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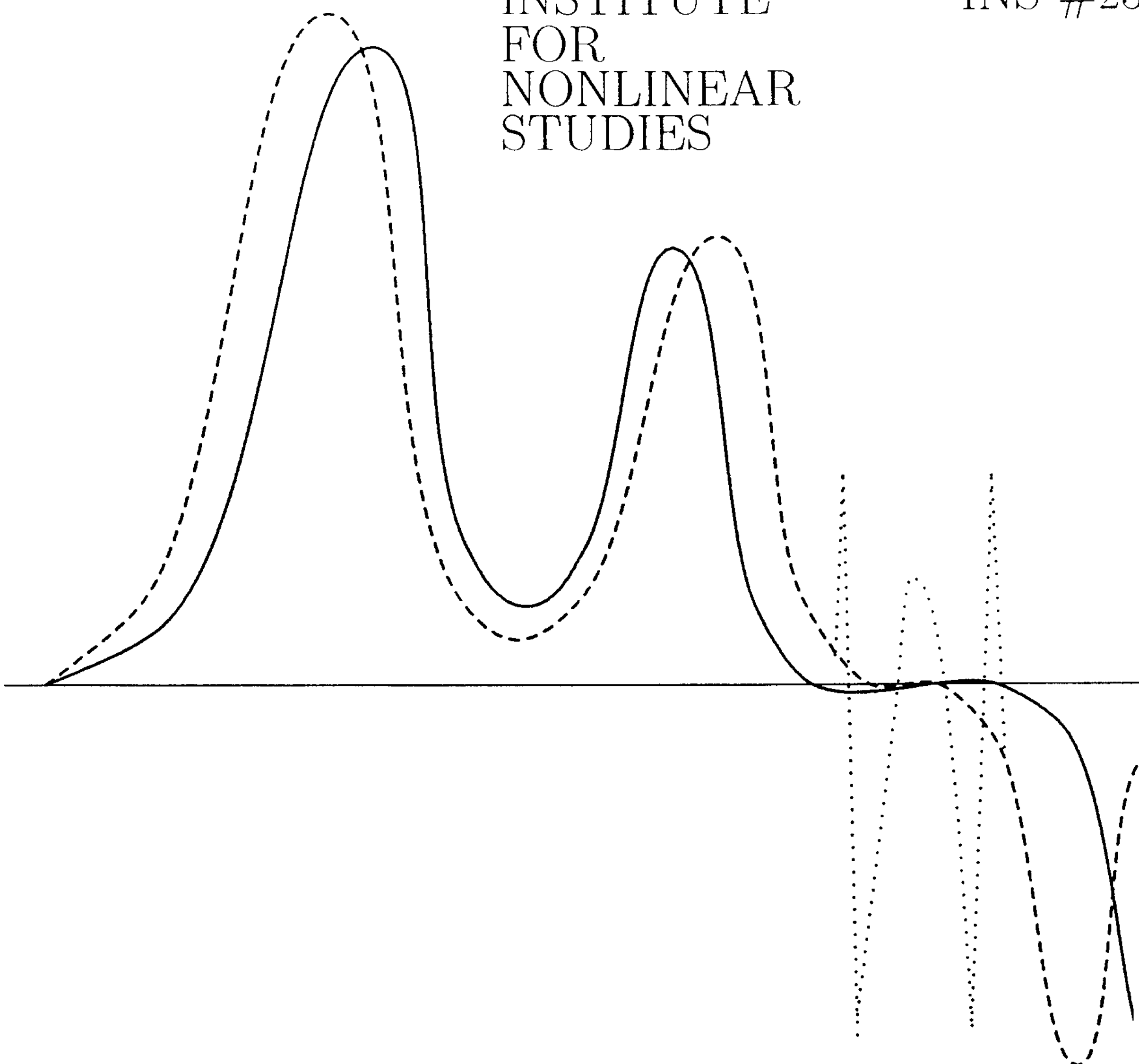

On the Asymptotic Analysis of the Painlevé Equations Via the Isomonodromy Method

by

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On the Asymptotic Analysis of the Painlevé Equations via the Isomonodromy Method

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Abstract

During the last ten years considerable progress in the theory of classical Painlevé equations has been achieved. This progress is based on the so-called Isomonodromy Method that makes the Painlevé transcendents such efficient tools of the modern analysis as the usual special functions are. In this paper, the problem of the rigorous justification of the global asymptotic results which are obtained via Isomonodromy Method, is considered. Taking the pure imaginary solutions of the second Painlevé equation as an example, we discuss in detail three different rigorous methodologies of their asymptotic analysis including derivation of the corresponding connection formulae.

Introduction

The isomonodromy method in the modern theory of Painlevé equations was introduced in [1] and [2], where it was realized that solving a Cauchy problem for a given Painlevé equation, is equivalent to solving the inverse monodromy problem for an associated system of linear ODE's with rational coefficients. We shall refer to this linear system as the λ-equation. It was shown in [3] that the inverse problem can be formulated as a matrix, singular, discontinuous Riemann-Hilbert (RH) problem defined on a complicated contour. A rigorous methodology for implementing the isomonodromy method was developed in [4] and has been applied to PI-PV [4], [5]. Using this method, it is possible to solve the initial value problem for Painlevé equations in terms of the Fredholm theory of linear integral equations.

Another important application of the isomonodromy method is that it yields a method for investigating global asymptotics of the solutions of Painlevé equations. This method, which can be thought of as a nonlinear analogue of the Laplace's method for solving linear ODE's, allows one to give in closed form the following: (i) A complete description of all types of asymptotic behavior for all solutions. (ii) The derivation of explicit connection formulae for the asymptotics in different domains. (iii) A complete description of the asymptotic behavior in the complex plane (nonlinear Stokes phenomena). (iv) The distributions of zeros and poles of all solutions. It should be emphasized that before the emergence of this theory, such problems had been solved only for the linear differential equations associated with the classical special functions. The first results about the evaluation of the connection formulae for the Painlevé transcendents appeared in [6]-[12], and have been summarized in the monograph [13]. By now such asymptotic results have been obtained for most of the Painlevé equations [14]-[26].

The idea of the asymptotic approach, presented in [13] (see also the review papers [27] and [28] is based on the fact that the so called monodromy data of the λ-equation are a complete set of first integrals of motion for the associated Painlevé equation. Of course, these integrals cannot be evaluated in closed form otherwise, this would mean that the Painlevé transcendents could be expressed in terms of known functions. However assuming a certain type of asymptotic behavior for the solution of the Painlevé equation, the corresponding monodromy data can be evaluated asymptotically. Since these data are constants of motion, this yields a connection formulae.

The derivation of connection formulae is one of the most interesting and unexpected results of the modern theory of the Painlevé transcendents that opens a new area in the theory of ODEs. This is perhaps why the research in this area suffers from the usual weakness of pioneer work, namely the lack of mathematical rigor. Finally, the time has come to make the asymptotic analysis of the Painlevé equations rigorous. It should be emphasized that this question is not a straightforward one. The formal derivation of the connection formulae required a new technique (the Isomonodromy

Method); similarly the rigorous justification of these formulae required the development of new analytical ideas.

Here we present our own pont of view on the current status of the rigorous justification of the asymptotic results obtained via the isomonodromy method. We will discuss three approaches to this problem. The first one, which was used in [13] and [12], is based on the independent local asymptotic analysis of the Painlevé equations. This approach is quite adequate when one deals with quasilinear asymptotics. The second approach, which was proposed by A.V. Kitaev in [29], does not depend on the specific form of the asymptotics. It is based on the unexpected use of the classical Brouwer fixed point theorem, and utilizes the general information about solvability of the inverse monodromy problem obtained in [4] and [30]. The third approach has been recently developed by P. Deift and X. Zhou [31]. This method, which is based on the direct asymptotic analysis of the relevant oscillatory RH problems, can be treated as a nonlinear analogue of the classical steepest descent method.

As an illustrative example we shall discuss the connection problem for pure imaginary solutions of the second Painlevé equation,

$$u_{xx} - xu - 2u^3 = 0, \quad x\epsilon\mathbb{R}, \quad u\epsilon i\mathbb{R}. \tag{1}$$

We restrict ourselves to this case because such solutions are smooth functions which simplifies our considerations.

In §1 we briefly summarize the derivation of the corresponding connection formulae first obtained in [21]. In §2 and §3 we present in details the rigorous justification of these formulae using first and second approaches mentioned above. We emphasize that Kitaev's approach has not been presented so far in all its important technical details (see theorem 3.1 and its proof in Appendix). This is perhaps the reason why, although this approach seems the most adequate one, it has so far been neglected. §4 is devoted to the Deift-Zhou method. We sketch the results of their recent paper [32] where the rigorous derivation of the connection problem for equation (1) is obtained. In all the constructions concerning the deformations of the initial RH problem presented in this section we follow directly the original paper [32]. We have included §4 mainly for completeness. Indeed, the Deift-Zhou method is the most self-consistent scheme; its significance, we believe, goes beyond the justification problem itself. Also, we discuss in §4 the connection between the Deift-Zhou method and the Kitaev method and make a few observations that could possibly technically simplify the implementation of the methodology of [32].

We conclude this introduction with two remarks:

1. The asymptotics of some special families of the solutions of the Painlevé equations can be investigated rigorously by linear integral equations of the Gel'fand-Levitan-Marchenko type [33], [34], [35].

2. It is possible to obtain certain rigorous global asymptotic results about the Painlevé equations without the isomonodromy method [36].

1 The Derivation of the Connection Formulae for Pure Imaginary Solutions of the Second Painlevé Equation

The λ-equation corresponding to the second Painlevé equation (1) has the form (see [1], [2])

$$\frac{d\Psi}{d\lambda} = \left[-(4i\lambda^2 + ix + 2iu^2)\sigma_3 - 4u\lambda\sigma_2 - 2w\sigma_1\right]\Psi, \tag{1.1}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

x is real and u, w are pure imaginary parameters. The set of the monodromy data of this equation consists of the six Stokes matrices,

$$S_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix}, \quad S_{2k+1} = \begin{pmatrix} 1 & 0 \\ s_{2k+1} & 1 \end{pmatrix}, \quad k = 1, 2, 3$$

defined through the equations

$$S_k = \Psi_k^{-1}\Psi_{k+1}, \quad k = 1, ..., b, \tag{1.2}$$

where $\Psi_1(\lambda), ..., \Psi_7(\lambda)$ are the canonical solutions of the equation (1.1). These solutions are fixed by the asymptotic condition (for more details see for example [13]),

$$\Psi_k(\lambda) = (I + O(\lambda^{-1}))\exp\left\{-\frac{4}{3}i\lambda^3\sigma_3 - ix\lambda\sigma_3\right\},$$

$$\lambda \to \infty, \quad \frac{\pi}{3}(k-2) < \arg\lambda < \frac{\pi}{3}k, \quad k = 1, ..., 7. \tag{1.3}$$

Note that $\Psi_7 = \Psi_1$. The following general constraints take place:

$$S_{k+3} = \sigma_2 S_k \sigma_2, \quad k = 1, 2, 3, \quad S_1 = \sigma_1 \bar{S}_6 \sigma_1, \tag{1.4}$$

$$S_1 S_2 ... S_6 = I.$$

Using equations (1.4), it follows that all six Stokes matrices can be parametrized by only one complex parameter s:

$$S_1 = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & \frac{s-\bar{s}}{1+|s|^2} \\ 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 \\ -\bar{s} & 1 \end{pmatrix}$$

$$S_4 = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}, \quad S_5 = \begin{pmatrix} 1 & 0 \\ \frac{\bar{s}-s}{1+|s|^2} & 1 \end{pmatrix}, \quad S_6 = \begin{pmatrix} 1 & \bar{s} \\ 0 & 1 \end{pmatrix}.$$

The Stokes multiplier s is the function of the coefficients of the system (1.1), i.e. $s = s(x, u, w)$. The essence of the Isomonodromy Method is that this complex valued function is the first integral of the nonlinear equation (1). More precisely (Flaschka-Newell theorem [1]),

$$s \equiv const \Leftrightarrow w = u_x, \quad u_{xx} - xu - 2u^3 = 0. \tag{1.5}$$

Moreover, as shown in [4], parameter s may assume any complex value and can be taken as an adequate parametrization of the pure imaginary solutions of (1):

$$u = u(x, s).$$

The utilization of this fact for the calculation of the asymptotics of the solutions of the second Painlevé equation is based on the following theorems proven in [37] and [21] (see also [13]):

Theorem 1.1 ([37]) Let u, w be pure imaginary functions of x defined on $(-\infty, x_-]$ and satisfying the conditions:

a)

$$u(x) = 0((-x)^{-1/4}), \quad w(x) = 0((-x)^{1/4}), \quad x \to -\infty. \tag{1.6a}$$

b) There exist constants $c > 0$ and $X_- \le x_-$ such that

$$|v(x)| \ge c \quad \forall x \epsilon (-\infty, X_-], \tag{1.6b}$$

where

$$v = \frac{i}{\sqrt{2}} e^{i\pi/4}[(-x)^{1/4}u + i(-x)^{-1/4}w].$$

Then for the corresponding monodromy data $s = s(x, u, w)$ the following asymptotic equation holds:

$$s = \frac{i}{v} \frac{\sqrt{2\pi}}{\Gamma(i|v|^2)} \exp\left\{\frac{2}{3}i(-x)^{3/2} + i\frac{3}{2}|v|^2 \log(-x) + i|v|^2 \log 8 + \frac{\pi}{2}|v|^2\right\} + o(1), \quad x \to -\infty, \tag{1.7}$$

where $\Gamma(z)$ is the Euler gamma-function.

