows that

$$\nabla p^\varepsilon = \Delta Q u^\varepsilon - Q\left(\operatorname{div}(u^\varepsilon \otimes u^\varepsilon) + \frac{3}{2}u^\varepsilon \operatorname{div} Q u^\varepsilon\right). \tag{3.3}$$

Now by choosing a test function $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ and taking into account (2.5), (2.20), and (2.31), we get, as $\varepsilon \downarrow 0$,

$$\begin{aligned}\langle u^\varepsilon \operatorname{div} Qu^\varepsilon, Q\varphi\rangle &\le \|Qu^\varepsilon\|_{L^2_tL^4_x}\|\nabla u^\varepsilon\|_{L^2_tL^2_x}\|Q\varphi\|_{L^\infty_tL^4_x}\\ &\quad+\|Qu^\varepsilon\|_{L^2_tL^4_x}\|u^\varepsilon\|_{L^\infty_tL^2_x}\|\nabla Q\varphi\|_{L^2_tL^4_x}\to 0, \qquad (3.4)\\ \langle \operatorname{div}(u^\varepsilon\otimes u^\varepsilon), Q\varphi\rangle &= \langle \operatorname{div}(Pu^\varepsilon\otimes Pu^\varepsilon), Q\varphi\rangle + \langle \operatorname{div}(Qu^\varepsilon\otimes Qu^\varepsilon), Q\varphi\rangle\\ &\quad+\langle \operatorname{div}(Pu^\varepsilon\otimes Qu^\varepsilon), Q\varphi\rangle + \langle \operatorname{div}(Qu^\varepsilon\otimes Qu^\varepsilon), Q\varphi\rangle\\ &\quad\to \langle \operatorname{div}(Pu\otimes Pu), Q\varphi\rangle = \langle Q\operatorname{div}((Pu\cdot\nabla)Pu),\varphi\rangle. \qquad (3.5)\end{aligned}$$

So as $\varepsilon \downarrow 0$ one has

$$\langle \nabla p^\varepsilon, \varphi\rangle \longrightarrow \langle \nabla\Delta^{-1}\operatorname{div}((u\cdot\nabla)u),\varphi\rangle. \qquad (3.6)$$

(v) The last thing to do now is to pass to the limit in the third equation of the system (1.10). By using the weak formulation and the weak lower semicontinuity of the weak limit we end up with the following integral inequality:

$$\int_0^T\int_{\mathbb{R}^3} \theta\partial_t\varphi + \theta u\cdot\nabla\varphi + \theta\Delta\varphi dxdt \le \int_0^T\int_{\mathbb{R}^3} \mathbb{S} : \nabla u\varphi dxdt,$$

for any $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^3)$, where $\mathbb{S} = \nabla u + \nabla u^t$.

4. Simplified Navier-Stokes-Fourier system. In this section we are going to consider the approximating system (1.15). In this model the temperature field is just advected by the velocity field u^ε and diffuses according to Fourier's law. This entails stronger bounds on θ^ε; in particular we will get the strong convergence of the sequence θ^ε.

4.1. *A priori bounds and strong convergence for θ^ε.*

Proposition 4.1. Let us consider the solution $(u^\varepsilon,\theta^\varepsilon,p^\varepsilon)$ of the Cauchy problem for the system (1.15). Let us assume that the hypotheses (1.16) hold; then it follows that

$$\begin{aligned}&\nabla\theta^\varepsilon && \text{is bounded in } L^2([0,T]\times\mathbb{R}^3), && (4.1)\\ &\theta^\varepsilon && \text{is bounded in } L^\infty([0,T];L^2(\mathbb{R}^3))\cap L^2([0,T];L^6(\mathbb{R}^3)), && (4.2)\\ &u^\varepsilon\nabla\theta^\varepsilon && \text{is bounded in } L^2([0,T];L^1(\mathbb{R}^3))\cap L^1([0,T];L^{3/2}(\mathbb{R}^3)), && (4.3)\\ &(\operatorname{div} u^\varepsilon)\theta^\varepsilon && \text{is bounded in } L^2([0,T];L^1(\mathbb{R}^3))\cap L^1([0,T];L^{3/2}(\mathbb{R}^3)). && (4.4)\end{aligned}$$

Proof. By multiplying the third equation of the system (1.15) we get the equality

$$\frac{1}{2}\int_{\mathbb{R}^3}|\theta^\varepsilon(x,t)|^2dx + \int_0^t\int_{\mathbb{R}^3}(|\nabla\theta^\varepsilon(x,s)|^2)dxds = \frac{1}{2}\int_{\mathbb{R}^3}|\theta_0^\varepsilon|^2dx. \qquad (4.5)$$

Assertion (4.1) follows from (4.5), while (4.2) follows from (4.5) and Sobolev's embedding theorems. Finally (4.3) and (4.4) come from (2.5) and (4.1). □

With the previous a priori estimates we are able to prove the following theorem.

