ANNALES
POLONICI MATHEMATICI
XLI (1983)

On the asymptotic behavior of perturbed linear systems

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Abstract. In this paper we shall study the existence and the asymptotic behavior of the solutions of the linear system $dy(t)/dt = A(t)y(t)$ and these of the nonlinear system $dx(t)/dt = A(t)x(t)+f(t, x(t))$. Several new results are obtained via the techniques introduced by Brauer and Wong [1] and Hallam and Heidel [4]. Theorem 2.1 is an improvement of a theorem of Hallam and Heidel [4] while Theorem 2.2 is related to a theorem of Brauer and Wong [1].

I. Introduction. In this section we shall be concerned with asymptotic relationships between the solutions of the system

$$\frac{dy(t)}{dt} = A(t)y(t), \quad t \geq 0, \tag{1.1}$$

and those of the nonlinear system

$$\frac{dx(t)}{dt} = A(t)x(t)+f(t, x(t)), \quad t \geq 0. \tag{1.2}$$

where x, y and f are n-vectors in R^n, $A(t)$ is a continuous $n \times n$ matrix in $R^{n\times n}$ for $t \geq 0$, and $f(t, x)$ is a continuous function of t and x for $t \geq 0$ and $\|x\| < \infty$. Here $\|\cdot\|$ denotes any appropriate vector (or matrix) norm. Denote by $\Phi(t)$ the fundamental matrix of (1.1) with initial condition $\Phi(0) = I$ (the identity $n \times n$ matrix). Throughout this paper we shall always call the following three conditions "Assumption A":

(i) $\alpha(t)$ and $v(t)$ are positive continuous functions on $J = [0, \infty)$;

(ii) $\Delta(t)$ is a nonsingular continuous $n \times n$ matrix on J;

and

(iii) $\omega(t, s)$ is nonnegative, continuous on $J \times J$, and is non-decreasing in s for $s > 0$ and fixed $t \in J$.

There are two types of problems to be studied here. First, suppose that a solution $y(t)$ of (1.1) is given. We are interested in knowing if there exists a solution $x(t)$ of (1.2) such that $\|\Delta(t)(x(t)-y(t))\| = O(\alpha(t))$ as $t \to \infty$ for some

* This paper was supported by the U.S. Army Research Office Grant DAAG29-78-G-0042 at Research Triangle Park, N.C.

given function $\alpha(t)$ and continuous $n \times n$ matrix $\Delta(t)$. Many papers have been devoted to a discussion of this problem (see [1], [3], [4], etc.). For example, Hallam and Heidel [4] obtained the following theorem, namely,

THEOREM A (Hallam and Heidel [4]). *Suppose that there exist $\alpha(t)$, $\Delta(t)$, and $\omega(t, s)$ satisfying the following conditions*:

(i) *Assumption* A;

(ii) $\|\Delta(t)\Phi(t)\| \leqslant \alpha(t)$;

(iii) $\|\Phi^{-1}(t)f(t, x)\| \leqslant \omega(t, \|\Delta(t)x\|\alpha^{-1}(t))$; *and*

(iv) *equation $dr/dt = \omega(t, r)$ has a positive solution which is bounded on the interval $t \geqslant t_0$.*

Given any solution $y(t) = \Phi(t)c$ of (1.1) *with $|c|$ sufficiently small, there exists a solution $x(t)$ of* (1.2) *such that*

$$\|\Delta(t)(x(t)-y(t))\| = o(\alpha(t)) \quad \text{as } t \to \infty.$$

Theorem 2.1 below deals with this type of problem. Our result (Theorem 2.1) is an improvement of Theorem A because we replace the condition $\|\Delta(t)\Phi(t)\| \leqslant \alpha(t)$ by a more general inequality and we do not restrict the initial condition of a given solution $y(t)$, to be sufficiently small.

Second we shall deal with the converse problem. Many papers have been devoted to a discussion of this problem (see [1], [3], etc.). Theorem 2.2 below deals with this type of problem. Our result is related to a theorem of Brauer and Wong [1]. In the last section we shall apply Theorem 2.1 to a given equation to obtain a criterion which is an improvement of a criterion from Theorem A.