Theorem 1.2 ([21]) Let u, w be pure imaginary functions of x defined on $[x_+, \infty)$ and satisfying the conditions

a)

$$u(x) = i\sqrt{\frac{x}{2}} + u_0(x), \quad u_0 = O(x^{-1/4}), \quad x \to +\infty,$$

$$w(x) = \frac{i}{2}\sqrt{\frac{1}{2x}} + w_0(x), \quad w_0 = O(x^{1/4}), \quad x \to +\infty. \tag{1.8a}$$

b) There exist constants $c > 0$ and $X_+ \geq x_+$, such that

$$|v_0(x)| \geq c, \quad \forall x \epsilon [X_+, \infty), \tag{1.8b}$$

where

$$v_0 = e^{i\pi/4}[(2x)^{1/4} u_0 + i(2x)^{-1/4} w_0].$$

Then for the corresponding monodromy data $s = s(x, u, w)$ the following asymptotic equations hold:

$$\frac{\bar{s} - s}{1 + |s|^2} = ie^{-\pi|v_0|^2} + o(1), \quad x \to +\infty, \tag{1.9}$$

$$\frac{1 + s^2}{1 + |s|^2} = -\frac{i}{v_0} \frac{\sqrt{2\pi}}{\Gamma(i|v_0|^2)} \exp\left\{-\frac{2i\sqrt{2}}{3} x^{3/2} + i\frac{3}{2}|v_0|^2 \log x + i\frac{7}{2}|v_0|^2 \log 2 - \frac{\pi}{2}|v_0|^2\right\} + o(1), \quad x \to +\infty.$$

The formal derivation of the asymptotics and the corresponding connection formulae for the pure imaginary solutions of the Painlevé equation (1) can be obtained as follows. Suppose that u and w are not arbitrary functions of x satisfying (1.6) and (1.8), but are such that $w = u_x$, and $u(x)$ is a solution of the Painlevé equation (1). Then, by (1.5), the parameter s is constant. Solving (1.7) and (1.9) in this case for $u(x)$, we obtain the following (differentiable in x) asymptotics for $u(x)$ as $x \to \pm\infty$:

$$u(x) = i\alpha(-x)^{-1/4} \sin\left\{\frac{2}{3}(-x)^{3/2} + \frac{3}{4}\alpha^2 \log(-x) + \varphi\right\} + o((-x)^{-1/4}), \quad x \to -\infty, \tag{1.10}$$

where

$$\alpha^2 = \frac{1}{\pi} \log(1 + |s|^2), \quad \alpha > 0, \quad \varphi = \frac{3}{2}\alpha^2 \log 2 - \frac{\pi}{4} - \arg \Gamma(i\frac{\alpha^2}{2}) - \arg s. \tag{1.11}$$

Also

$$u(x) = i\sqrt{\frac{x}{2}} + i(2x)^{-1/4} \rho \cos\left\{\frac{2\sqrt{2}}{3} x^{3/2} - \frac{3}{2}\rho^2 \log x + \theta\right\} + o(x^{-1/4}), \quad x \to +\infty \tag{1.12}$$

where

$$\rho^2 = \frac{1}{\pi} \log \frac{1 + |s|^2}{2|Ims|}, \quad \rho > 0, \tag{1.13}$$

and

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \log 2 + \arg \Gamma(i\rho^2) + \arg(1 + s^2).$$

(In the original paper [21] there is an arithmetic mistake in the derivation of the formula for θ from (1.9). We are grateful to P. Deift for pointing that out.) Since the parameter s in (1.13) and in

(1.11) is the **same**, we can eliminate it and obtain the explicit **connection formulae** between the asymptotic parameters (α, φ) at $-\infty$ and the asymptotic parameters (ρ, θ) at $+\infty$:

$$\rho^2 = \alpha^2 - \frac{1}{\pi} \log 2(e^{\pi\alpha^2} - 1)^{1/2} |\sin \Psi(\alpha, \varphi)|,$$

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \log 2 + \arg \Gamma(i\rho^2) + \arg \left(1 + (e^{\pi\alpha^2} - 1)e^{2i\Psi(\alpha,\varphi)}\right), \tag{1.14}$$

where

$$\Psi(\alpha, \varphi) = \frac{3}{2}\alpha^2 \log 2 - \frac{\pi}{4} - \arg \Gamma(i\alpha^2/2) - \varphi. \tag{1.15}$$

The asymptotics (1.10) and (1.12) satisfy the conditions (1.6) and (1.8) respectively, therefore on the formal level the above derivation is self-consistent. On the other hand the formulae (1.10)-(1.15) do not cover the entire class of all pure imaginary solutions of (1). In fact, these formulae cover only "half" of the solutions: taking into account (1.9a), it follows that the asymptotic behavior given by (1.12) corresponds to those solutions $u(x, s)$ which satisfy

$$Ims < 0 \Leftrightarrow \sin \Psi(\alpha, \varphi) < 0. \tag{1.16}$$

To obtain the other "half" of the solution manifold it is enough to notice that the transformation $u \to \bar{u}$, and $w \to \bar{w}$ corresponds to the transformation $\Psi(\lambda) \to \sigma_1 \bar{\Psi}(\bar{\lambda})\sigma_1$, which in turn implies $s \to -s$. This means that in order to obtain the asymptotic behavior as $x \to \pm\infty$ for $u(x, s)$ with $Ims > 0$ $(\sin \Psi(\alpha, \varphi) > 0)$, one simply has to replace i by $-i$ in (1.12) and keep (1.10), (1.11), and (1.13) unchanged.

The special case when $Ims = 0$ contradicts formula (1.9a). Therefore, $u(x, s)$ with $Ims = 0$ does not have the asymptotic behavior given by (1.8a). However, in this case the second and the fifth of Stokes matrices become trivial,

$$s_2 = s_5 = I.$$

This allows one (see [13]) to extract the asymptotic behavior of u as $x \to +\infty$ quite easily directly from the corresponding RH problem. The answer is

$$Ims = 0: \quad u(x, s) = \frac{is}{2\sqrt{\pi}} x^{-1/4} e^{-(2/3)x^{3/2}} (1 + o(1)), \quad x \to +\infty. \tag{1.17}$$

In terms of the parameters α, φ (all the formulae associated with $x \to -\infty$ are valid for any s!), the condition $Ims = 0$ can be written as

$$\varphi = \frac{3}{2}\alpha^2 \log 2 - \frac{\pi}{4} - \arg \Gamma(i\alpha^2/2)(mod\pi). \tag{1.18}$$

This is the Ablowitz-Segur [7] one parametric family of solutions of (1).

Equations (1.10)-(1.18) provide a complete description of the asymptotic behavior and of the connection formulated of all pure imaginary solutions of the second Painlevé equation (1). This description was first obtained in [21] and certainly **is not a rigorous one**. To make it rigorous, one has to prove that:

a) Any solution $u(x,s)$ has the asymptotic (1.10) as $x \to -\infty$, where the parameters α and φ satisfy (1.11).

b) Any solution $u(x,s)$ with $Ims < 0$, has the asymptotic (1.12) as $x \to +\infty$, where the parameters ρ and θ satisfy (1.13).

We expect that this should be true because first, we have rigorously proven theorems 1.1 and 1.2, and secondly, because equations (1.11) and (1.13) make sense for **all** s (for $Ims \neq 0$ in the case of (1.13)). In the next two sections we will show how this expectation can be transformed into a rigorous mathematical fact.

2 Justification of the Asymptotic Formulae at $x \to -\infty$ Based on the Local Asymptotic Analysis

Here we are going to justify formulae (1.10), (1.11) for **any** pure imaginary solution $u(x,s)$ of the Painlevé equation (1). To do this we will use the following fact from the **conventional** ODE theory established by A.S. Abdullaev [38]:

Theorem 2.1 ([38])
Given any $\alpha > 0$ and $\varphi \epsilon [0, 2\pi)$ there exists a unique pure imaginary solution $u(x;\alpha,\varphi)$ of equation (1) that has the (differentiable in x) asymptotics (1.10) as $x \to -\infty$.

The proof of theorem 2.1 given in [38] is based on a local analysis of equation (1) near $x \sim -\infty$. It uses the natural fact that for decreasing $u(x)$ the second Painlevé equation can be approximated by the Airy equation

$$u_{xx} - xu = 0.$$

This proof does not use the isomonodromy method. In fact in [38] a more general (nonintegrable) type of nonlinearity is considered, and an asymptotic result similar to Theorem 2.1 is established. To make this result global, i.e. to guarantee that **any** solution of (1) has the asymptotics (1.10) for some α, φ as $x \to -\infty$, we need to use the integrability of equation (1). Tis means we have to use the results listed in Theorem 2.1 as well as the "isomonodromy" Theorem 1.1.

Let $u(x)$, $x \epsilon \mathbb{R}$ be an arbitrary pure imaginary solution of the second Painlevé equation (1), and let s be the corresponding monodromy data of the λ-equation (1.1); that is $u(x) = u(x,s)$. Suppose also that $s \neq 0$; otherwise, because of the uniqueness of the solution of the inverse monodromy problem for (1.1) (see [4]) we get a trivial situation , $u(x) \equiv 0$. Defining $\alpha = \alpha(s)$ and $\varphi = \varphi(s)$ by equations (1.11), consider the corresponding solution $u(x;\alpha(s),\varphi(s))$. Because of Theorem 2.1, this solution exists. Now, to get the desired justification of the asymptotic formulae (1.10) and (1.11) for any pure imaginary solution of (1), it is enough to prove that

$$u(x,s) = u(x;\alpha(s), \quad \varphi(s)). \tag{2.1}$$

To prove equation (2.1) one has to show that

$$\tilde{s} = s, \tag{2.2}$$

where we denote $\tilde{s}$ by the monodromy parameter corresponding to the solution $u(x;\alpha(s),\varphi(s))$.

The proof of (2.2) follows from Theorem 1.1. In fact, $u = u(x;\alpha(s),\varphi(s))$ and $w = u_x(x;\alpha(s),\varphi(s))$ satisfy condition (1.6) of this theorem. Therefore, the monodromy data $\tilde{s}$ satisfy the asymptotic equation (1.7). But we use precisely this equation to derive the basic formulae (1.11). This means that equation (1.7) becomes

$$\tilde{s} = s + o(1), \quad x \to -\infty. \tag{2.3}$$

In (2.3) neither s nor $\tilde{s}$ depend on x (core point of IM!). This means equality (2.2). Hence, we have

Theorem 2.2 Any pure imaginary solution $u(x)$ of the second Painlevé equation (1) has asymptotics (1.10) as $x \to -\infty$. The corresponding asymptotic parameters α, φ can be represented as the one-to-one function (1.11) of the relevant monodromy data s.

Theorem 2.2 completes our discussion of the first approach mentioned in the introduction. As it has already been mentioned, this scheme was used in the first works devoted to the asymptotic analysis of the Painlevé equations via the isomonodromy method. The reader should not be mislead by its apparent simplicity. First, we skipped over the proof of Theorem 2.1, which is by no means elementary. Furthermore, for more general situations such a proof does not exist. This is precisely the case with the "right" asymptotics (1.12). It is relatively straightforward to guess the crude estimates (1.8), but to write down formally the next oscillatory term in (1.12) using only the local perturbation analysis of equation (1), is already quite complicated. It is quite amazing that the crude estimate (1.8) is sufficient for obtaining the asymptotic solution of the direct monodromy problem (Theorem 1.2). This immediately provides us with the explicit expression for the oscillatory term in (1.12), but it does not provide us with the rigorous proof of (1.12), (1.13). In the next section, we shall show how this proof can be derived using the general approach proposed by A.V. Kitaev [29] coupled with the general results concerning the solvability of the corresponding inverse monodromy problem obtained in [4].

3 The Justification of the Asymptotic Formulae at $x \to +\infty$ Based on the Kitaev's Method

In order to avoid nonessential technical difficulties, we will prove the asymptotic formulae (1.12), (1.13) assuming that $s \neq -i$.

The central technical point of this method is the following "uniform" extension of the results of Theorem 1.2.

Theorem 3.1 Let $u(x)$ and $w(x)$ be defined by

$$u(x) = i\sqrt{\frac{x}{2}} + i(2x)^{-1/4}\rho\cos\{\theta(x)\} \equiv \hat{u}(x,s), \quad w(x) = \hat{u}_x \equiv \hat{w}(x,s), \tag{3.0}$$

where

$$\rho^2 = \frac{1}{\pi}\log\frac{1+|s|^2}{2|Ims|}, \quad \rho > 0,$$

and

$$\theta(x) = \frac{2\sqrt{2}}{3}x^{3/2} - \frac{3}{2}\rho^2 \ln x + \theta_0, \quad \theta_0 = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \ln 2 + \arg \Gamma(i\rho^2) + \arg(1+s^2).$$

Assume that

$$s\epsilon D(s_0, \varepsilon) \equiv \{s : |s - s_0| \le \varepsilon\}, \quad Im s_0 < 0, \quad s_0 \ne -i, \quad 0 < \varepsilon < \min\{|Im s_0|, \ |i + s_0|\}.$$

Let $\hat{s}(x, s)$ be the corresponding monodromy data of system (1.1). Then for any δ, $0 < \delta < 3/16$, there exist constants $C(s_0, \varepsilon) > 0$ and $x_0(s_0, \varepsilon, \delta) \ge 1$ depending only on s_0, ε, and δ such that

$$|\hat{s}(x, s) - s| < \frac{C(s_0, \varepsilon)}{x^\delta}, \quad x \ge x_0(s_0, \varepsilon, \delta), \quad s\epsilon D(s_0, \varepsilon).$$

The detailed proof of this theorem is given in Appendix.

The method of Kitaev [29] involves the following steps. Define $\hat{u}(x, s)$ and $\hat{w}(x, s)$ by equations (3.0). Use these functions to compute the associated monodromy data $\hat{s}$. Since $\hat{u}(x, s)$ is not an exact solution of Painlevé II, $\hat{s}$ will depend on s and on x, i.e. $\hat{s} = \hat{s}(x, s)$. Allow s to depend on x and on the constant s_0 in such a way that $\hat{s}(x, s(x, s_0)) = s_0$ and that $s(x, s_0) \to s_0$ as $x \to \infty$. Then $\hat{u}(x, s(x, s_0))$ is an exact solution of Painlevé II, since its monodromy data s_0 is constant; moreover, it has desired asymptotics since $s(x, s_0) \to s_0$. The **uniform** estimate for the monodromy data $\hat{s}(x, s)$ given by Theorem 3.1 and the Brouwer fixed point theorem are used to establish the solvability of $\hat{s}(x, s) = s_0$ and $s(x, s_0) \to s_0$ as $x \to \infty$. Below we present all details of this scheme following [29] with some minor modifications.

Theorem 3.2 For every s_0, $s_0 \ne -i$, $Im s_0 < 0$, the corresponding solution $u(x, s_0)$ of the second Painlevé equation (1) possesses the following asymptotic behavior as $x \to +\infty$:

$$u(x, s_0) = \hat{u}(x, s_0) + o(x^{-1/4}). \tag{3.1}$$

Remark: The existence of $u(x, s_0)$ and its smoothness as $x\epsilon\mathbb{R}$ have been proved in [4] (see also [39]).

Proof of Theorem 3.2. Consider the closed disk $D(s_0, \varepsilon)$, the coefficient functions $\hat{u}(x, s)$, $\hat{w}(x, s)$, and the corresponding monodromy data $\hat{s}(x, s)$; where $s\epsilon D(s_0, \varepsilon)$. The known result by Shibuya [30] (see also [2]) imply that the canonical solutions Ψ_k are smooth functions of x, $\hat{u}$, $\hat{w}$. This yields:

Statement 3.1. $\hat{s}(x, s)$ is a continuous function on $[1; \infty) \times D(s_0, \varepsilon)$.