THEOREM 4.2. Let us consider the solution $(u^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ of the Cauchy problem for the system (1.2). Assume that the hypotheses (1.16) hold. Then as $\varepsilon \downarrow 0$,

$$\nabla\theta^\varepsilon \rightharpoonup \nabla\theta \qquad \text{weakly in } L^2((0,T)\times\mathbb{R}^3), \tag{4.6}$$

$$\theta^\varepsilon \longrightarrow \theta \qquad \text{strongly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3)). \tag{4.7}$$

Proof. Statement (4.6) is an immediate consequence of (4.1). In order to prove the strong convergence (4.7) we need to use Theorem A.2 provided we are able to show that θ^ε is equicontinuous in time, namely that the following inequality holds:

$$\|\theta^\varepsilon(t+h) - \theta^\varepsilon(t)\|_{L^2([0,T]\times\mathbb{R}^3)} \le C_T h^{1/5}. \tag{4.8}$$

First of all, to simplify the notation, let us set $w^\varepsilon = \theta^\varepsilon(t+h) - \theta^\varepsilon(t)$; then we have

$$\begin{aligned}\|\theta^\varepsilon(t+h) - \theta^\varepsilon(t)\|^2_{L^2([0,T]\times\mathbb{R}^3)} &= \int_0^T\int_{\mathbb{R}^3} dtdx w^\varepsilon\cdot(w^\varepsilon - w^\varepsilon * j_\alpha)\\ &\quad + \int_0^T\int_{\mathbb{R}^3} dtdx w^\varepsilon\cdot(w^\varepsilon * j_\alpha) = I_1 + I_2.\end{aligned} \tag{4.9}$$

By using (A.2) we can estimate I_1 in the following way:

$$\begin{aligned} I_1 &\le \|w^\varepsilon\|_{L^\infty_t L^2_x}\int_0^T \|w^\varepsilon(t) - (w^\varepsilon * j_\alpha)(t)\|_{L^2_x} dt\\ &\lesssim \alpha T^{1/2}\|\theta^\varepsilon\|_{L^\infty_t L^2_x}\|\nabla\theta^\varepsilon\|_{L^2_{t,x}}. \end{aligned} \tag{4.10}$$

Let us reformulate w^ε in integral form by using the equation $(1.15)_2$; hence

$$I_2 \le \left|\int_0^T dt\int_{\mathbb{R}^3} dx\int_t^{t+h} ds(\Delta\theta^\varepsilon - u^\varepsilon\cdot\nabla\theta^\varepsilon - \frac{1}{2}\theta^\varepsilon(\operatorname{div} u^\varepsilon)(s,x)\cdot(w^\varepsilon * j_\alpha)(t,x)\right|. \tag{4.11}$$

Then, integrating by parts and using (A.3) with $s=0$, $p=\infty$ and $q=2$, we deduce that

$$\begin{aligned} I_2 &\le h\|\nabla\theta^\varepsilon\|^2_{L^2_{t,x}} + C\alpha^{-3/2}T^{1/2}\|\theta^\varepsilon\|_{L^\infty_t L^2_x}\left(h\int_t^{t+h}\|u^\varepsilon\cdot\nabla\theta^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon)\theta^\varepsilon\|^2_{L^1_x} ds\right)^{1/2}\\ &\le h\|\nabla\theta^\varepsilon\|^2_{L^2_{t,x}} + C\alpha^{-3/2}T^{1/2}h\|u^\varepsilon\|_{L^\infty_t L^2_x}\|u^\varepsilon\cdot\nabla\theta^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon)\theta^\varepsilon\|_{L^2_t L^1_x}. \end{aligned} \tag{4.12}$$

Adding I_1 and I_2 and taking into account (2.5), (4.1), (4.3) and (4.4), we have

$$\|\theta^\varepsilon(t+h) - \theta^\varepsilon(t)\|^2_{L^2([0,T]\times\mathbb{R}^3)} \le C(\alpha T^{1/2} + h\alpha^{-3/2}T^{1/2} + h), \tag{4.13}$$

and by choosing $\alpha = h^{2/5}$, we end up with (4.8). □

4.2. *Proof of Theorem* 1.3.