II. Theorems. Before stating and proving our main theorems let us first study some properties of $\omega(t, r)$ in Theorem A.

LEMMA 2.1. *Suppose that $\omega(t, r)$ satisfies Assumption* A. *Then the following three statements are equivalent*:

(1) *Given any number $r_0 > 0$ there exists a $t_0 \geqslant 0$ and a solution $r(t, t_0, r_0)$ of the equation $dr/dt = \omega(t, r)$ such that*

$$\lim_{t \to \infty} r(t, t_0, r_0) = \infty.$$

(2) $\int^{\infty} \omega(t, \lambda)\,dt < \infty$ *for all λ satisfying $0 \leqslant \lambda < \infty$.*

(3) $\lim_{t \to \infty} \int_t^{\infty} \omega(s, \lambda)\,ds = 0$ *for all λ satisfying $0 \leqslant \lambda < \infty$.*

Proof. The equivalent relationship between (1) and (2) was proved in [4]. It is clear that (2) and (3) are equivalent. This proves the lemma.

Now we shall prove the following theorem via a technique introduced by Hallam and Heidel [4].

THEOREM 2.1. *Let $y(t)$ be an arbitrary nontrivial solution of* (1.1) *and $\beta(t)$*

be a positive continuous function on J. Suppose that there exist $\alpha(t)$, $v(t)\Delta(t)$, *and* $\omega(t, s)$ *satisfying the following four conditions*:

(i) *Assumption* A;

(ii) *for an arbitrary positive constant* $\varepsilon < 1$, *there exists* $t_0 > 0$ *such that*

$$\|\Delta(t)y(t)\| \leqslant (1-\varepsilon)\alpha(t), \qquad t \geqslant t_0;$$

(iii) $\|\Delta(t)\Phi(t)\Phi^{-1}(s)f(s, x)\| \leqslant v(t)\omega(s, \|\Delta(s)x\|\alpha^{-1}(s))$, *for* $t_0 \leqslant t \leqslant s$;

and

(iv) $\limsup_{t\to\infty} \frac{v(t)}{\gamma(t)} \int_t^\infty \omega(s, 1)\,ds = 0$; $\gamma(t) = \min\{\alpha(t), \beta(t)\}$.

Then there exists a solution $x(t)$ *of* (1.2) *such that*

$$\|\Delta(t)(x(t)-y(t))\| = o(\beta(t)) \quad \text{as } t\to\infty. \tag{2.1}$$

and

$$\|\Delta(t)x(t)\| \leqslant \alpha(t) \quad \text{as } t\to\infty. \tag{2.2}$$

Proof. For a given positive constant ε in hypothesis (ii), hypothesis (iv) implies that there exists a large T_0 ($> t_0$) such that

$$\frac{v(t)}{\gamma(t)} \int_t^\infty \omega(s, 1)\,ds < \varepsilon \quad \text{for } t \geqslant T_0. \tag{2.3}$$

Via the Schauder–Tychonoff theorem (see [3], p. 9) we will establish the existence of a solution of the integral equation

$$x(t) = \Phi(t)c - \Phi(t)\int_t^\infty \Phi^{-1}(s)f(s, x(s))\,ds, \qquad t \geqslant T_0,$$

where $\Phi(t)c = y(t)$. Consider the set

$$F = \{u\colon u(t) = \alpha^{-1}(t)\Delta(t)x(t), \text{ where } x(t) \text{ is continuous on } J_0 = [T_0, \infty) \text{ and } \|u(t)\| \leqslant 1 \text{ for } t \geqslant T_0\}$$

and define the operator T by

$$Tu(t) = \frac{\Delta(t)\Phi(t)c}{\alpha(t)} - \frac{\Delta(t)\Phi(t)}{\alpha(t)}\int_t^\infty \Phi^{-1}(s)f(s, \alpha(s)\Delta^{-1}(s)u(s))\,ds. \tag{2.4}$$