Let us introduce the function $g(x, s)$ by

$$\hat{s}(x, s) = s + g(x, s). \tag{3.2}$$

Statement 3.1 with Theorem 3.1 imply:

Statement 3.2. The function $g(x,s)$ is continuous on $[1,\infty)\times D(s_0,\varepsilon)$. Given $\delta, 0<\delta<\frac{3}{16}$ there exist constants $C=C(s_0,\varepsilon)>0$ and $x_0=x_0(s_0,\varepsilon,\delta)\geq 1$ such that

$$|g(x,s)|<\frac{C}{x^{\delta}},\quad \forall x\geq x_0,\quad \forall s\epsilon D(s_0,\varepsilon). \tag{3.3}$$

Now, to define the function $s(x,s_0)$ consider the equation

$$s+g(x,s)=s_0,\quad s\epsilon D(s_0,\varepsilon). \tag{3.4}$$

Introducing the new complex variable $\tau=s_0-s$ one can rewrite equation (3.4) as

$$\tilde{g}(x,\tau)=\tau;\quad \tilde{g}(x,\tau)\equiv g(x,s_0-\tau),\quad \tau\epsilon D(0,\varepsilon)\equiv\{\tau:|\tau|\leq\varepsilon\}. \tag{3.5}$$

Picking any $x\geq\max\{x_0,(C/\varepsilon)^{1/\delta}\}\equiv x_1$, we conclude that the function $\tilde{g}(x,\cdot)$ is a **continuous mapping from the compact disk** $D(0,\varepsilon)$ **into itself.** Thus the Brouwer Fixed Point Theorem implies that for each $x\geq x_1$ there exists at least one solution of equation (3.5), therefore for each $x\geq x_1$ there exists at least one solution of equation (3.4).

Statement 3.3 The solution of (3.4) is unique for $x\geq\max\{x_1,x_2\}$, where

$$x_2=\left(\frac{1}{\pi}\ln\frac{1+(|s_0|+\varepsilon)^2}{2(|Ims_0|-\varepsilon)}\right)^{2/3}3^{4/3}2^{-1/3}.$$

Proof. Let $x\geq x_1$ and suppose we have two complex numbers s, $\tilde{s}$ satisfying (3.4). Consider the corresponding functions $\hat{u}(x,s)$, $\hat{w}(x,s)$ and $\hat{u}(x,\tilde{s})$, $\hat{w}(x,\tilde{s})$. For the corresponding monodromy data $\hat{s}(x,s)$, $\hat{s}(x,\tilde{s})$ we have

$$\hat{s}(x,s)=s+g(x,s)=s_0,\quad \hat{s}(x,\tilde{s})=\tilde{s}+g(x,\tilde{s})=s_0.$$

So, $\hat{s}(x,s)=\hat{s}(x,\tilde{s})$. Since the inverse monodromy problem under the pure imaginary reduction is uniquely solvable [4], one gets that

$$\hat{u}(x,s)=\hat{u}(x,\tilde{s}),\quad \hat{w}(x,s)=\hat{w}(x,\tilde{s}).$$

This means

$$\rho\cos\theta(x)=\tilde{\rho}\cos\tilde{\theta}(x)$$

$$\rho\sin\theta(x)\left(1-\frac{3\rho^2}{(2x)^{3/2}}\right)=\tilde{\rho}\sin\tilde{\theta}(x)\left(1-\frac{3\tilde{\rho}^2}{(2x)^{3/2}}\right).$$

Introducing the complex variables $Z=\rho e^{i\theta}$, $\tilde{Z}=\tilde{\rho}e^{i\tilde{\theta}}$, these equations become

$$Z-\frac{3i}{(2x)^{3/2}}|Z|^2ImZ=\tilde{Z}-\frac{3i}{(2x)^{3/2}}|\tilde{Z}|^2Im\tilde{Z}.$$

This yields to the following inequality:

$$|Z - \tilde{Z}| \leq \frac{9C'}{(2x)^{3/2}} |Z - \tilde{Z}|,$$

where C' is defined by

$$C' = \max_{s \epsilon D(s_0, \varepsilon)} \rho^2 \leq \frac{1}{\pi} \ln \frac{1 + (|s_0| + \varepsilon)^2}{2(|Ims_0| - \varepsilon)} \equiv \frac{\sqrt{2}}{9} x_2^{3/2}.$$

If $x \geq x_2$ one concludes that $Z = \tilde{Z}$, i.e. $\rho = \tilde{\rho}$ and $\theta_0 = \tilde{\theta}_0$, or

$$\frac{1 + |s|^2}{2|Ims|} = \frac{1 + |\tilde{s}|^2}{2|Im\tilde{s}|}, \quad \arg(1 + s^2) = \arg(1 + \tilde{s}^2).$$

These equations imply $s = \tilde{s}$ (see [13]), which completes the proof of Statement 3.3.

Assuming $x \geq X(s_0, \varepsilon, \delta) \equiv \max\{x_1, x_2\}$, we will denote the **unique solution** of (3.4) as $s(x, s_0)$. This function:

(a) Is defined for $x \geq X$.

(b) For all $x \geq X$ satisfies the equation

$$s(x, s_0) + g(x, s(x, s_0)) = s_0, \quad s(x, s_0) \epsilon D(s_0, \varepsilon).$$

(c) For all $x \geq X$ satisfies

$$|s(x, s_0) - s_0| < \frac{C(s_0, \varepsilon)}{x^\delta} < \varepsilon, \quad 0 < \delta < 3/16.$$

Now, the last step in the proof. Let us define

$$\tilde{u}(x, s_0) \equiv \hat{u}(x, s(x, s_0)), \quad \tilde{w}(x, s_0) \equiv \hat{w}(x, s(x, s_0)), \quad x \geq X.$$

Taking these functions as the coefficients in the λ-equation, we find that the corresponding monodromy data s' satisfies the identity

$$s' = \hat{s}(x, s(x, s_0)) = s(x, s_0) + g(x, s(x, s_0)) \equiv s_0$$

for all $x \geq X$ and hence does not depend on x. This means that the functions $\tilde{u}(x, s_0)$, and $\tilde{w}(x, s_0)$, are, for $x \geq X$, a solution of the inverse monodromy problem with associated corresponding monodromy data s_0. However, it is known [4] (see also [39]) that this solution is unique and is given by the solution $u(x, s_0)$ of the second Painlevé equation (1). This implies the equation

$$u(x, s_0) = \hat{u}(x, s(x, s_0)), \quad x \geq X, \tag{3.6}$$

which completes the proof of the Theorem 3.2. In fact, it remains to use estimate (c) for $s(x, s_0)$ and the smoothness of the functions

$$\rho = \rho(Res, Ims), \quad \theta_0 = \theta_0(Res, Ims)$$

for $s \epsilon D(s_0, \varepsilon)$, and $0 < \varepsilon < \min\{|Ims_0|, |i + s_0|\}$.

Remark Absolutely in a similar way one can obtain the justification of the asymptotics (1.10), (1.11), independently on the local analysis presented in §2. One only needs an "uniform" extension of the results of Theorem 1.1 (the analogy of Theorem 3.1 for $x \to -\infty$).

4 The Method of Deift and Zhou

The central idea of the isomonodromy method is (see [1], [3]) the interpretation of the basic monodromy relation (1.2) as a Riemann Hilbert problem for the sectionally analytic function $Y(\lambda)$:

$$Y(\lambda) = \Psi_k(\lambda)e^{\frac{4i}{3}\lambda^3\sigma_3 + ix\lambda\sigma_3} \equiv Y_k(\lambda), \quad \frac{\pi}{3}(k-2) + \frac{\pi}{6} \leq \arg\lambda \leq \frac{\pi}{3}k - \frac{\pi}{6}, \quad k = 1, ..., 6$$

on the anti-Stokes rays, $\arg\lambda = \pi k/3 - \pi/6$, $k = 1, ..., 6$. The matrices,

$$e^{-\theta(\lambda)\sigma_3}S_k e^{\theta(\lambda)\sigma_3}, \quad \theta(\lambda) = \frac{4i}{3}\lambda^3 + ix\lambda,$$

are the jump matrices:

$$Y_{k+1} = Y_k e^{-\theta(\lambda)\sigma_3}S_k e^{\theta(\lambda)\sigma_3}. \tag{4.1}$$

This RH-problem is depicted in Figure 1. The corresponding solution $u(x,s)$ of the second Painlevé equation (1) can be determined from $Y(\lambda)$ by the equation

$$u(x;s) = 2\lim_{\lambda\to\infty}(\lambda Y_{12}). \tag{4.2}$$

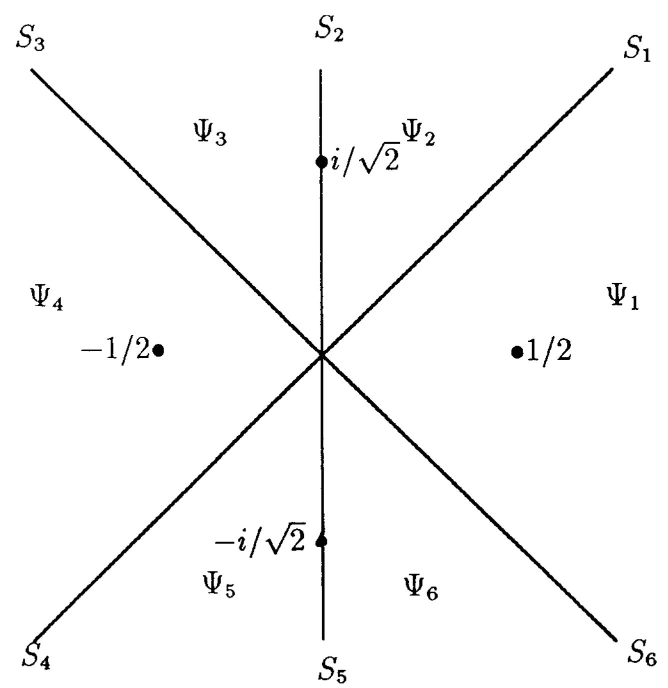

Figure 1

So far, in our exposition of the asymptotic analysis of equation (1) we have made only an indirect use of the RH problem (4.1). The crucial observation of P. Deift and X. Zhou [31-32] is that the RH problem can be used for the **deductive** derivation of the asymptotic formulae (1.10)-(1.13). Their method is the analogue of the classical steepest descent method for the asymptotic evaluation of oscillatory integrals. In analogy with the classical method, one has to deform the initial conjugating contours to the steepest descent (anti-Stokes) lines and then to solve explicitly the model RH problems that arise in the course of this deformation. We consider this part of the Deift-Zhou method as the most important step of their procedure. We will outline Deift-Zhou deformation technique in this section following closely their paper [32].

1. The deformation of the RH problem associated with the asymptotics of $x \to -\infty$. (P. Deift, X. Zhou [32])

In analogy with the classical steepest descent method we perform the natural change of variable,

$$\lambda \to z(-x)^{1/2}, \quad x \to t = (-x)^{3/2}.$$

Then, the characteristic exponent, $\exp(\theta(\lambda))$, is replaced by the exponent

$$e^{t\theta(z)}, \quad \theta(z) = \frac{4i}{3}z^3 - iz.$$

Thus the stationary phase points are $z_\pm = \pm 1/2$. The anti-Stokes lines are represented by broken lines in Figure 2.

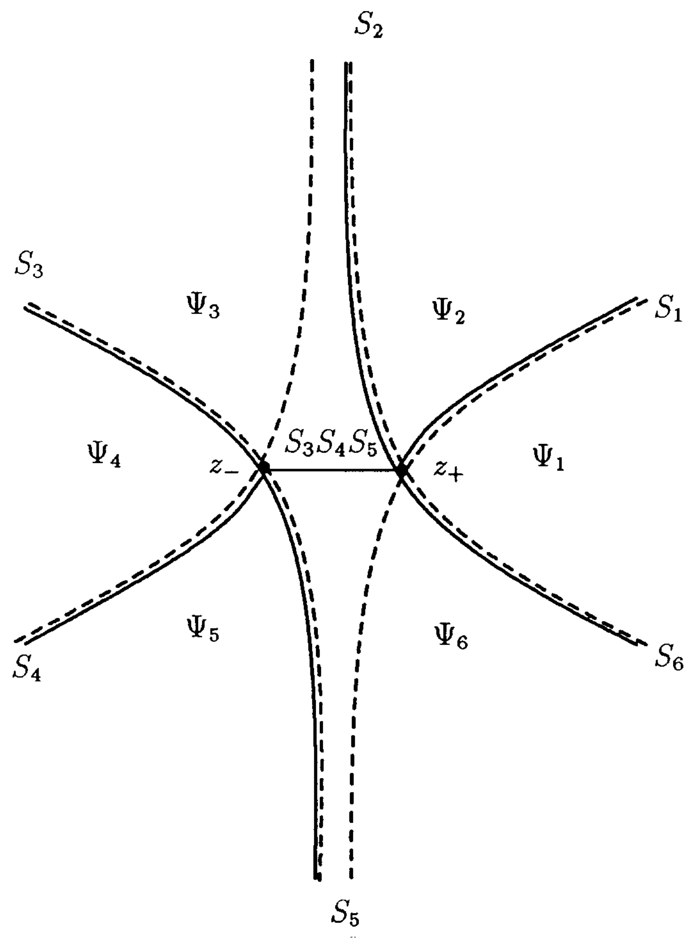

Figure 2

Our aim is to deform the initial RH problem, to a RH problem formulated on these anti-Stokes lines. We first note that the initial RH problem can be immediately rewritten as the problem on the solid lines of Figure 2. Suppose for a moment that along the segment $[z_-, z_+]$ we had the jump $S_U S_L^{-1}$ instead of $S_3 S_4 S_5$, where S_U and S_L are some upper and lower triangular matrices. Then,

$$\Psi_6 S_L = \Psi_3 S_U \equiv \Psi_0$$

and the jump along $[z_-, z_+]$ could be eliminated. In fact, in this case the resulting RH problem would have been represented by Figure 3 (all our RH problems are read in the counter-clockwise direction).

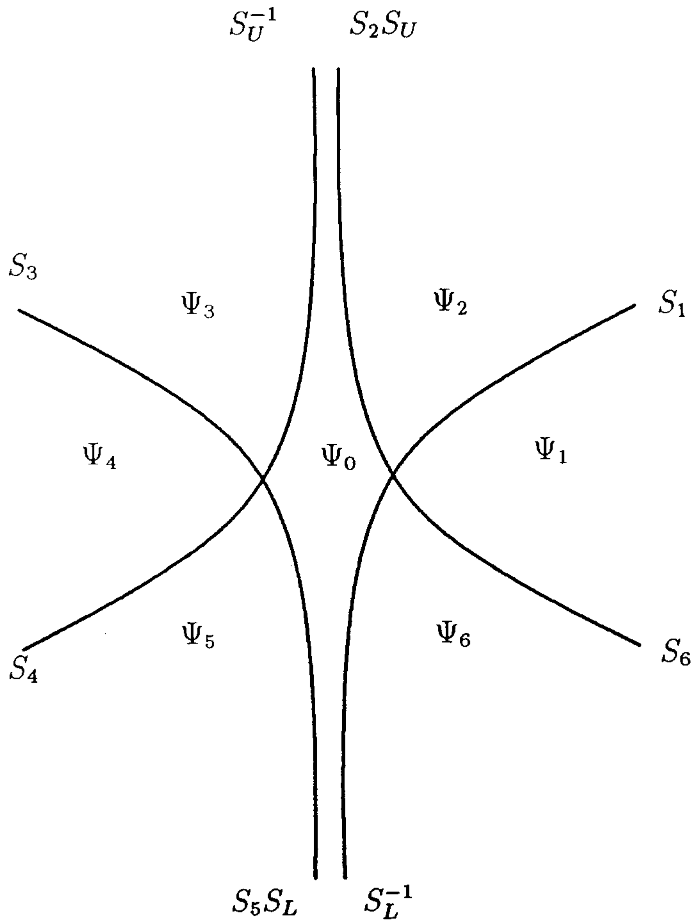

Figure 3

Figure 3 shows that the triangular structure of all jump matrices is consistent with the exponent $e^{t\theta(z)}$ in the Y-formulation (see (4.1)) all jump matrices approach exponentially the identity away from the stationary phase points $\pm 1/2$ as $t \to \infty$.