Proof. The proofs of **(i)**, **(ii)**, **(iii)** and **(v)** of Theorem 1.3 are exactly as the proofs of **(i)**, **(ii)**, **(iii)** and **(iv)** of Theorem 1.1. The convergence statements in **(iv)** are established in Theorem 4.2. Finally, to pass to the limit in system (1.15) the convergence of the nonlinear term $u^\varepsilon\cdot\nabla\theta^\varepsilon$ deserves a little discussion. By using again for u^ε the associated Hodge decomposition, namely $u^\varepsilon = Pu^\varepsilon + Qu^\varepsilon$, and by taking into account

(2.20), (2.31), (4.6) and (4.7) one can prove that, for any $p \in [4, 6)$,

$$Qu^\varepsilon \cdot \nabla\theta^\varepsilon \longrightarrow 0 \qquad \text{strongly in } L^1([0,T]; L^{\frac{2p}{p+2}}(\mathbb{R}^3)), \tag{4.14}$$

$$Pu^\varepsilon \cdot \nabla\theta^\varepsilon \longrightarrow u \cdot \nabla\theta \qquad \text{strongly in } L^1([0,T] \times \mathbb{R}^3). \tag{4.15}$$

Now we can pass to the limit inside the system (1.15) and get that u and θ satisfy the following equations in $\mathcal{D}'([0,T] \times \mathbb{R}^3)$:

$$P(\partial_t u - \Delta u + (u \cdot \nabla)u) = 0, \tag{4.16}$$

$$\partial_t \theta - \Delta\theta + u \cdot \nabla\theta = 0. \tag{4.17}$$

Finally we prove the energy inequality. By using the weak lower semicontinuity of the weak limits, assuming the hypotheses (1.16) and denoting by χ the weak limit of $\sqrt{\varepsilon}p^\varepsilon$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{2}\left(|\chi(x,t)|^2 + |u(x,t)|^2 + |\theta(x,t)|^2\right) dx \\
&+ \int_0^t \int_{\mathbb{R}^3} \left(|\nabla u(x,s)|^2 + |\nabla\theta(x,s)|^2\right) dxds \\
&\le \liminf_{\varepsilon\to 0} \left(\frac{1}{2}\int_{\mathbb{R}^3} (\varepsilon|p^\varepsilon(x,t)|^2 + |u^\varepsilon(x,t)|^2 + |\theta^\varepsilon(x,t)|^2) dx\right) \\
&+ \liminf_{\varepsilon\to 0} \left(\int_0^t \int_{\mathbb{R}^3} (|\nabla u^\varepsilon(x,s)|^2 + |\nabla\theta^\varepsilon(x,s)|^2) dxds\right) \\
&= \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} \frac{1}{2}\left(\varepsilon|p_0^\varepsilon(x)|^2 + |u_0^\varepsilon(x)|^2 + |\theta_0^\varepsilon(x)|^2\right) dx \\
&= \int_{\mathbb{R}^3} \frac{1}{2}(|u_0(x)|^2 + |\theta_0(x)|^2) dx.
\end{aligned} \tag{4.18}$$

In this way we have proved **(vi)**. □

Appendix A. Preliminaries and notation. For the convenience of the reader we establish some notation and recall some basic facts that will be useful in the main part of the paper.