First we will establish that $TF \subset F$. Taking the norm to both sides of (2.4) and using hypotheses (ii) and (iii) and (2.3), we obtain

$$\begin{aligned}\|Tu(t)\| &\leqslant \frac{\|\Delta(t)\Phi(t)c\|}{\alpha(t)} + \frac{1}{\alpha(t)}\int_t^\infty \|\Delta(t)\Phi(t)\Phi^{-1}(s)f(s, \Delta^{-1}(s)u(s)\alpha(s))\|\,ds \\ &\leqslant \frac{\|\Delta(t)y(t)\|}{\alpha(t)} + \frac{v(t)}{\alpha(t)}\int_t^\infty \omega(s, \|u(s)\|)\,ds \\ &\leqslant \frac{\|\Delta(t)y(t)\|}{\alpha(t)} + \varepsilon \leqslant 1.\end{aligned}$$

It is clear that $\alpha(t)\Delta^{-1}(t)\, Tu(t)$ is continuous on $J_0 = [T_0, \infty)$. This proves $TF \subset F$.

Second we will show that T is continuous. Suppose that the sequence $\{u_n\}$ in F converges uniformly to u in F on every compact subinterval of J_0. We claim that Tu_n converges uniformly to Tu on every compact subinterval J_1 of J_0. Let ε_1 be a small positive number satisfying $\varepsilon_1 < 1$. Hypothesis (iv) implies that there exists $T_1 > T_0$ so that for $t \geqslant T_1$

$$\frac{v(t)}{\gamma(t)} \int_t^\infty \omega(s, 1)\, ds < \varepsilon_1/4. \tag{2.5}$$

Then using (2.4) we obtain the following inequalities, for $t \in J_0$.

$$\begin{aligned} \|Tu_n(t) - Tu(t)\| &\leqslant \frac{1}{\alpha(t)} \Big\| \int_t^\infty \Delta(t)\Phi(t)\Phi^{-1}(s) f\big(s, \alpha(s)\Delta^{-1}(s)u_n(s)\big)\, ds - \\ &\quad - \int_t^\infty \Delta(t)\Phi(t)\Phi^{-1}(s) f\big(s, \alpha(s)\Delta^{-1}(s)u(s)\big)\, ds \Big\| \\ &\leqslant \frac{\|\Delta(t)\Phi(t)\|}{\alpha(t)} \cdot \int_t^{T_1} \big[\|\Phi^{-1}(s)\| \cdot \|f\big(s, \alpha(s)\Delta^{-1}(s)u_n(s)\big) - \\ &\quad - f\big(s, \alpha(s)\Delta^{-1}(s)u(s)\big)\|\big]\, ds + \\ &\quad + \frac{1}{\alpha(t)} \int_{T_1}^\infty \big[\|\Delta(t)\Phi(t)\Phi^{-1}(s) f\big(s, \Delta^{-1}(s)u_n(s)\alpha(s)\big)\| + \\ &\quad + \|\Delta(t)\Phi(t)\Phi^{-1}(s) f\big(s, \alpha(s)\Delta^{-1}(s)u(s)\big)\|\big]\, ds. \end{aligned} \tag{2.6}$$

Now using hypothesis (iii) and (2.5), the second integral on the right-hand side of (2.6) satisfies

$$\begin{aligned} &\frac{1}{\alpha(t)} \int_{T_1}^\infty \big[\|\Delta(t)\Phi(t)\Phi^{-1}(s) f\big(s, \alpha(s)\Delta^{-1}(s)u_n(s)\big)\| + \\ &\qquad + \|\Delta(t)\Phi(t)\Phi^{-1}(s) f\big(s, \alpha(s)\Delta^{-1}(s)u(s)\big)\|\big]\, ds \\ &\leqslant \frac{v(t)}{\alpha(t)} \int_{T_1}^\infty \big[\omega\big(s, \|u_n(s)\|\big) + \omega\big(s, \|u(s)\|\big)\big]\, ds \\ &\leqslant \frac{2v(t)}{\alpha(t)} \int_{T_1}^\infty \omega(s, 1)\, ds < \frac{\varepsilon_1}{2}. \end{aligned} \tag{2.7}$$