It turns out that it is possible to transform $S_3 S_4 S_5$ into the product $S_U S_L^{-1}$. This is the central idea of [31-32]. Let $\delta(z)$ be an analytic function on the complex z-plane cut along $[z_-, z_+]$. This function tends to 1 as $z \to \infty$; furthermore along the cut it satisfies the jump condition

$$\delta_- = (1 + s_3 s_4)\delta_+ = (1 + |s|^2)\delta_+ \qquad -\tfrac{1}{2} \; \bullet\underset{+}{\overset{-}{\rule{3cm}{0.4pt}}}\bullet \; \tfrac{1}{2} \tag{4.3}$$

Let us introduce the function $\Psi(z)$ by

$$\Psi = \hat{\Psi} W, \quad W = diag(\delta, \delta^{-1})$$

and let us denote by $\hat{S}$, its jump matrices, i.e. $\hat{S} = WSW^{-1}$. Then for $z \epsilon [-1/2, 1/2]$

$$\hat{\Psi}_6 = \hat{\Psi}_3 W_- S_3 S_4 S_5 W_+^{-1} = \hat{\Psi}_3 \hat{S}_U \hat{S}_L^{-1},$$

where

$$\hat{S}_U = \begin{pmatrix} 1 & \frac{s_4 \delta_-^2}{1+s_3 s_4} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-s}{1+|s|^2}\delta_-^2 \\ 0 & 1 \end{pmatrix},$$

$$\hat{S}_L = \begin{pmatrix} 1 & 0 \\ -\delta_+^{-2}\left(\frac{s_3}{1+s_3 s_4} + s_5\right) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{s}{1+|s|^2}\delta_+^{-2} & 1 \end{pmatrix}.$$

Therefore, introducing

$$\hat{\Psi}_0 = \hat{\Psi}_6 \hat{S}_L = \hat{\Psi}_3 \hat{S}_U$$

we obtain for the function $\hat{\Psi}$, the RH problem depicted on Figure 3, where $\Psi, S \to \hat{\Psi}, \hat{S}$.

Note, that one can easily verify the continuity conditions (see [44], [45]) for this RH problem at the crossing points $z = \pm 1/2$. The function $\delta(z)$ is given explicitly by

$$\delta(z) = \left(\frac{z - 1/2}{z + 1/2}\right)^{\nu}, \quad \nu = \frac{1}{2\pi i}\log(1 + |s|^2). \tag{4.4}$$

Let

$$\hat{Y}(z) = \hat{\Psi}(z) e^{t\theta(z)\sigma_3}, \quad \hat{Y}(z) \to I, \quad z \to \infty.$$

As it has already been mentioned, all jump matrices for $\hat{Y}$ approach exponentially the identity (away from the turning points $\pm 1/2$) as $t \to \infty$. Thus in analogy with the classical steepest descent method one expects that the main contributions of the asymptotics of $\hat{Y}(z)$ as $t \to \infty$, comes from the neighborhood of the points $\pm 1/2$. More precisely, taking into account the symmetry $z \to -z$ one can expect the following asymptotic representation for $\hat{Y}(z)$:

$$\hat{Y}(z) \approx \Phi(z)\sigma_2 \Phi(-z)\sigma_2 \tag{4.5}$$

where $\Phi(z)$ is the solution of the **model** RH problem about the turning point $+1/2$ depicted on Figure 4 ($\Phi(\infty) = I$). The notations δ_0 and θ_0 denote the master terms (compare again with the classical steepest descent method!) of the functions $\delta(z)$ and $\theta(z)$ near the stationary phase point $z = 1/2$:

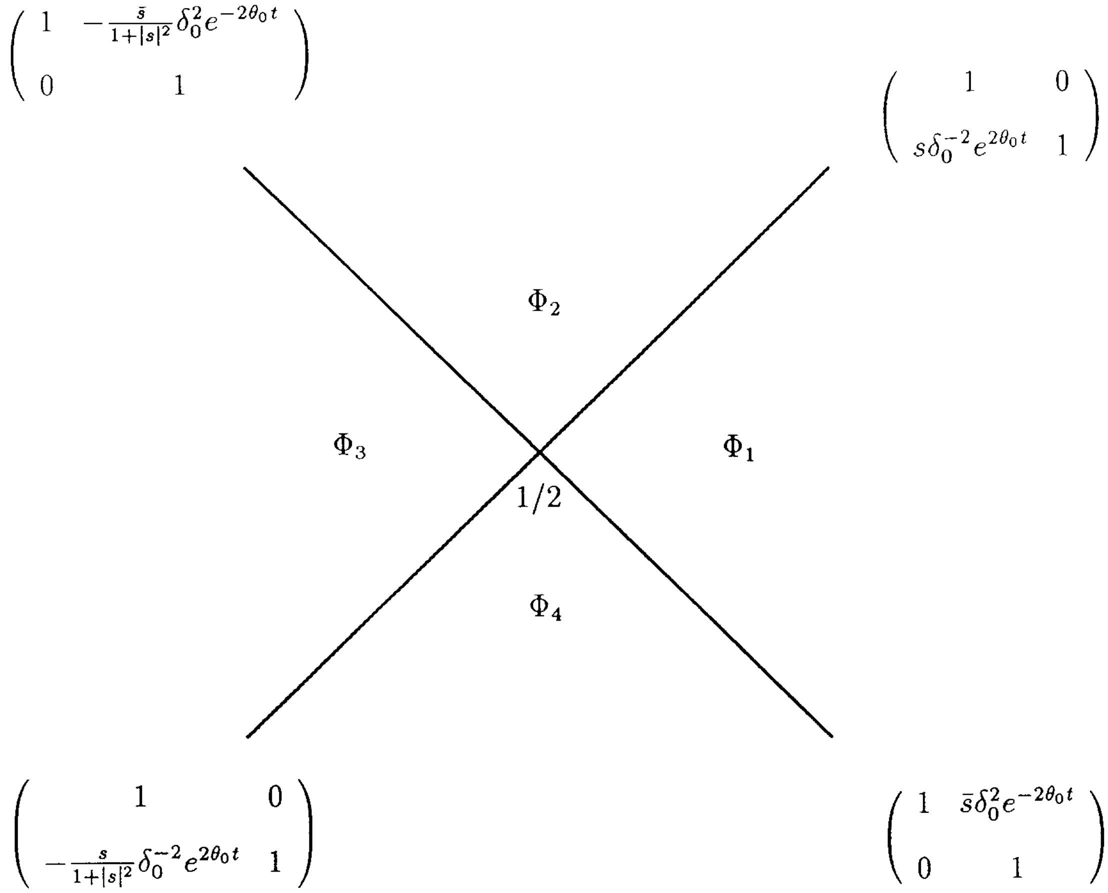

Figure 4

$$\delta_0(z) = (z - \frac{1}{2})^{\nu}, \quad \theta_0(z) = -\frac{i}{3} + 2i(z - \frac{1}{2})^2.$$

The function $\delta_0(z)$ is determined on the z-plane cut along $(-\infty, \frac{1}{2}]$.

It turns out ([40], [37], [31]) that the model RH problem can be solved explicitly in terms of the parabolic cylinder functions D_ν. In particular,

$$[\Phi_2(z)]_{12} = -i\frac{\sqrt{2\pi}}{s\Gamma(-\nu)} e^{\frac{i\pi\nu}{4}} (8t)^{-\frac{\nu}{2}} e^{\frac{2i}{3}t} D_{-1-\nu}\left(e^{-\frac{i\pi}{4}}\sqrt{8t}(z - \frac{1}{2})\right)(z - \frac{1}{2})^{\nu} e^{-2it(z-\frac{1}{2})^2}. \tag{4.6}$$

Taking into account the known asymptotic expansions for the parabolic cylinder functions, symmetry $z \to \bar{z}$, as well as equations (4.2) and (4.5), one can easily obtain the asymptotic formulae (1.10), (1.11) for the solution $u(x,s)$ of the Painlevé equation (1) as $x \to -\infty$.

To justify rigorously the asymptotics of $u(x,s)$, it is enough to prove that

$$\hat{Y}(z) = \left(I + o(t^{-\frac{1}{2}})\right)\Phi(z)\sigma_2\Phi(-z)\sigma_2, \quad t \to +\infty, \quad |z^2 - \frac{1}{4}| \geq \varepsilon_0 > 0. \tag{4.7}$$

To this end let us consider the ratio

$$\hat{\chi}(z) = \hat{Y}(z)\sigma_2\Phi^{-1}(-z)\sigma_2\Phi^{-1}(z).$$

The jump matrices for $\hat{\chi}$ still approach exponentially the identity away from the turning points $\pm 1/2$, while in the neighborhoods of the turning points they behave as

$$I + O(t^{-1/2}). \tag{4.8}$$

These properties of the jump matrices imply, due to the boundness of the corresponding Cauchy operator (for details see [32]), the estimate

$$\hat{\chi}(z) = I + o(t^{-1/2}), \quad |z^2 - \frac{1}{4}| \geq \varepsilon_0 > 0,$$

which yields (4.7).

2. The deformation of the RH problem associated with the asymptotics of $x \to +\infty$. (P. Deift, X. Zhou [32])

We begin with the usual scaling

$$\lambda \to zx^{1/2}, \quad x \to t = x^{3/2}$$

and the RH problem defined on Figure 1. The jumps along $[0, \frac{i}{\sqrt{2}}]$ and $[0, -\frac{i}{\sqrt{2}}]$ are given by S_2 and $S_5 = \sigma_2 S_2 \sigma_2$ respectively. The first step involves replacing these jumps by the constant jump $i\sigma_1$, where

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.8}$$

Let $\delta(z)$ be an analytic function in the z-plane cut along $[-\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}]$, and such that $\delta(z) \to 1$ as $z \to \infty$. Along the cut $\delta(z)$ satisfies the jump conditions

$$\delta_+\delta_- = \left[\begin{array}{ll} \frac{1}{is_2}, & Im z > 0, \\ is_2, & Im z < 0, \end{array}\right. \qquad \begin{array}{c} i/\sqrt{2} \\ +|- \\ -i/\sqrt{2} \end{array}. \tag{4.9}$$

We introduce the function $\hat{\Psi}$ by

$$\Psi = \hat{\Psi}W, \quad W \equiv diag(\delta, \delta^{-1})$$

and we denote by $\hat{S}$ its jump matrices.

Then for $z \epsilon [0, \frac{i}{\sqrt{2}}]$:

$$\hat{S}_2 = W_- S_2 W_+^{-1} = \begin{pmatrix} \frac{\delta_-}{\delta_+} & s_2\delta_-\delta_+ \\ 0 & \frac{\delta_+}{\delta_-} \end{pmatrix} = \begin{pmatrix} \frac{\delta_-}{\delta_+} & -i \\ 0 & \frac{\delta_+}{\delta_-} \end{pmatrix},$$

where we have used equation (4.9). The matrix $\hat{S}_2$ can be written as

$$z\epsilon[0;\frac{i}{\sqrt{2}}]: \quad \hat{S}_2 = -G_1 i\sigma_1 G_3, \quad G_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{\delta_-^2 s_2} & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 \\ \frac{1}{\delta_+^2 s_2} & 1 \end{pmatrix}. \tag{4.10}$$

Similarly

$$z\epsilon[-\frac{i}{\sqrt{2}},0]: \quad \hat{S}_5 = G_4 i\sigma_1 G_6, \quad G_4 = \begin{pmatrix} 1 & -\delta_+^2 s_2^{-1} \\ 0 & 1 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 1 & -\delta_-^2 s_2^{-1} \\ 0 & 1 \end{pmatrix}. \tag{4.11}$$

Using equation (4.10) and (4.11) and introducing the "intermediate" functions

$$\hat{\Psi}_7 = \hat{\Psi}_2 G_1, \quad \hat{\Psi}_8 = \hat{\Psi}_3 G_3^{-1}, \quad \hat{\Psi}_9 = \hat{\Psi}_5 G_4, \quad \hat{\Psi}_{10} = \hat{\Psi}_6 G_6^{-1},$$

the RH problem can be transformed to the one depicted on Figure 5.

The matrices $\hat{S}_j$ are defined by

$$\hat{S}_{2k+1} = \begin{pmatrix} 1 & 0 \\ s_{2k+1}\delta^{-2} & 1 \end{pmatrix}, \quad \hat{S}_{2k} = \begin{pmatrix} 1 & s_{2k}\delta^2 \\ 0 & 1 \end{pmatrix},$$

while the matrices G_j are defined in equations (4.10) and (4.11). We note that

$$\hat{\Psi}_8 = -\hat{\Psi}_7 i\sigma_1, \quad \Psi_3 = \hat{\Psi}_8 G_3, \quad \Psi_2 = \hat{\Psi}_7 G_1^{-1},$$

which imply

$$\hat{\Psi}_3 = -\hat{\Psi}_2 G_1 i\sigma_1 G_3 = \hat{\Psi}_2 \hat{S}_2.$$

Similarly,

$$\hat{\Psi}_{10} = \hat{\Psi}_9 i\sigma_1, \quad \Psi_5 = \hat{\Psi}_9 G_4^{-1}, \quad \hat{\Psi}_6 = \hat{\Psi}_{10} G_6,$$

which imply

$$\hat{\Psi}_6 = \hat{\Psi}_5 G_4 i\sigma_1 G_6 = \hat{\Psi}_5 \hat{S}_5.$$

These relations mean that the $\hat{\Psi}$-RH problem is continuous (see [44], [45]) at the crossing point $\pm\frac{i}{\sqrt{2}}$. One can easily check that at the crossing point $z = 0$ the continuity condition also holds.

The explicit solution of the scalar RH problem (4.9) is given by

$$\delta(z) = (\Delta(z))^\nu, \quad \Delta(z) = \frac{(z^2+\frac{1}{2})^{1/2} - \frac{1}{\sqrt{2}}}{(z^2+\frac{1}{2})^{1/2} + \frac{1}{\sqrt{2}}}, \quad \nu = -\frac{1}{2\pi i}\ln is_2, (Ims_2 < 0!).$$

The branch of $(z^2+\frac{1}{2})^{1/2}$ and the branch of w^ν are fixed as indicated in Figure 6a and Figure 6b respectively.