A.1. *Notation and technical tools.* We will denote by $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+)$ the space of test function $C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, by $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$ the space of Schwartz distributions and by $\langle \cdot, \cdot \rangle$ the duality bracket between $\mathcal{D}'$ and $\mathcal{D}$. Moreover $W^{k,p}(\mathbb{R}^d) = (I - \Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ denote the nonhomogeneous Sobolev spaces for any $1 \le p \le \infty$ and $k \in \mathbb{R}$, while $\dot{W}^{k,p}(\mathbb{R}^d) = (-\Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $\dot{H}^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ denote the homogeneous Sobolev spaces. The expressions $L_t^p L_x^q$ and $L_t^p W_x^{k,q}$ will stand for the spaces $L^p([0,T]; L^q(\mathbb{R}^d))$ and $L^p([0,T]; W^{k,q}(\mathbb{R}^d))$ respectively.
The operators Q and P denote the Leray projectors on the space of gradient vector fields and on the space of divergence-free vector fields, respectively. Hence, in the sense of distribution, one has

$$Q = \nabla\Delta^{-1}\,\mathrm{div}, \qquad P = I - Q. \tag{A.1}$$

Let us remark that Q and P can be expressed in terms of Riesz multipliers, therefore they are bounded linear operators on every $W^{k,p}$ $(1 < p < \infty)$ space (see [28]).

Next we recall here two technical tools. The first one is related to interpolation theory and Young type inequalities and is useful for getting L^p type estimates by means of estimates in $W^{-k,p}$ space. Specifically the following lemma holds.

LEMMA A.1. Let us consider a smoothing kernel $j \in C_0^\infty(\mathbb{R}^d)$ such that $j \geq 0$ and $\int_{\mathbb{R}^d} j dx = 1$, and define the Friedrichs mollifiers as

$$j_\alpha(x) = \alpha^{-d} j\left(\frac{x}{\alpha}\right).$$

Then for any $f \in \dot{H}^1(\mathbb{R}^d)$, one has

$$\|f - f * j_\alpha\|_{L^p(\mathbb{R}^d)} \leq C_p \alpha^{1-\sigma} \|\nabla f\|_{L^2(\mathbb{R}^d)}, \tag{A.2}$$

where

$$p \in [2, \infty) \quad \text{if } d = 2, \quad p \in [2, 6] \quad \text{if } d = 3 \ \text{ and } \quad \sigma = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

Moreover the following Young type inequality holds:

$$\|f * j_\alpha\|_{L^p(\mathbb{R}^d)} \leq C\alpha^{-s-d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{W^{-s,q}(\mathbb{R}^d)}, \tag{A.3}$$

for any $p, q \in [1, \infty]$ with $q \leq p$, $s \geq 0$, and $\alpha \in (0, 1)$.

The second result concerns how to get compactness in L^p spaces (see [25]).

THEOREM A.2. Let $\mathcal{F} \subset L^p([0, T]; B)$, with $1 \leq p < \infty$ and B a Banach space. Then $\mathcal{F}$ is relatively compact in $L^p([0, T]; B)$ for $1 \leq p < \infty$ or in $C([0, T]; B)$ for $p = \infty$ if and only if

(i) $\left\{\int_{t_1}^{t_2} f(t)dt, \ f \in B\right\}$ is relatively compact in B for $0 < t_1 < t_2 < T$,

(ii) $\lim_{h \to 0} \|f(t + h) - f(t)\|_{L^p([0,T-h];B)} = 0$ uniformly for any $f \in \mathcal{F}$.

A.2. *Strichartz type estimates for wave equations.* Let us recall that if w is a (weak) solution of the wave equation

$$\begin{cases} \left(-\frac{\partial^2}{\partial t} + \Delta\right) w(t, x) = F(t, x) \\ w(0, \cdot) = f, \quad \partial_t w(0, \cdot) = g, \end{cases}$$

in the space $[0, T] \times \mathbb{R}^d$ for some data f, g, F and time $0 < T < \infty$, then w satisfies the following Strichartz estimates (see [13], [15]):

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_x^\gamma} + \|g\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{A.4}$$

where (q, r) and $(\tilde{q}, \tilde{r})$ are *wave admissible* pairs, i.e. they satisfy

$$\frac{2}{q} \leq (d-1)\left(\frac{1}{2} - \frac{1}{r}\right), \qquad \frac{2}{\tilde{q}} \leq (d-1)\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right),$$

and moreover the following conditions hold:

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} - 2.$$

We shall use (A.4) in the case of $d = 3$ and $(\tilde{q}', \tilde{r}') = (1, 3/2)$, so that one has $\gamma = 1/2$ and $(q, r) = (4, 4)$; in particular, the following estimate holds:

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{-1/2}_x} + \|F\|_{L^1_t L^{3/2}_x}. \tag{A.5}$$