By the uniform convergence there is an $N = N(\varepsilon, T_1)$ such that if $n \geqslant N$, then

$$\|f\big(t, \alpha(t)\Delta^{-1}(t)u_n(t)\big) - f\big(t, \alpha(t)\Delta^{-1}(t)u(t)\big)\| < \frac{\varepsilon_1}{2M_1 M_2 (T_1 - T_0)}, \tag{2.8}$$

where

$$M_1 = \sup_{T_0 \leqslant t \leqslant T_1} \|\Phi^{-1}(t)\| \quad \text{and} \quad M_2 = \sup_{t \in J_1} \frac{\alpha(t)}{\|\Delta(t)\,\Phi(t)\|}.$$

Combining (2.6), (2.7), and (2.8) yields for $t \in J_1$

$$\|Tu(t) - Tu_n(t)\| < \varepsilon_1 \quad \text{for } n \geqslant N.$$

This shows that Tu_n converges uniformly to Tu on compact subintervals J_1 of J_0. Hence T is continuous.

Third we claim that the functions in the image set TF are equicontinuous and bounded at every point of J_0. Since $TF \subset F$, it is clear that the functions in TF are uniformly bounded. Now we show that they are equicontinuous at each point of J_0. For each $u \in F$, the function $z(t) = \alpha(t)\Delta^{-1}(t)\,Tu(t)$ is a solution of the linear system below

$$\frac{dv}{dt} = A(t)v + f\big(t,\ \alpha(t)\,\Delta^{-1}(t)\,u(t)\big).$$

Since $\|z(t)\| \leqslant \alpha(t)\,\|\Delta^{-1}(t)\|\,\|Tu(t)\| \leqslant \alpha(t)\,\|\Delta^{-1}(t)\|$ and $\big\|f\big(t,\ \alpha(t)\,\Delta^{-1}(t)\,u(t)\big)\big\|$ is uniformly bounded for $u \in F$ on any finite t interval, we see that dv/dt is uniformly bounded on any finite interval. Therefore, the set of all such z is equicontinuous on any finite interval. To see that the functions in TF are equicontinuous at every point in J_0, consider

$$\text{(2.9)} \quad \|Tu(t_1) - Tu(t_2)\| = \|\alpha^{-1}(t_1)\,\Delta(t_1)\,z(t_1) - \alpha^{-1}(t_2)\,\Delta(t_2)\,z(t_2)\|$$
$$\leqslant \|\alpha^{-1}(t_1)\,\Delta(t_1)\|\,\|z(t_1) - z(t_2)\| + \|\alpha^{-1}(t_1)\,\Delta(t_1) - \alpha^{-1}(t_2)\,\Delta(t_2)\| \cdot \|z(t_2)\|,$$

where t_1, t_2 are in some finite interval. The right-hand side of (2.9) can be made small by virtue of the equicontinuity of the family $\{z(t)\}$ and the continuity of $\alpha^{-1}(t)\,\Delta(t)$. Thus the functions in TF are equicontinuous at each point of J_0.

All of the hypotheses of the Schauder–Tychonoff theorem are satisfied. Thus there exists a $u \in F$ such that $u(t) = Tu(t)$; that is, there exists a solution $x(t)$ of

$$x(t) = y(t) - \Phi(t) \int_t^{\infty} \Phi^{-1}(s) f\big(s,\ x(s)\big)\,ds.$$

Therefore $x(t)$ is a solution of (1.2) and possesses the asymptotic behavior of (2.1) and (2.2). This proves Theorem 2.1.

Remark 2.1. We here replaced the condition $\|\Delta(t)\Phi(t)\| \leqslant \alpha(t)$ in Theorem A by the more general condition $\|\Delta(t)\,y(t)\| \leqslant (1-\varepsilon)\,\alpha(t)$ in Theorem 2.1. Here $y(t) = \Phi(t)\,c$ for some vector c.