Let

$$\hat{Y}(z) = \hat{\Psi}(z)e^{itg(z)\sigma_3}, \tag{4.12}$$

where

$$g(z) = \frac{4}{3}(z^2 + \frac{1}{2})^{3/2}.$$

Since $g(z) \to \frac{4}{3}z^3 + z$, as $z \to \infty$, function $\hat{Y}(z)$ is normalized by

$$\hat{Y}(z) \to I, \quad z \to \infty.$$

The off-diagonal entries of the jump matrices of the $\hat{Y}$-RH problem depend on t and z through the factors $e^{\pm 2itg(z)}$. The crucial point is that all of the jump lines on Figure 5, except for the interval $[-i/\sqrt{2}, i/\sqrt{2}]$, are the anti-Stokes lines of $g(z)$. Moreover, the triangular structure of the jump matrices corresponding to these lines is such that all these matrices approach exponentially the identity away from the turning points $z = 0, \ \pm i/\sqrt{2}$. Thus one expects that the main contribution of the solution $\hat{Y}(z)$ of the RH-problem is given by the function $\check{Y}(z)$ that has only a jump on $[-i/\sqrt{2}, i/\sqrt{2}]$:

$$\check{Y}_-(z) = \check{Y}_+(z) \cdot e^{-itg_-(z)\sigma_3} i\sigma_1 e^{itg_+(z)\sigma_3} = \check{Y}_+(z)\sigma_1, \tag{4.13}$$

$$z \epsilon \left[-\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right].$$

This RH-problem can be solved explicitly:

$$\check{Y}(z) = \frac{1}{2}\begin{pmatrix} 1 + \sqrt{\frac{z - i/\sqrt{2}}{z + i/\sqrt{2}}} & 1 - \sqrt{\frac{z - i/\sqrt{2}}{z + i/\sqrt{2}}} \\ 1 - \sqrt{\frac{z - i/\sqrt{2}}{z + i/\sqrt{2}}} & 1 + \sqrt{\frac{z - i/\sqrt{2}}{z + i/\sqrt{2}}} \end{pmatrix} \cdot \left(\frac{z - i/\sqrt{2}}{z + i/\sqrt{2}}\right)^{1/4}, \tag{4.14}$$

This suggests that

$$u(x,s) \approx i\sqrt{\frac{x}{2}}, \quad as \quad x \to \infty.$$

After that, to get a hint about the next term of the asymptotics, one can replace equation (4.12) by

$$\hat{Y}(z) = \hat{\Psi}(z)e^{itg(z)\sigma_3}\check{Y}^{-1}(z). \tag{4.15}$$

The function $\hat{Y}(z)$ has no jump on $[-i/\sqrt{2}, i/\sqrt{2}]$, and all of its jump matrices approach exponentially the identity away from the turning points $z = 0, \ \pm i/\sqrt{2}$. This motivates one to consider the model RH problems corresponding to the turning points $z = 0, \ \pm i/\sqrt{2}$. All three of them can be again solved explicitly in terms of the parabolic cylinder functions ($z = 0$) and Airy functions ($z = \pm i/\sqrt{2}$). This yields the ansatz given by (1.12) and (1.13).

The rigorous justification of the asymptotic equation

$$u(x,s) = \hat{u}(x,s) + o(x^{-1/4}), \quad x \to \infty \tag{4.16}$$

is now much more complicated than it was in the case $x \to -\infty$. The main difficulty is that the model solutions of the RH problems around the simple turning points $z = \pm i/\sqrt{2}$, are not bounded as $x \to +\infty$ (see [32]). To overcome this difficulty, P. Deift and X. Zhou replace the function $\check{Y}$ in (4.15) by the solution of the direct monodromy problem corresponding $u = i\sqrt{x/2}$, $w = 0$. The details of the corresponding rather skillful calculations can be found in the original paper [32]. In what follows we present a different idea of handling this problem.

Denoting, as in the previous section, the monodromy data corresponding to $\hat{u}(x,s)$ by $\hat{s}$, we obtain for the corresponding jump matrices $\hat{\hat{S}}$ the asymptotic equation

$$\hat{\hat{S}}^{-1}\hat{S} = I + O(x^{-\delta}), \quad 0 < \delta < \frac{3}{16}. \tag{4.17}$$

This equation is actually a weaker version of the Theorem 3.1 of the previous section (now we do not need uniform estimates for s). The corresponding function $\hat{\hat{\Psi}}$ has the same jump $i\sigma_1$ on the interval $[-i/\sqrt{2}, i/\sqrt{2}]$. This allows us to introduce the following correction to the equation (4.15):

$$\hat{Y}(z) = \hat{\Psi}(z)\hat{\hat{\Psi}}^{-1}(z). \tag{4.18}$$

The function $\hat{Y}(z)$ has again no jump on $[-i/\sqrt{2}, i/\sqrt{2}]$. The other jump matrices of the $\hat{Y}$-RH problem have the form

$$\hat{\hat{\Psi}}(z)\hat{\hat{S}}^{-1}\hat{S}\hat{\hat{\Psi}}^{-1}(z), \tag{4.19}$$

and are under control. In fact, in analogy with the results of Appendix, one can obtain a uniform (in z!) estimate of $\hat{\hat{\Psi}}(z)$ along the anti-Stokes lines depicted in Figure 5. Away from the turning points, $\hat{\hat{\Psi}}(z)$ is represented by the WKB-formulae that is basically

$$\check{Y}(z) \cdot e^{-itg(z)\sigma_3}.$$

Near the turning points, $\hat{\hat{\Psi}}(z)$ is described either by parabolic cylinder functions ($z = 0$) or by Airy functions ($z = \pm i/\sqrt{2}$) (for details see [41]). This means that again the jump matrices of the $\hat{Y}$-RH problem approach exponentially the identity away the turning points as $x \to +\infty$. Moreover, just like in the case $x \to -\infty$, the jump matrices (4.19) behave as (4.8) in the neighborhood of the double turning point $z = 0$. Unfortunately, the situation is not so good in the neighborhoods of the simple turning points $z = \pm i/\sqrt{2}$; the Airy-representation of $\hat{\hat{\Psi}}$ is growing as $x^{1/2}$ when $x \to +\infty$. This is the reflection of the general problem mentioned earlier with the model solutions near $z = \pm i/\sqrt{2}$. In the framework of the proposed modification of the Deift-Zhou scheme, this singularity can be cancelled as follows: Using the basic ODE (1), one can in principle, obtain as many terms in the formal asymptotic expansion for $u(x,s)$ starting with $\hat{u}(x,s)$, as one wishes. Proceeding with the **improved** asymptotic ansatz an **improved** version of the asymptotic formula (4.17) for the monodromy data $\hat{s}$ can be obtained. If one takes enough additional terms in $\hat{u}(x,s)$, one expects that the $x^{1/2}$-growth in (4.19) near $z = \pm i/\sqrt{2}$, will be cancelled by $\hat{\hat{S}}^{-1}\hat{S} - I$. The corresponding detailed calculations will be published elsewhere.

Comparing the method outlined above with the one presented in §3, we would like to suggest the following scheme as the most economic approach to the asymptotic analysis of the Painlevé equations. Using the Deift-Zhou steepest descent method, one deforms the given RH-problem to the model problem and then one obtains the master terms of the asymptotic. One then uses Kitaev's method to justify this asymptotics.

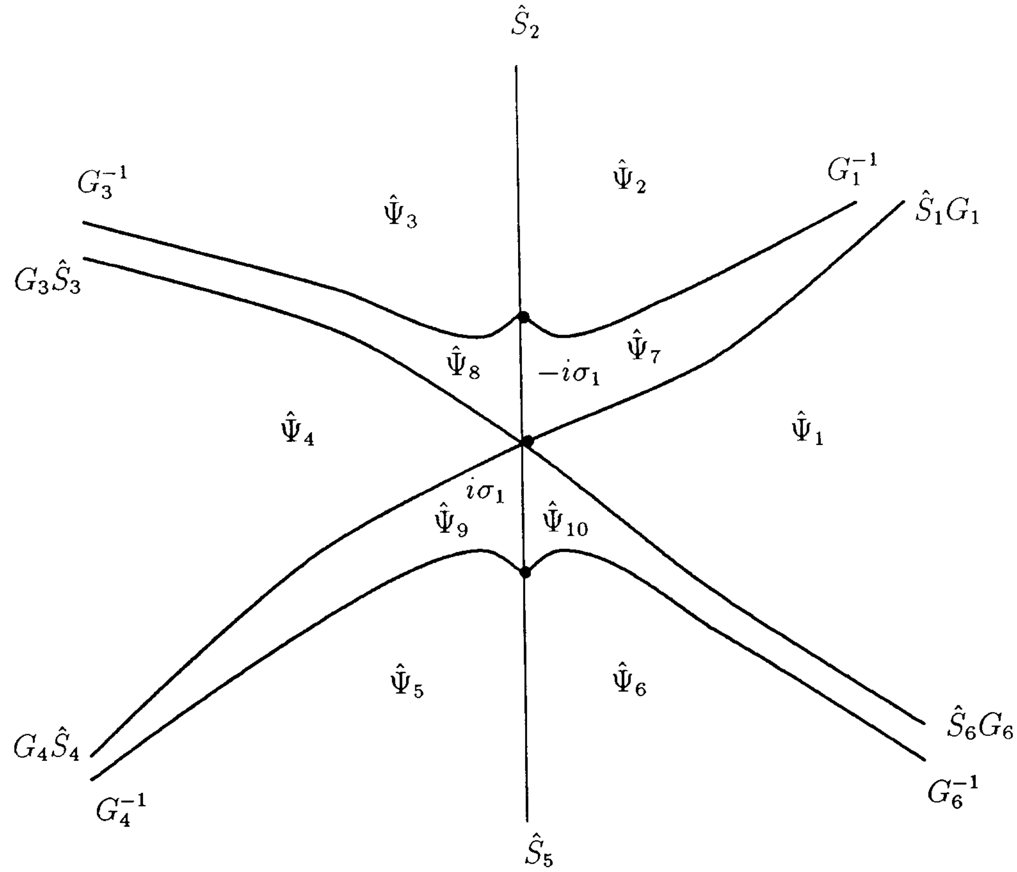

Figure 5

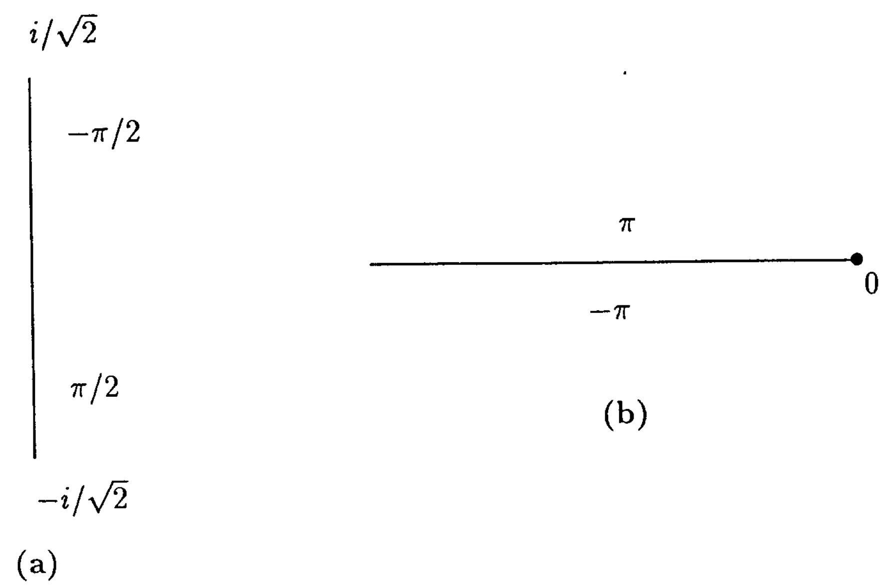

Figure 6

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Appendix: The Proof of Theorem 3.1

Our aim is to calculate asymptotically as $x \to +\infty$ the monodromy data $\hat{s}$ for the system (1.1) with $u = \hat{u}(x,s)$ and $w = \hat{w}(x,s)$, where $\hat{u}$ and $\hat{w}$ are given by equations (3.0). Performing the natural scaling, $\lambda \to z = \lambda x^{-1/2}$, one can rewrite the system (1.1) as

$$\frac{d\Psi}{dz} = tA_0(z,t)\Psi, \quad t = x^{3/2}, \tag{A.1}$$

where

$$A_0 = -(4iz^2 - c)\sigma_3 - z(i2\sqrt{2} - 4id)\sigma_2 - g\sigma_1,$$

$$c = 2^{5/4} i\rho\cos\theta\frac{1}{t^{1/2}} + i\sqrt{2}\rho^2\cos^2\theta\frac{1}{t}, \quad d = -2^{-1/4}\rho\cos\theta\frac{1}{t^{1/2}}, \tag{A.2}$$

$$g = -i2^{5/4}\rho\sin\theta\frac{1}{t^{1/2}} + \frac{i}{\sqrt{2}}\frac{1}{t} + 3i2^{-1/4}\rho^3\sin\theta\frac{1}{t^{3/2}} - i2^{-5/4}\rho\cos\theta\frac{1}{t^{3/2}}.$$

In equation (A.1), t is a large parameter and for the matrix $A_0(z,t)$ we have that

$$A_0(z,t) = O(1), \quad t \to \infty,$$

if z is fixed. In other words we get the situation which is typical in the WKB-method (see, for example [42]); moreover, since

$$\det A_0 \approx 16z^4 + 8z^2,$$

we expect three turning points: $z = 0$ (double turning point) and $z = \pm i/\sqrt{2}$ (simple turning points). In what follows we will proceed according to the following scheme (compare with [13,21]):

1. Calculation of WKB-solutions $\Psi_{\pm}^{WKB}(z)$ for $z \gtrless 0$, away from the double turning point $z=0$.

2. Calculation of the solution $\overset{T_p}{\Psi}(z)$ near the turning point $z=0$.

3. Calculation of the asymptotics of the matrices connecting the WKB-solutions and the canonical solutions Ψ_k, $k=1,4$:

$$C_{\pm} = [\Psi_{\pm}^{WKB}]^{-1}\Psi_{1,4}.$$

4. Calculation of the asymptotics of the matrices connecting the WKB-solutions and the turning-point solution:

$$N_{\pm} = \overset{T_p}{\Psi}{}^{-1}\ \Psi_{\pm}^{WKB}.$$

Having the asymptotic representations of the matrices $C_{\pm}$, $N_{\pm}$, we can immediately derive the desired asymptotic formula for the monodromy data $\hat{s}$ using,

$$\begin{pmatrix} \frac{1+\hat{s}^2}{1+|\hat{s}|^2} & -\frac{\hat{s}-\bar{\hat{s}}}{1+|\hat{s}|^2} \\ -\frac{\hat{s}-\bar{\hat{s}}}{1+|\hat{s}|^2} & \frac{1+\bar{\hat{s}}^2}{1+|\hat{s}|^2} \end{pmatrix} = \hat{S}_3^{-1}\hat{S}_2^{-1}\hat{S}_1^{-1} = C_-^{-1}N_-^{-1}N_+C_+. \tag{A.3}$$

Here we denote by $\hat{S}_k$ the Stokes matrices corresponding to the choice $u=\hat{u}(x,s)$ and $w=\hat{w}(x,s)$ for the coefficients of the system (1.1). We also take into account that the Stokes matrices are not affected by the rescaling $\lambda \to z$, since $x>0$. Technically, we are going to perform the same analysis that has already been done in [21] for proving Theorem 1.2, but with paying special attention to the fact that all estimates must be uniform for $s \epsilon D(s_0;\varepsilon)$.