Besides the Strichartz estimate (A.4) or (A.5) in the case of $d = 3$ (see [27]), there exists a non-standard estimate which follows from an earlier linear Strichartz [29] estimate. This inequality can also be deduced from the bilinear estimates of Klainerman and Machedon [16] or Foschi and Klainerman [11], namely

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{1/2}_x} + \|F\|_{L^1_t L^2_x}. \tag{A.6}$$

A.3. *Renormalized solutions for parabolic equations.* Let us consider a parabolic equation of the type

$$\partial_t f + u \cdot \nabla f - \Delta f = F \qquad \text{in } \Omega \times (0, T), \tag{A.7}$$
$$f(x, 0) = f_0(x) \qquad \text{in } \Omega, \tag{A.8}$$
$$\frac{\partial f}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{A.9}$$

where $f_0 \in L^1(\Omega)$, $F \in L^1(\Omega \times (0, T))$ and Ω is a bounded smooth open domain in $\mathbb{R}^N$. Furthermore assume that

$$\left.\begin{array}{l} u \in L^2(\Omega \times (0, T)), \\ \operatorname{div} u = 0 \quad \text{in } \mathcal{D}'(\Omega \times (0, T)), \\ u \cdot n \quad \text{on } \partial\Omega \times (0, T). \end{array}\right\} \tag{A.10}$$

Without loss of generality we can also assume $F \geq 0$. If we want to solve the problem (A.7)-(A.9) simply by using distribution theory, we would need to define the product uf (taking into account that $u\nabla f = \operatorname{div}(uf)$, since $\operatorname{div} u = 0$). Since we only assume that $u \in L^2(\Omega \times (0, T))$ we would need to know that $f \in L^2(\Omega \times (0, T))$. Unfortunately since $F \in L^1$ we cannot expect to have $f \in L^2(\Omega \times (0, T))$. For this reason it is necessary to use the notion of renormalized solutions introduced by R. J. DiPerna and P.-L. Lions [6] in the context of Focker-Planck-Boltzmann equations and by D. Blanchard and F. Murat [1] for parabolic equations (see also P.-L. Lions [20]). So following P.-L. Lions [20], Appendix E, we recall the following theorem.

Theorem A.3. Assume that f is a solution of (A.7)-(A.9) and that (A.10) is satisfied; then:

- **(R1)** $f \in C([0, T]; L^1(\Omega)) \cap L^1([0, T]; L^q(\Omega))$ for all $q \in \left[1, \frac{N}{N-2}\right)$.
- **(R2)** $T_R(f) \in L^2([0, T]; H^1(\Omega))$ for all $r \in (0, \infty)$ and $\lim_{R \to +\infty} \frac{1}{R} \int_\Omega dx \int_0^T dt |\nabla T_R(f)|^2 = 0$, where $T_R(f)(t) = \max(\min(t, R), -R)$ for $t \in \mathbb{R}$, $R \in (0, +\infty)$.
- **(R3)** $\nabla F \in L^r(\Omega \times (0, T))$ for all $r \in \left[1, \frac{N+1}{N}\right)$.

(R4) For all $\beta \in C_0^\infty(\Omega)$ and for all $\varphi \in C^1(\overline{\Omega} \times [0,T])$,

$$\frac{d}{dt}\int_\Omega \beta(f(x,t))\varphi(x,t)dx + \int_\Omega \beta(f)\left(-\frac{\partial\varphi}{\partial t} - u\cdot\nabla\varphi\right)\nabla\beta(f)\cdot\nabla\varphi + \beta''(f)|\nabla f|^2\varphi - \beta'(f)F\varphi dx = 0 \quad \text{in } \mathcal{D}'(0,T),$$

$$\beta(f) \in C([0,T];L^1(\Omega)), \quad \beta(f)|_{t=0} = \beta(f_0) \quad \text{a.e. in } \Omega.$$

If f satisfies (R1)-(R4) we say that f is a *renormalized solution* of (A.7)-(A.9).

REMARK A.4. The same result as in Theorem A.3 holds when $\Omega = \mathbb{R}^N$ or, in the periodic case, when the Neumann boundary conditions are replaced by Dirichlet boundary conditions; it holds also for more general conditions on u and $\operatorname{div} u$ such as those in (A.10).

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