If we take $v(t) = \|\Delta(t)\,\Phi(t)\|$ and $\alpha(t) = \beta(t)$, Theorem 2.1 implies the following corollary.

COROLLARY 2.1. *Let $y(t)$ be an arbitrary nontrivial solution of* (1.1). *Suppose that there exist $\alpha(t)$, $\Delta(t)$, and $\omega(t, s)$ satisfying Assumption* A *and for some positive $\varepsilon < 1$ there exists t_0 such that for $t \geqslant t_0$*

$$\frac{\|\Delta(t)\Phi(t)\|}{\alpha(t)} < 1-\varepsilon, \qquad \|\Phi^{-1}(t)f(t, x)\| \leqslant \omega\big(t, \|\alpha^{-1}(t)\Delta(t)x(t)\|\big)$$

and

$$\lim_{t\to\infty}\int_t^\infty \omega(s, 1)\,ds = 0.$$

Then there exists a solution $x(t)$ of (1.2) *such that* (2.1) *with $\beta(t) = \alpha(t)$ holds.*

Proof. Since $\|\Delta(t)y(t)\| \leqslant \|\Delta(t)\Phi(t)\|$, Corollary 2.1 follows from Theorem 2.1.

Remark 2.2. From Lemma 2.1, Corollary 2.1 is an improvement of Theorem A. Moreover, the given solution $y(t)$ in the above corollary does not require a sufficiently small initial condition.

If we let the coefficient $A(t)$ in (1.1) be constant, $\alpha(t) = \beta(t)$, and $\Delta(t) = I$, Theorem 2.1 implies the following corollary.

COROLLARY 2.2. *Suppose that $A(t)$ is a constant $n\times n$ matrix. Let $y(t)$ be an arbitrary nontrivial solution of* (1.1). *Suppose also that there exist $\alpha(t)$ and $\omega(t, s)$ satisfying*

(i) *Assumption* A;

(ii) $\|y(t)\| \leqslant \alpha(t)$, $t \geqslant t_0$;

(iii) $\|f(t, x)\| \exp(\|A\|t) \leqslant \omega\big(t, \|x\|\alpha^{-1}(t)\big)$;

and

(iv) $\lim_{t\to\infty}\int_t^\infty \omega(s, 1)\,ds = 0.$

Then there exists a solution $x(t)$ of (1.2) *such that $\|x(t)-y(t)\| = o\big(\alpha(t)\big)$ and $\|x(t)\| \leqslant \alpha(t)$ as $t\to\infty$.*

Proof. Since $\Phi(t)\Phi^{-1}(s) = \exp\big(A(t-s)\big)$, using hypothesis (iii) we obtain for $t \leqslant s$,

$$\begin{aligned}\|\Phi(t)\Phi^{-1}(s)f(s, x)\| &\leqslant \|\Phi(t)\Phi^{-1}(s)\|\,\|f(s, x)\| \leqslant \exp\big(\|A\|(s-t)\big)\|f(s, x)\|\\ &\leqslant \exp(-\|A\|t)\cdot(\|A\|s)\cdot\|f(s, x)\|\\ &\leqslant v(t)\,\omega\big(s, \|x\|\alpha^{-1}(s)\big).\end{aligned}$$

Here we choose $v(t) = \exp(-\|A\|t)$. Since $\|y(t)\| \leqslant \alpha(t)$ and $y(t) = \Phi(t)c$ for some vector c, we obtain $\alpha(t) \geqslant \exp(-\|A\|t)$. Thus hypothesis (iv) in Theorem 2.1 becomes

$$\limsup_{t\to\infty}\frac{v(t)}{\gamma(t)}\int_t^\infty \omega(s, 1)\,ds \leqslant \lim_{t\to\infty}\int_t^\infty \omega(s, 1)\,ds = 0.$$

All hypotheses of Theorem 2.1 hold. This proves Corollary 2.2.