We will now break up the proof of Theorem 3.1 into four lemmas; each lemma is associated with each step mentioned above.

Lemma A.1 (WKB-Lemma) Let

$$T_0(z) = (I + ia(z)\sigma_1)\frac{1}{\sqrt{1+a^2(z)}},$$

and

$$\mu(z) = i\left[16z^4 + 8z^2 - c^2 - \left(g + \frac{ia_z}{t(1+a^2)}\right)^2\right]^{1/2}, \tag{A.4}$$

where

$$a(z) = \frac{2\sqrt{2}z}{4z^2 + 4\sqrt{z^4 + z^2/2}}, \tag{A.5}$$

and the branches of the multivalued functions are chosen such that

$$\sqrt{z^4 + \frac{z^2}{2}} \geq 0, \quad \sqrt{1 + a(z)} > 0, \quad \forall z \epsilon \mathbb{R}, \tag{A.6}$$

$$\mu(z) = 4i\sqrt{z^4 + \frac{z^2}{2}} + o(1), \quad t \to +\infty, \quad |z| \geq 1.$$

Then, given $\gamma > 0$, $\delta > 0$, there exist positive constants $t_1 = t_1(s_0, \varepsilon, \gamma, \delta) \geq 1$, $Q_1 = Q_1(s_0, \varepsilon, \gamma)$, and solutions $\Psi_{\pm}^{WKB}(z)$ of equation (A.1), such that

$$\Psi_{\pm}^{WKB}(z) = T_0(z)\{I + R_{\pm}(z)\}e^{-t\int_0^z \mu(\eta)d\eta\sigma_3}, \quad z \gtrless 0 \tag{A.7}$$

where

$$\|R_{\pm}(z)\| \leq \frac{\gamma}{t^{1/2}|z|}Q_1 \leq \frac{1}{t^{\delta}}Q_1, \tag{A.8}$$

for all

$$t \geq t_1, \quad |z| \geq \gamma t^{-1/2+\delta}, \quad s \epsilon D(s_0, \varepsilon).$$

Proof of the Lemma A.1 Defining the function $\Phi(z)$ by

$$\Phi(z) = T_0^{-1}(z)\Psi(z),$$

one can rewrite equation (A.1) as

$$\frac{d\Phi(z)}{dz} = tA_1(z)\Phi(z), \tag{A.9}$$

where

$$A_1(z) = T_0^{-1}A_0T_0 - \frac{1}{t}T_0^{-1}\frac{dT_0}{dz} = -4i\sqrt{z^4 + \frac{z^2}{2}}\sigma_3 + \alpha(z)\sigma_3 + \beta(z)\sigma_2 + \omega(z)\sigma_1, \tag{A.10}$$

$$\begin{aligned} \alpha(z) &= \tfrac{1-a^2}{1+a^2}c + 8i\tfrac{za}{1+a^2}d, \\ \beta(z) &= 4iz\tfrac{1-a^2}{1+a^2}d - \tfrac{2a}{1+a^2}c, \\ \omega(z) &= -\tfrac{i}{t}\tfrac{a_z}{1+a^2} - g, \end{aligned} \tag{A.11}$$

and c, d, g are defined in (A.2). The advantage of (A.9) is that the zero-order term of the matrix A_1 is a diagonal matrix. Thus we can seek an asymptotic solution of (A.9) in the form (see []),

$$\Phi(z) \approx T(z)e^{-t\int_0^z \mu(\eta)d\eta\sigma_3}, \tag{A.12}$$

where

$$\mu(z) = i\sqrt{\det A_1(z)} = i[16z^4 + 8z^2 - c^2 - \omega^2]^{1/2}$$

and $T(z)$ is a matrix that diagonalizes the matrix A_1, $T^{-1}A_1T = -\mu\sigma_3$. We will take T in the form

$$T(z) = \frac{1}{8i\sqrt{z^4+\frac{z^2}{2}}}\left[\left(4i\sqrt{z^4+\frac{z^2}{2}} - \alpha + \mu\right) I - i\beta\sigma_1 + i\omega\sigma_2\right],$$

and will assume the convention (A.6) about the choice of the branches of the multivalued functions.

Before justifying the asymptotic equation (A.12), we need some uniform estimates for μ and T. To this end we note that

$$\mu(z) = 4i\sqrt{z^4+\frac{z^2}{2}}\left(1 + \frac{\mu_0}{z^4+\frac{z^2}{2}}\right)^{1/2},$$

where

$$\mu_0 = \frac{\sqrt{2}\,\rho^2}{4}\frac{}{t} + \frac{\rho}{\sqrt{t}}2^{-7/4}(c_1\cos\theta - g_1\sin\theta) + c_1^2 + g_1^2 - \frac{ig}{8t}\frac{a_z}{1+a^2} + \frac{a_z^2}{16t^2(1+a^2)^2}, \qquad (A.13)$$

$$c_1 = \sqrt{2}\rho^2\cos^2\theta\frac{1}{t},\quad g_1 = \frac{1}{\sqrt{2t}} + 3\cdot 2^{-1/4}\rho^3\sin\theta\frac{1}{t^{3/2}} - 2^{-5/4}\rho\cos\theta\frac{1}{t^{3/2}}.$$

Taking into account the continuity of the mapping

$$D(s_0,\varepsilon) \ni s \to \sqrt{\frac{1}{\pi}\log\frac{1+|s|^2}{2|Ims|}} = \rho,$$

the compactness of $D(s_0,\varepsilon)$, and the boundness of the function $a_z/1+a^2$, $z\epsilon\mathbb{R}$, we conclude that there exists a positive constant $C = C(s_0,\varepsilon)$ such that

$$|\mu_0| \le \frac{1}{t}C(s_0,\varepsilon),\quad \forall s\epsilon D(s_0,\varepsilon),\quad \forall t\ge 1,\quad \forall z\epsilon\mathbb{R}. \qquad (A.14)$$

This implies that under the same conditions on $s,t,z\neq 0$,

$$\left|\frac{\mu_0}{z^4+\frac{z^2}{2}}\right| \le \frac{2}{tz^2}C(s_0,\varepsilon).$$

Now, given $\gamma>0$, $\delta>0$ and supposing that

$$z\epsilon\mathbb{R},\quad |z|\ge\gamma t^{\delta-1/2},$$

$$t\ge t_0(s_0,\varepsilon,\gamma,\delta)\equiv\max\left\{\left(\frac{4C(s_0,\varepsilon)}{\gamma^2}\right)^{1/2\delta},1\right\}$$

we obtain

$$\left|\frac{\mu_0}{z^4+\frac{z^2}{2}}\right| \le \frac{1}{2}. \qquad (A.15)$$

This provides us with the following estimate for the function $\mu(z)$:

$$\mu(z) = 4i\sqrt{z^4 + \frac{z^2}{2}} + \mu_1(z) \tag{A.16}$$

where

$$|\mu_1| \leq \frac{1}{t\sqrt{z^4+\frac{z^2}{2}}}C(s_0,\varepsilon), \quad \forall t \geq t_0, \quad \forall s\epsilon D(s_0,\varepsilon), \quad \forall z\epsilon\mathbb{R}, \quad |z| \geq \gamma t^{\delta-1/2} \tag{A.17}$$

and we have absorbed the factor $4/(1+\sqrt{1/2})$ in $C(s_0,\varepsilon)$. Similar arguments give us similar estimates for the derivative μ_{0z} and for the function $\mu^{-1}(z)$:

$$|\mu_{0z}| \leq \frac{1}{t^{3/2}}C(s_0,\varepsilon), \quad \forall s\epsilon D(s_0,\varepsilon), \quad \forall t \geq 1, \quad \forall z\epsilon\mathbb{R} \tag{A.18}$$

and

$$\mu^{-1}(z) = \frac{1}{4i\sqrt{z^4+\frac{z^2}{2}}} + \mu_2,$$

$$|\mu_2| \leq \frac{1}{t(z^4+\frac{z^2}{2})^{3/2}}C(s_0,\varepsilon), \quad \forall s\epsilon D(s_0,\varepsilon), \quad \forall t \geq t_0, \quad \forall z\epsilon\mathbb{R}, \quad |z| \geq \gamma t^{\delta-1/2}. \tag{A.19}$$

In the estimates (A.14), (A.17), (A.18) and (A.19) we use the same notation $C(s_0,\varepsilon)$ for the relevant constants since we are not interested in their concrete numerical values. We will always follow this convention during the proof.

Consider now the matrix $T(z)$. Because of (A.16), (A.17),

$$T(z) = I + \frac{1}{8i\sqrt{z^4+\frac{z^2}{2}}}[(\mu_1-\alpha)I - i\beta\sigma_1 + i\omega\sigma_2] = I + T_1(z) + T_2(z) + T_3(z) \tag{A.20}$$

where T_1, T_2, T_3 are smooth functions of $z\epsilon\mathbb{R}$ satisfying the estimates

$$\begin{aligned}
\|T_1(z)\| &\leq \tfrac{1}{t^{1/2}}\tfrac{|z|}{\sqrt{z^4+\frac{z^2}{2}}}C(s_0,\varepsilon) \leq \tfrac{1}{t^{1/2}|z|}C(s_0,\varepsilon) \leq \tfrac{1}{\gamma t^{\delta}}C(s_0,\varepsilon) \\
\|T_2(z)\| &\leq \tfrac{1}{t^{1/2}}\tfrac{1}{\sqrt{z^4+\frac{z^2}{2}}}C(s_0,\varepsilon) \leq \tfrac{\sqrt{2}}{t^{1/2}|z|}C(s_0,\varepsilon) \leq \sqrt{2}\tfrac{C(s_0,\varepsilon)}{\gamma t^{\delta}} \\
\|T_3(z)\| &\leq \tfrac{1}{t}\tfrac{1}{z^4+\frac{z^2}{2}}C(s_0,\varepsilon) \leq \tfrac{2}{tz^2}C(s_0,\varepsilon) \leq 2\tfrac{C(s_0,\varepsilon)}{\gamma^2 t^{2\delta}}
\end{aligned} \tag{A.21}$$

for all $s\epsilon D(s_0,\varepsilon)$, $t \geq t_0(s_0,\varepsilon,\gamma,\delta)$, $z\epsilon\mathbb{R}$, $|z| \geq \gamma t^{\delta-1/2}$. Here, evaluating $|\alpha(z)|$, $|\beta(z)|$ and $|w(z)|$ we again use the continuity of the map $s \to \rho$ and the boundness of the functions $(1-a^2)/1+a^2$, $za/1+a^2$, $a/1+a^2$, and $a_z/1+a^2$. Assuming that

$$t \geq \max\left\{t_0, \left(6\sqrt{2}\frac{C}{\gamma}\right)^{1/\delta}, \left(\frac{12C}{\gamma^2}\right)^{1/2\delta}\right\} \equiv t_1(\varepsilon_0,\varepsilon,\gamma,\delta),$$

we obtain from (A.21) that

$$\|T_1(z) + T_2(z) + T_3(z)\| \leq \frac{1}{2}. \qquad (A.22)$$

This implies the following representation for $T^{-1}(z)$:

$$T^{-1}(z) = I + T_4(z), \quad \|T_4(z)\| \leq 2(\|T_1(z)\| + \|T_2(z)\| + \|T_3(z)\| \leq 1 \qquad (A.23)$$

$$\forall t \geq t_1, \quad s\epsilon D(s_0, \varepsilon), \quad |z| \geq \gamma t^{\delta - 1/2}, \quad z\epsilon\mathbb{R}.$$

In addition to $T(z)$ and $T^{-1}(z)$, we will also need $\frac{d}{dz}T(z)$. From the definition of $T(z)$ it follows that

$$\frac{d}{dz}T(z) = -\frac{1}{16i}\frac{4z^3 + z}{(z^4 + \frac{z^2}{2})^{3/2}}[(-\alpha + \mu_1)I - i\beta\sigma_1 + i\omega\sigma_2]$$

$$+\frac{1}{8i\sqrt{z^4 + \frac{z^2}{2}}}\left[\left(\frac{2i\mu_{0z}}{\sqrt{z^4 + \frac{z^2}{2}}} - \alpha_z - 8\mu_2(4z^3 + z + \mu_{0z})\right) I - i\beta_z\sigma_1 + i w_z\sigma_2\right].$$

This equation, the estimates (A.16)-(A.19), the explicit formulae for α, β, ω, the continuity of the map $s \rightarrow \rho$, and the boundness of the functions

$$\frac{1 - a^2}{1 + a^2}, \quad \frac{za}{1 + a^2}, \quad \frac{2a}{1 + a^2}, \quad \frac{a_z}{1 + a^2}$$

and of their derivatives, allow us to evaluate $\frac{dT}{dz}$ as

$$\left|\frac{dT}{dz}\right| \leq \frac{1}{t^{1/2}z^2}C(s_0, \varepsilon, \gamma), \quad \forall s\epsilon D(s_0, \varepsilon), \quad \forall t \geq t_1, \quad \forall z\epsilon\mathbb{R}, \quad |z| \geq \gamma t^{\delta - 1/2}. \qquad (A.24)$$

Combining (A.24) with (A.23) we conclude that there exist positive constants $t_1(s_0, \varepsilon, \gamma, \delta)$ and $C(s_0, \varepsilon, \gamma)$ such that

$$\left|T^{-1}(z)\frac{dT}{dz}(z)\right| \leq \frac{C(s_0, \varepsilon, \gamma)}{t^{1/2}z^2}, \quad \forall s\epsilon D(s_0, \varepsilon), \quad t \geq t_1, \quad z\epsilon\mathbb{R}, \quad |z| \geq \gamma t^{1/2 - \delta}. \qquad (A.25)$$

Now we are ready to justify the WKB-formula (A.12) and then to complete the proof of Lemma A.1. Putting

$$\Phi(z) = T(z)e^{-t\int_0^z \mu(\eta)d\eta}\chi(z),$$

one can rewrite (A.9) as integral equations of the Volterra type:

$$\chi_\pm(z) = I - \int_z^{\pm\infty} R(\eta)\chi_\pm(\eta)d\eta, \qquad (A.26)$$

where $R(z)$ is a smooth function defined by

$$R(z) = -e^{t\int_0^z \mu(\eta)d\eta\sigma_3}T^{-1}(z)\frac{dT(z)}{dz}e^{-t\int_0^z \mu(\eta)d\eta\sigma_3}.$$

Taking into account (A.25) and the fact that $\mu(z)$ is purely imaginary for $z\epsilon\mathbb{R}$, we have that

$$\|R(z)\| \le \frac{C(s_0,\varepsilon,\gamma)}{t^{1/2}z^2}, \quad \forall s\epsilon D(s_0,\varepsilon), \quad \forall t\ge t_1, \quad \forall z\epsilon\mathbb{R}, \quad |z|\ge \gamma t^{\delta-1/2}.$$

This inequality yields to the existence of the C^1-solutions $\chi_\pm$ of equation (A.26) that satisfies

$$\|\chi_\pm(z) - I\| \le e^{\pm\int_z^{\pm\infty}|R(\eta)|d\eta} - 1 \le e^{C/t^{1/2}|z|} - 1 \le \frac{\gamma}{t^{1/2}|z|}Q \le$$

$$\le \frac{1}{t^\delta}Q, \quad \forall t\ge t_1, \quad \forall z \gtrless 0, \quad |z|\ge \gamma t^{\delta-1/2}, \quad \forall s\epsilon D(s_0,\varepsilon), \tag{A.27}$$

where Q is a positive constant depending only on s_0,ε, and γ, i.e. $Q = Q(s_0,\varepsilon,\gamma)$. The inequality $z > 0 (z < 0)$ corresponds to $\chi_+(\chi_-)$. Coming back to the initial function $\Psi(z) = T_0(z)\Phi(z)$ and taking into account (A.20) and (A.21), we find that the inequality (A.27) is equivalent to the statement of the Lemma A.1 (again, for real z the exponent in (A.7) is purely imaginary!). This completes the proof.

