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On the associated graded ring of a local Cohen-Macaulay ring

By

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It seldom happens that properties of a local ring $(R, \boldsymbol{m})$ are carried over to its associated graded ring $G(R)$. Here we investigate when the property of being Cohen-Macaulay is transferred from R to $G(R)$.

First, we set up some notation. $(R, \boldsymbol{m})$ denotes a local Cohen-Macaulay ring of dimension $d>0$ and multiplicity $e(R)$. $G(R)$ denotes its associated graded ring. $G(R)=R/\boldsymbol{m}\oplus\boldsymbol{m}/\boldsymbol{m}^2\oplus\boldsymbol{m}^2/\boldsymbol{m}^3\oplus\cdots$. We denote the embedding dimension of R, that is the number of generators in a minimal basis of $\boldsymbol{m}$, by $v(\boldsymbol{m})$.

It is true that $d\leqslant v(\boldsymbol{m})\leqslant e(R)+d-1$. The first inequality is the Krull principal ideal theorem and the second inequality is a result of Abhyankar [A]. We investigate whether $G(R)$ is Cohen-Macaulay in terms of $v(\boldsymbol{m})$. If $v(\boldsymbol{m})=d$, R is regular and $G(R)$ is also regular. The fact that $G(R)$ is Cohen-Macaulay if $v(\boldsymbol{m})=d+1$ is also well known (see, for example, [B-S]) but for the convenience of the reader we sketch a proof here. We may assume that R is a complete Cohen-Macaulay local ring and thus a homomorphic image of a $d+1$ dimensional complete regular local ring S. The kernel C must be a height 1 unmixed, hence principal, ideal of S. Say $C=fS$. Then $G(R)=G(S)/\bar{f}G(S)$, where $\bar{f}$ is the initial form of f in $G(S)$. Since $G(S)$ is a polynomial ring over a field, $G(R)$ is CohenMacaulay.

We will show that if $v(\boldsymbol{m})=e(R)+d-1$ then $G(R)$ is Cohen-Macaulay and that this is the only other case where $G(R)$ is Cohen-Macaulay for all such $(R, \boldsymbol{m})$.

Theorem 1. *Assume that $R/\boldsymbol{m}$ is an infinite field. Then there exist elements $x_1, x_2, \cdots, x_d$ in $\boldsymbol{m}$ such that $\boldsymbol{m}^2=(x_1, x_2, \cdots, x_d)\boldsymbol{m}$ if and only if $v(\boldsymbol{m})=e(R)+d-1$.*

Proof. By the results of Northcott and Rees [N-R] on minimal reductions of an ideal, there exist elements $x_1, x_2, \cdots, x_d$ in $\boldsymbol{m}$ with $\boldsymbol{m}^n=\boldsymbol{x}\boldsymbol{m}^{n+1}$ for some positive integer n, where $\boldsymbol{x}=(x_1, x_2, \cdots, x_d)$. Tensor the exact sequence

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$$0\to \boldsymbol{m}\to R\to R/\boldsymbol{m}\to 0$$

by $R/\boldsymbol{x}$ to obtain the exact sequence:

$$0\to \mathrm{Tor}_1^R(R/\boldsymbol{m}, R/\boldsymbol{x})\to \boldsymbol{m}/\boldsymbol{xm}\to R/\boldsymbol{x}\to R/\boldsymbol{m}\to 0.$$

Let $\lambda_R(A)$ denote length of the R-module A. We have that $\lambda_{R/m}(\mathrm{Tor}_1^R(R/\boldsymbol{m}, R/\boldsymbol{x}))=d$ because $\boldsymbol{x}$ is a regular sequence. By the properties of minimal reductions, $e(R)=\lambda_{R/x}(R/\boldsymbol{x})$. Hence, $\lambda_{R/m}(\boldsymbol{m}/\boldsymbol{xm})=e(R)+d-1$. From this it follows immediately that $v(\boldsymbol{m})\leqslant e(R)+d-1$, and that $v(\boldsymbol{m})=e(R)+d-1$ if and only if $\boldsymbol{m}^2=\boldsymbol{xm}$.