Lemma A.2 (Turning-point Lemma) Let

$$\Psi_0(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} D_{-i\rho^2}(i\xi) & D_{i\rho^2-1}(\xi) \\ \dot{D}_{-i\rho^2}(i\xi) & \dot{D}_{i\rho^2-1}(\xi) \end{pmatrix}, \tag{A.28}$$

where

$$\xi = 2^{5/4}e^{-i\pi/4}t^{1/2}z, \tag{A.29}$$

$D_\nu(\xi)$ is the parabolic cylinder function, and $\dot{f}(\xi)$ is defined by

$$\dot{f}(\xi) = \frac{f_\xi - \frac{\xi}{2}f}{\rho}e^{i\theta-\frac{3i\pi}{4}}. \tag{A.30}$$

Then, given

$$\gamma_1 > 0, \quad 0 < \delta_1 < \frac{1}{6},$$

there exist a positive constant $Q_2 = Q_2(s_0,\varepsilon,\gamma_1)$ and solution $\overset{T_p}{\Psi}(z)$ of equation (A.1) such that

$$\overset{T_p}{\Psi}(z) = \{I + R_0(z)\}\Psi_0(z), \tag{A.31}$$

where

$$\|R_0(z)\| \le \frac{1}{t^{1/2-3\delta_1}}Q_2 \tag{A.32}$$

for all

$$t\ge 1, \quad |z|\le \gamma_1 t^{-1/2+\delta_1}, \quad \arg z = 0,\pi, \quad s\epsilon D(s_0,\varepsilon).$$

Proof of the Lemma A.2 In the region

$$\Omega = \{z : |z| \leq \gamma_1 t^{-1/2+\delta_1}\},$$

the matrix A_0 of the system (A.1) can be represented as

$$A_0 = c_0\sigma_3 - zi2\sqrt{2}\sigma_2 - g_0\sigma_1 + R(z), \tag{A.33}$$

where

$$c_0 = 2^{5/4}i\rho\cos\theta\frac{1}{t^{1/2}}, \quad g_0 = -i2^{5/4}\rho\sin\theta\frac{1}{t^{1/2}}$$

and the remainder R satisfies (again using the continuity of the map $s \to \rho$) the estimate

$$\|R(z)\| \leq \frac{1}{t^{1-2\delta_1}}C(s_0, \varepsilon, \gamma_1), \quad \forall s \epsilon D(s_0, \varepsilon), \quad t \geq 1, \quad z\epsilon\Omega. \tag{A.34}$$

This motivates us to consider the system

$$\frac{d\Psi_0}{dz} = t[c_0\sigma_3 - zi2\sqrt{2}\sigma_2 - g_0\sigma_1]\Psi_0 \tag{A.35}$$

as a model system for $z\epsilon\Omega$.

The system (A.35) has an explicit solution (compare with [13,21]) that is given by (A.28)-(A.30). Note that since $s_0 \neq -i$ and $0 < \varepsilon < \min\{|Ims_0|, |s_0+i|\}$, it follows that the continuous on $D(s_0, \varepsilon)$ function $\rho(s)$ satisfies the inequality

$$\rho(s) > 0, \quad \forall s\epsilon D(s_0, \varepsilon),$$

i.e. $inf\rho = \min\rho > 0$ on $D(s_0, \varepsilon)$. This means that ρ is not only uniformly bounded from above, but it is uniformly different than zero as well. In other words, there exist positive constants

$$\rho_1(s_0, \varepsilon), \quad \rho_2(s_0, \varepsilon)$$

such that

$$0 < \rho_1 \leq \rho \leq \rho_2, \quad \forall s\epsilon D(s_0, \varepsilon). \tag{A.36}$$

The functions

$$D_{-i\rho^2}(i\xi), \quad \dot{D}_{-i\rho^2}(i\xi), \quad D_{i\rho^2-1}(\xi), \quad \dot{D}_{i\rho^2-1}(\xi)$$

are bounded if $\arg\xi = -\pi/4, 3\pi/4$ uniformly with respect of $(\rho, \theta)\epsilon[\rho_1, \rho_2] \times [0, 2\pi)$ (see the known contour integrals or the known asymptotic expansions for the parabolic cylinder functions [43]). This yields immediately to estimate

$$\|\Psi_0(z)\| \leq C \equiv C(s_0, \varepsilon), \quad \forall s\epsilon D(s_0, \varepsilon), \quad z\epsilon\mathbb{R}, \quad \arg z = 0, \pi. \tag{A.37}$$

Moreover, since

$$\det\Psi_0 = 2e^{\pi\rho^2/2}\frac{1}{\rho}e^{i\theta - i\pi/4}$$

we conclude that the same estimate (A.37) is valued for $\Psi_0^{-1}(z)$.

Now, we are ready to write down an integral equation. Letting

$$\Psi(z) = \chi(z)\Psi_0(z)$$

we obtain

$$\chi(z) = I + t\int_0^z \Psi_0(z)\Psi_0^{-1}(\eta)R(\eta)\chi(\eta)\Psi_0(\eta)\Psi_0^{-1}(z)d\eta. \qquad (A.38)$$

This is an equation of Volterra type with regular kernel. Thus, it has a unique solution $\chi(z)$ in the whole domain Ω, and this solution is analytic with respect to $z(\Psi_0(z),\ \Psi_0^{-1}(z),\ R(z)$ are analytic on Ω). Suppose that

$$z\epsilon\Omega \cap \mathbb{R}(\arg z = 0, \pi).$$

Then using (A.34) and (A.37), it follows that

$$\sigma(z) \equiv \pm t\int_0^z \|\Psi_0(z)\Psi_0^{-1}(\eta)\|\|R(\eta)\|\|\Psi_0(\eta)\Psi_0^{-1}(z)\| \det \le$$

$$\le \pm C^4(s_0,\varepsilon)t\int_0^z \|R(\eta)\|d\eta \le \frac{C(s_0,\varepsilon,\gamma_1)}{t^{1/2-3\delta_1}} \qquad (A.39)$$

for all $s\epsilon D(s_0,\epsilon)$, $t \ge 1$, $z\epsilon\Omega\cap\mathbb{R}$; here "+" corresponds to $\arg z = 0$ while "−" corresponds to $\arg z = \pi$. Recalling that $0 < \delta_1 < 1/6$, equation (A.39) implies the following estimate for the solution $\chi(z)$:

$$\|\chi(z) - I\| \le e^{\sigma(z)} - 1 \le \frac{Q_2}{t^{1/2-3\delta_1}}, \quad \forall s\epsilon D(s_0,\varepsilon), \quad t \ge 1, \quad |z| \le \gamma_1 t^{-1/2+\delta_1}, \quad \arg z = 0, \pi$$

where $Q_2 = Q_2(s_0,\varepsilon,\gamma_1) > 0$ is a constant depending only on s_0, ε, and γ_1. This estimate completes the proof of Lemma A.2.

Lemma A.3 ($C_\pm$-Lemma) Let $\Psi_k(z)$, $k = 1, ..., 6$ be the canonical solutions of the system (A.1) and let

$$C_\pm = [\Psi_\pm^{WKB}]^{-1}\Psi_{1,4}$$

be matrices connecting these canonical solutions with the WKB-solutions (these matrices are independent of z). Then, given $0 < \delta < 1/6$ there exist positive constants $t_3 = t_3(s_0,\varepsilon,\delta) \ge 1$, $Q_3 = Q_3(s_0,\varepsilon)$ such that

$$C_\pm = \exp\{\pm(-i\frac{\sqrt{2}}{3}t + b)\sigma_3\}(1 + R_1^\pm), \qquad (A.40)$$

where

$$\|R_1^\pm\| \le \frac{1}{t^{\delta'}}Q_3 \le \frac{1}{2}, \quad \delta' = \min\{\delta, 1-6\delta\}, \quad \forall t \ge t_3, \quad \forall s\epsilon D(s_0,\varepsilon) \qquad (A.41)$$

and

$$b = \frac{i}{2}\rho^2 - \frac{i\rho^2}{2}\log\frac{\sqrt{2}}{16}\frac{\rho^2}{t}.$$

Proof of the Lemma A.3 The relations

$$\Psi_4(z) = \sigma_2\Psi_1(-z)\sigma_2, \quad \Psi_-^{WKB}(z) = \sigma_2\Psi_+^{WKB}(-z)\sigma_2,$$

imply

$$C_- = \sigma_2 C_+ \sigma_2.$$

Hence, it is enough to prove (A.40) for C_+ only. In fact, C_+ can be determined by equation,

$$C_+ = \lim_{z\to+\infty} \exp\left\{\left[t\int_0^z \mu(\eta)d\eta - t(\frac{4i}{3}z^3 + iz)\right]\sigma_3\right\}. \tag{A.42}$$

This means that we have to evaluate asymptotically the integral $\int_0^z \mu(\eta)d\eta$.

Assuming that $\tilde{t}_0(s_0,\varepsilon,\delta) = t_0(s_0,\varepsilon,1,\delta)$ where $t_0(s_0,\varepsilon,\gamma,\delta)$ is the same as in the proof of Lemma A.1, and using the estimates (A.14), (A.15) we obtain

$$\mu(z) = 4i\sqrt{z^4 + \frac{z^2}{2}} + 2i\frac{\mu_0(z)}{\sqrt{z^4+\frac{z^2}{2}}} + \mu_3(z), \tag{A.43}$$

where

$$|\mu_3| \le \frac{4}{3+2\sqrt{2}}\frac{|\mu_0|^2}{(z^4+\frac{z^2}{2})^{3/2}} \le \frac{C(s_0,\varepsilon)}{t^2z^3}, \quad \forall s\epsilon D(s_0,\varepsilon), \quad t\ge\tilde{t}_0, \quad z\ge t^{-1/2+\delta}.$$

Noting (see (A.13)) that

$$\left|\mu_0 - \frac{\sqrt{2}}{4}\frac{\rho^2}{t}\right| \le \frac{1}{t^{3/2}}C(s_0,\varepsilon), \quad s\epsilon D(s_0,\varepsilon), \quad t\ge 1, \tag{A.44}$$

equation (A.43) yields

$$\int_{t^{-1/2+\delta}}^z \mu(\eta)d\eta = 4i\int_{t^{-1/2+\delta}}^z\sqrt{z^4+\frac{z^2}{2}}dz + i\frac{\sqrt{2}}{2}\frac{\rho^2}{t}\int_{t^{-1/2+\delta}}^z\frac{dz}{\sqrt{z^4+\frac{z^2}{2}}} + I_0(z),$$

$$|I_0| \le \frac{1}{t^{1+\delta}}C(s_0,\varepsilon), \quad \forall s\epsilon D(s_0,\varepsilon), \quad t\ge\tilde{t}_0, \quad z\ge t^{-1/2+\delta}.$$

Evaluating the integrals in the RHS of this equation, one finds

$$\int_{t^{-1/2+\delta}}^z \mu(\eta)d\eta = \frac{4i}{3}(z^2+\frac{1}{2})^{3/2} + \frac{i}{2}\frac{\rho^2}{t}\log\frac{\sqrt{z^2+1/2}-1/\sqrt{2}}{\sqrt{z^2+1/2}+1/\sqrt{2}} - \frac{i}{3}\sqrt{2} - i\sqrt{2}t^{-1+2\delta} - \frac{i}{2}\frac{\rho^2}{t}\log\frac{t^{-1+2\delta}}{2} + I_1(z) \tag{A.45}$$

where

$$|I_1| \le C(s_0,\varepsilon)\left(\frac{1}{t^{1+\delta}} + \frac{1}{t^{2-4\delta}}\right)$$

$$\forall s \epsilon D(s_0,\varepsilon), \quad t \ge \tilde{t}_3(s_0,\varepsilon,\delta) = \max\left\{\tilde{t}_0(s_0,\varepsilon,\delta), 4^{1/1-2\delta}\right\}, \quad z \ge t^{-1/2+\delta}.$$

Consider now the complementary integral, $\int_0^{t^{-1/2+\delta}} \mu(\eta)d\eta$. Note that for small z the function $\mu(z)$ can be represented as follows:

$$\mu(z) = 4i\sqrt{\frac{z^2}{2} + \frac{\sqrt{2}}{4}\frac{\rho^2}{t}}(1+\mu_3(z))^{1/2},$$

where

$$|\mu_3| \le C(s_0,\varepsilon)\left(\frac{1}{t^{1/2}} + \frac{1}{t^{1-4\delta}}\right), \quad s\epsilon D(s_0,\varepsilon), \quad 0 \le z \le t^{-1/2+\delta}, \quad t \ge 1.$$

This means that there exists $\tilde{\tilde{t}}_3(s_0,\varepsilon,\delta) \ge \tilde{t}_3(s_0,\varepsilon,\delta)$, such that

$$|\mu_3| \le \frac{1}{2}, \quad \forall s\epsilon D(s_0,\varepsilon), \quad 0 \le z \le t^{-1/2+\delta}, \quad t \ge \tilde{\tilde{t}}_3.$$

This implies the equation

$$\mu(z) = 4i\sqrt{\frac{z^2}{2} + \frac{\sqrt{2}}{4}\frac{\rho^2}{t}} + \mu_4(z), \tag{A.46}$$

where

$$|\mu_4| \le C(s_0,\varepsilon)\left(\frac{1}{t^{1-\delta}} + \frac{1}{t^{3/2-5\delta}}\right), \quad \forall s\epsilon D(s_0,\varepsilon), \quad t \ge \tilde{\tilde{t}}_3, \quad 0 \le z \le t^{-1/2+\delta}.$$