Theorem 2. *If $v(\boldsymbol{m})=e(R)+d-1$, then $G(R)$ is Cohen-Macaulay.*

Proof. We may assume that $R/\boldsymbol{m}$ is an infinite field. By Theorem 1, there exist elements $x_1, x_2, \cdots, x_d$ in $\boldsymbol{m}$ such that $\boldsymbol{m}^2=\boldsymbol{xm}$. We prove by induction on d, that $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_d$, the images of $x_1, x_2, \cdots, x_d$ in $G(R)$, form a regular sequence. By [H-R], [M-R] this is sufficient to prove that $G(R)$ is Cohen-Macaulay. If $d=1$, we have $\boldsymbol{m}^2=x_1\boldsymbol{m}$. $\bar{x}_1$ is not a zero divisor for $x_1y\in \boldsymbol{m}^{t+1}=x_1\boldsymbol{m}^t$, for any $t>1$, implies that $y\in \boldsymbol{m}^t$ because x_1 is not a zero divisor in R. Assume that $d>1$. We first check that $\bar{x}_1$ is not a zero divisor in $G(R)$. If $x_1y\in \boldsymbol{m}^{t+1}=(x_1, \cdots, x_d)^t\,\boldsymbol{m}$, where $t>1$, we must show that $y\in \boldsymbol{m}^t=(x_1, \cdots, x_d)^{t-1}\boldsymbol{m}$. $x_1y=g(x_1, \cdots, x_d)x_1+f(x_2, \cdots, x_d)$, where $g(x_1, \cdots, x_d)$ is a homogeneous polynomial of degree $t-1$ in $x_1, \cdots, x_d$ with coefficients in $\boldsymbol{m}$ and $f(x_2, \cdots, x_d)$ is a homogeneous polynomial of degree t in $x_2, \cdots, x_d$ with coefficients in $\boldsymbol{m}$. Hence $hx_1=f(x_2, \cdots, x_d)$ with $h=y-g(x_1, \cdots, x_d)$. Since $x_1, \cdots, x_d$ is a regular sequence in R, the associated graded ring of R with respect to the ideal $\boldsymbol{x}=(x_1, \cdots, x_d)$ is a polynomial ring in d variables over $R/\boldsymbol{x}$. Hence $h\in (x_1, \cdots, x_d)^t$ and $y\in (x_1, \cdots, x_d)^{t-1}\boldsymbol{m}$. Pass to the Cohen-Macaulay ring R/x_1R. $e(R/x_1R)=e(R)$, $\dim R/x_1R=d-1$ and $v(\boldsymbol{m}/x_1R)=v(\boldsymbol{m})-1$. The induction hypothesis applies to R/x_1R so that $\bar{x}_2, \cdots, \bar{x}_d$ form a regular sequence in $G(R/x_1R)=G(R)/\bar{x}_1G(R)$. Hence $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_d$ is a regular sequence in $G(R)$.

Remark. The case $d=1$ is also proved in [D].

Corollary 3. *If $e(R)\leqslant 3$, then $G(R)$ is Cohen-Macaulay.*

To show that $v(\boldsymbol{m})=d, d+1$ and $e(R)+d-1$ are the only cases where $G(R)$ is Cohen-Macaulay for all local Cohen-Macaulay rings R, we construct a 1-dimensional local Cohen-Macaulay ring $(R, \boldsymbol{m})$ with $v(\boldsymbol{m})=d+2=e(R)+d-2$ and with $G(R)$ not Cohen-Macaulay. M. Hochster showed me that if k is a field, then $k[[t^5, t^8, t^{27}]]$ is an axample of a 1-dimensional complete local domain R with $G(R)$ not Cohen-Macaulay. The following examples show that such a domain can be constructed for any multiplicity $e(R)\geqslant 4$. Let k be a field. Let e be any integer $\geqslant 4$. The conductor of the numerical semigroup generated by e and $e+1$ is $(e-1)e$. Let $R=k[[t^e, t^{e+1}, t^{(e-1)e-1}]]$. Then $\boldsymbol{m}=(t^e, t^{e+1}, t^{(e-1)e-1})$ and $v(\boldsymbol{m})=3$. Now $t^{(e-1)e-1}\boldsymbol{m}\subseteq \boldsymbol{m}^3$; for $t^{(e-1)e-1}t^e=(t^{e+1})^{e-1}$, $t^{(e-1)e-1}t^{e+1}=t^{e^2}$ and $(t^{(e-1)e-1})^2=(t^e)^{e-1}(t^{e+1})^{e-2}$. Thus the maximal homogeneous ideal of $G(R)$ belongs to 0 in $G(R)$. In particular, if $e=4$, then $R=k[[t^4, t^5,$

$t^{11}]]$ has $v(\boldsymbol{m})=3=d+2=e(R)+d-2$ and $G(R)$ is not Cohen-Macaulay.

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