Equation (A.46) yields the formula

$$\int_0^{t^{-1/2+\delta}} \mu(\eta)d\eta = \frac{4i}{\sqrt{2}}\int_0^{t^{-1/2+\delta}} \sqrt{\eta^2 + \frac{\sqrt{2}}{2}\frac{\rho^2}{t}}d\eta + I_2,$$

where

$$|I_2| \le C(s_0,\varepsilon)\left(\frac{1}{t^{3/2-2\delta}} + \frac{1}{t^{2-6\delta}}\right), \quad \forall s\epsilon D(s_0,\varepsilon), \quad t \ge \tilde{\tilde{t}}_3.$$

Evaluating the integral of the RHS of the above equation, one finds

$$\int_0^{t^{-1/2+\delta}} \mu(\eta)d\eta = i\sqrt{2}t^{-1+2\delta} + \frac{i\rho^2}{2t} + \frac{i\rho^2}{2t}\log t^{-1+2\delta} - \frac{i\rho^2}{2t}\log\frac{\sqrt{2}}{8}\frac{\rho^2}{t} + I_3, \tag{A.47}$$

where

$$|I_3| \le C(s_0,\varepsilon)\left(\frac{1}{t^{1+2\delta}} + \frac{1}{t^{3/2-2\delta}} + \frac{1}{t^{2-6\delta}}\right), \quad \forall s\epsilon D(s_0,\varepsilon), \quad t \ge t_3(s_0,\varepsilon,\delta)$$

and $t_3(s_0,\varepsilon,\delta)$ is determined by the conditions

$$t_3 \geq \tilde{\tilde{t}}_3, \quad \left|\frac{\sqrt{2}}{2}\frac{\rho^2}{t_3^{2\delta}}\right| \leq \frac{1}{2}, \quad \forall s \epsilon D(s_0, \varepsilon).$$

Summing up equations (A.45) and (A.47) we find the desired estimate for $\int_0^z \mu(\eta)d\eta$:

$$\int_0^z \mu(\eta)d\eta = \frac{4i}{3}(z^2 + \frac{1}{2})^{3/2} - \frac{i}{3}\sqrt{2} + \frac{i}{2}\frac{\rho^2}{t}\log\frac{\sqrt{z^2 + 1/2} - 1/\sqrt{2}}{\sqrt{z^2 + 1/2} + 1/\sqrt{2}} + \frac{1}{t}b + I_4(z),$$

where

$$|I_4| \leq C(s_0, \delta)\left(\frac{1}{t^{1+\delta}} + \frac{1}{t^{2-6\delta}}\right), \quad \forall s \epsilon D(s_0, \delta), \quad t \geq t_3(s_0, \varepsilon, \delta), \quad z \geq t^{-1/2+\delta} \qquad (A.48)$$

and

$$b = \frac{i\rho^2}{2} - \frac{i\rho^2}{2}\log\frac{\sqrt{2}}{16}\frac{\rho^2}{t}.$$

Substituting this formula into equation (A.42), we obtain the estimate (A.40) for C_+ (note that the limit in the rhs of (A.42) exists aprior). Increasing if needed the bound t_3 we can always achieve $\|R_1^{\pm}\| \leq 1/2$. This completes the proof of Lemma A.3.

Lemma A.4 ($N_{\pm}$-Lemma) Let

$$N_{\pm} = \overset{T_p}{\Psi}{}^{-1} \Psi_{\pm}^{WKB}, \qquad (A.49)$$

be the matrices connecting the turning-point and the WKB-solutions (these matrices are independent of z). Then, given $0 < \delta < 1/6$ there exist positive constants $t_4 = t_4(s_0, \varepsilon, \delta)$, $Q_4 = Q_4(s_0, \varepsilon)$ such that

$$N_+ = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-\pi\rho^2/2} & 0 \\ 0 & \frac{\rho}{\sqrt{2}}e^{-i\theta + \pi i/4} \end{pmatrix} N_0(I + R_2^+), \qquad (A.50)$$

and

$$N_-^{-1} = N_0 \begin{pmatrix} \frac{2\sqrt{\pi}}{\rho\Gamma(i\rho^2)}e^{i\theta - \frac{\pi\rho^2}{2} + \frac{3i\pi}{4}}, & -\frac{\sqrt{2}}{\rho}e^{i\theta - \pi\rho^2 - \frac{3i\pi}{4}} \\ \sqrt{2}ie^{\frac{-\pi\rho^2}{2}}, & -\frac{2\sqrt{\pi}}{\rho^2}\frac{1}{\Gamma(-i\rho^2)} \end{pmatrix}(I + R_2^-), \qquad (A.51)$$

where

$$\|R_2^{\pm}\| \leq \frac{1}{t^{\delta''}}Q_4, \quad \delta'' = \min\{\delta, \frac{1}{2} - 3\delta\}, \quad \forall t \geq t_4, \quad \forall s \epsilon D(s_0, \varepsilon), \qquad (A.52)$$

and

$$N_0 = \exp\left\{\sigma_3\left(\frac{i\rho^2}{2}\log 8\sqrt{2}t + \frac{\pi}{4}\rho^2 - b\right)\right\}.$$

Proof of the Lemma A.4 Let γ, δ and γ_1, δ_1 be the parameters from Lemma A.1 and Lemma A.2 respectively. Assume that

$$\gamma = \frac{1}{2}, \quad \gamma_1 = 1, \quad \delta = \delta_1, \quad 0 < \delta < \frac{1}{6}$$

and

$$\frac{1}{2}t^{-\frac{1}{2}+\delta} \le |z| \le t^{-\frac{1}{2}+\delta}, \tag{A.53}$$

$$t \ge t_1(s_0, \varepsilon, \frac{1}{2}, \delta).$$

Then we can use in equation (A.49) the asymptotic representations (A.7-8) and (A.31-32) simultaneously. To proceed with the corresponding calculations we need first to evaluate the integral $\int_0^z \mu(\eta)d\eta$ under the conditions (A.53).

Let $z > 0$ ($\arg z = 0$) and $\tilde{t}_3(s_0, \varepsilon, \delta)$ is the same as in the Proof of Lemma A.3. Repeating literally the derivation of (A.47) we come to the equation

$$t\int_0^z \mu(\eta)d\eta = i\sqrt{2}tz^2 + i\rho^2 \log z + \frac{i}{2}\rho^2 - \frac{i\rho^2}{2}\log\frac{\sqrt{2}}{8}\frac{\rho^2}{t} + tI_3, \tag{A.54}$$

where

$$|I_3| \le C(s_0,\varepsilon)\left(\frac{1}{t^{1+2\delta}} + \frac{1}{t^{3/2-2\delta}} + \frac{1}{t^{2-6\delta}}\right), \quad \forall s \epsilon D(s_0,\varepsilon), \quad \frac{1}{2}t^{-1/2+\delta} \le z \le t^{-1/2+\delta}, \quad t \ge \tilde{t}_4(s_0,\varepsilon,\delta),$$

where $\tilde{t}_4(s_0,\varepsilon,\delta)$ is determined by conditions

$$\tilde{t}_4 \ge \max\{t_1(s_0,\varepsilon,\frac{1}{2},\delta), \ \tilde{t}_3(s_0,\varepsilon,\delta)\}, \quad \left|2\sqrt{2}\frac{\rho^2}{t_4^{2\delta}}\right| \le \frac{1}{2}, \quad \forall s \epsilon D(s_0,\varepsilon).$$

Introducing the variable ξ (see (A.29)), we can rewrite (A.54) as

$$t\int_0^z \mu(\eta)d\eta = -\frac{\xi^2}{4} + i\rho^2\log\xi - \frac{i\rho^2}{2}\log 8\sqrt{2}t - \frac{\pi}{4}\rho^2 + b + I_5, \tag{A.55}$$

where

$$|I_5| \le C(s_0,\varepsilon)\left(\frac{1}{t^{2\delta}} + \frac{1}{t^{1/2-2\delta}} + \frac{1}{t^{1-6\delta}}\right),$$

$$\forall s \epsilon D(s_0,\varepsilon), \quad \frac{1}{2}t^{-1/2+\delta} \le z \le t^{-1/2+\delta}, \quad t \ge \tilde{t}_4$$

and b is the same as in (A.40).

The asymptotic representation of $t\int_0^z \mu(\eta)d\eta$ for $\arg z = \pi$, can be obtained immediately from (A.55) using the symmetry $\mu(-z) = \mu(z)$.

Equation (A.55) means that the asymptotics (A.7) of Ψ_+^{WKB}, in the positive part of transition domain (A.53), can be rewritten as

$$\Psi_+^{WKB} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}[I + \tilde{R}_+]\exp\left\{\left(\frac{\xi^2}{4} - i\rho^2\log\xi + \frac{i\rho^2}{2}\log 8\sqrt{2}t + \frac{\pi}{4}\rho^2 - b\right)\sigma_3\right\},$$

$$\|\tilde{R}_+\| \le \tilde{Q}_1(s_0,\varepsilon)\left(\frac{1}{t^\delta} + \frac{1}{t^{1-6\delta}}\right), \tag{A.56}$$

$$\forall s \epsilon D(s_0,\varepsilon),\ \frac{1}{2}t^{-1/2+\delta} \le z \le t^{-1/2+\delta},\ \ t \ge \tilde{\tilde{t}}_4(s_0,\varepsilon,\delta),$$

where $\tilde{\tilde{t}}_4(s_0,\varepsilon,\delta)$ is determined by the conditions

$$\tilde{\tilde{t}}_4 \ge \max\left\{\tilde{t}_4, (2\sqrt{2})^{\frac{2}{1-2\delta}}\right\}$$

(an additional bound comes from the estimate of $T_0(z)$). At the same time, taking into account the known asymptotic formulae for the parabolic cylinder function, one can rewrite formulae (A.31) in the positive part of domain (A.53) as

$$\overset{T_p}{\Psi} = (I + \tilde{R}_0)\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\exp\left\{\left(\frac{\xi^2}{4} - i\rho^2\log\xi\right)\sigma_3\right\}\begin{pmatrix} e^{-\frac{\pi\rho^2}{2}} & 0 \\ 0 & \rho e^{-i\theta - \frac{i\pi}{4}} \end{pmatrix}^{-1}, \tag{A.57}$$

$$\|\tilde{R}_0\| \le \tilde{Q}_2(s_0,\varepsilon)\left(\frac{1}{t^{1/2-3\delta}} + \frac{1}{t^\delta}\right),\ \ \forall s\epsilon D(s_0,\varepsilon),\ \ \frac{1}{2}t^{-1/2+\delta} \le z \le t^{-1/2+\delta},\ \ t \ge t_4(s_0,\varepsilon,\delta).$$

For this estimate we have used the uniformity of the asymptotics of the involved parabolic cylinder functions with respect to $(\rho,\theta)\ \epsilon[\rho_1,\rho_2] \times [0,2\pi]$. The bound $t_4 \ge \tilde{\tilde{t}}_4$ is determined by the condition $\|\tilde{R}_0\| < 1/2$.

Substituting (A.56) and (A.57) into (A.49) and taking into account that all relevant matrices involved are uniformly bounded for $s\epsilon D(s_0,\varepsilon)$, $t \ge t_4$, we find equation (A.50) for the connection matrix N_+. Equation (A.51) is obtained absolutely in a similar way. This completes the proof of Lemma A.4.

Using the above Lemmas it is now straightforward to prove Theorem 3.1. In fact, substituting (A.40), (A.50) and (A.51) into equation (A.3) and again taking into account that relevant matrices are uniformly bounded for $s\epsilon D(s_0,\varepsilon)$, $t \ge \max\{t_3,t_4\}$, we obtain that

$$\begin{pmatrix} \frac{1+\hat{s}^2}{1+|\hat{s}|^2} & -\frac{\hat{s}-\bar{\hat{s}}}{1+|\hat{s}|^2} \\ -\frac{\hat{s}-\bar{\hat{s}}}{1+|\hat{s}|^2} & \frac{1+\bar{\hat{s}}^2}{1+|\hat{s}|^2} \end{pmatrix} = \begin{pmatrix} \frac{1+s^2}{1+|s|^2} & -\frac{s-\bar{s}}{1+|s|^2} \\ -\frac{s-\bar{s}}{1+|s|^2} & \frac{1+\bar{s}^2}{1+|s|^2} \end{pmatrix} + \tilde{R}(s,t), \tag{A.58}$$

where

$$\|\tilde{R}\| \le C(s_0,\varepsilon)\frac{1}{t^{\delta''}},\ \ \delta'' = \min\{\delta, \frac{1}{2} - 3\delta\},$$

$$\forall s\epsilon D(s_0,\varepsilon),\ \ t \ge t_5(s_0,\varepsilon,\delta) = \max\{t_3,t_4\}. \tag{A.59}$$

In particular, this means

$$\frac{\hat{s}-\bar{\hat{s}}}{1+|\hat{s}|^2} = \frac{s-\bar{s}}{1+|s|^2} - \tilde{R}_{21}.$$

Taking into account that there exist positive constants $m(s_0,\varepsilon)$, $M(s_0,\varepsilon)$ such that

$$0 < m \leq \left|\frac{s-\bar{s}}{1+|s|^2}\right| \leq M, \quad \forall s\epsilon D(s_0,\varepsilon),$$

one can conclude the existence of positive constants $\tilde{m}(s_0,\varepsilon)$, $\tilde{M}(s_0,\varepsilon)$, and $t_6(s_0,\varepsilon,\delta) \geq t_5(s_0,\varepsilon,\delta)$ such that

$$0 < \tilde{m} \leq \left|\frac{\hat{s}-\bar{\hat{s}}}{1+|\hat{s}|^2}\right| \leq \tilde{M}, \quad \forall s\epsilon D(s_0,\varepsilon), \quad t \geq t_6. \tag{A.60}$$

This inequality in turn implies the uniform boundness of $\hat{s}(x,s)$ as $s\epsilon D(s_0,\varepsilon)$, $x^{3/2} \geq t_6(s_0,\varepsilon,\delta)$. Hence, all matrices $\hat{S}_k^{\pm 1}$ are uniformly bounded as $s\epsilon D(s_0,\varepsilon)$, $t \geq t_6$. Obviously the same is true about the matrices $S_k^{\pm 1}$ constructed by the parameter s. Therefore, one can rewrite (A.58) as

$$\hat{S}_1^{-1}S_1 = \hat{S}_2\hat{S}_3S_3^{-1}S_2^{-1} + R'(s,t)$$

where R' satisfies the same estimates (A.59) as $\tilde{R}$. The last matrix equation leads in particular the scalar formula

$$(\bar{s}-\bar{\hat{s}})\frac{\hat{s}-\bar{\hat{s}}}{1+|\hat{s}|^2} = -R'_{11}.$$

Taking into account the inequality (A.60) we conclude then that

$$|\hat{s}-s| \leq \frac{1}{t^{\delta''}}C, \quad \forall s\epsilon D(s_0,\varepsilon), \quad t \geq t_6,$$

for some positive constant $C = C(s_0,\varepsilon)$. Returning to the initial variable $x = t^{2/3}$, and subjecting the exponent δ to the additional restriction

$$\delta < \frac{1}{8} \Rightarrow \delta'' = \delta,$$

we come to the statement of the Theorem 3.1. This completes its proof.