The following theorem deals with the converse problem to that considered in Theorem 2.1.

THEOREM 2.2. *Let $x(t)$ be an arbitrary solution of* (1.2) *and $\beta(t)$ be a positive continuous function on J. Suppose that there exist $\alpha(t)$, $v(t)$, $\Delta(t)$, and $\omega(t, s)$ satisfying*

(i) *Assumption* A;

(ii) $\|\Delta(t)x(t)\| \leqslant \alpha(t)\beta(t)$ *for* $t \geqslant t_0$;

(iii) $\|\Phi^{-1}(t)f(t, x)\| \leqslant \omega(t, \|\Delta(t)x(t)\|\beta^{-1}(t))$;

(iv) $\|\Delta(t)\Phi(t)\Phi^{-1}(s)f(s, x)\| \leqslant v(t)\omega(s, \|\Delta(s)x(s)\|\beta^{-1}(s))$;

(v) $\int^{\infty}\omega(t, \alpha(t))dt < \infty$;

and

(vi) $\limsup_{t\to\infty}\frac{v(t)}{\beta(t)}\int_t^{\infty}\omega(s, \alpha(s))ds = 0$.

Then there exists a solution $y(t)$ of (1.1) *such that* (2.1) *holds.*

Proof. Using the variation of constant formula, we can represent any solution $x(t)$ of (1.2) by the integral equation

$$x(t) = \Phi(t)x(t_0)+\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)f(s, x(s))ds. \tag{2.10}$$

Next, consider the expression

$$x(t_0)+\int_{t_0}^{t}\Phi^{-1}(s)f(s, x(s))ds.$$

Using hypotheses (i), (ii), (iii), and (v), we obtain

$$\begin{aligned}\int_{t_0}^{t}\|\Phi^{-1}(s)f(s, x(s))\|ds &\leqslant \int_{t_0}^{t}\omega(s, \|\Delta(s)x(s)\|\beta^{-1}(s))ds\\ &\leqslant \int_{t_0}^{t}\omega(s, \alpha(s))ds < \infty.\end{aligned}$$

As a consequence of the Lebesgue dominated convergence theorem, we have

$$c = \lim_{t\to\infty}\int_{t_0}^{t}\Phi^{-1}(s)f(s, x(s))ds+x(t_0). \tag{2.11}$$

Substituting (2.11) into (2.10) shows

$$x(t) = \Phi(t)c-\Phi(t)\int_t^{\infty}\Phi^{-1}(s)f(s, x(s))ds$$

and then

$$(2.12)\qquad \frac{\Delta(t)x(t)}{\beta(t)}=\frac{\Delta(t)\Phi(t)c}{\beta(t)}-\frac{\Delta(t)\Phi(t)}{\beta(t)}\int_t^\infty \Phi^{-1}(s)f(s,\,x(s))\,ds.$$

Let $y(t)=\Phi(t)c$. It is clear that $y(t)$ is a solution of (1.1). Thus it follows from hypothesis (iv), (2.11) and (2.10) that

$$(2.13)\qquad \begin{aligned}\frac{\|\Delta(t)(x(t)-y(t))\|}{\beta(t)} &\leqslant \frac{1}{\beta(t)}\int_t^\infty \|\Delta(t)\Phi(t)\Phi^{-1}(s)f(s,\,x(s))\|\,ds\\ &\leqslant \frac{v(t)}{\beta(t)}\int_t^\infty \omega(s,\,\|\Delta(s)x(s)\|\,\beta^{-1}(s))\,ds\\ &\leqslant \frac{v(t)}{\beta(t)}\int_t^\infty \omega(s,\,\alpha(s))\,ds.\end{aligned}$$

Therefore, the theorem follows from (2.13) and hypothesis (vi).

Corresponding to Corollary 2.1 if we take $v(t)=\|\Delta(t)\Phi(t)\|$ and $\alpha(t)=1$ in Theorem 2.2, we obtain the following result.

COROLLARY 2.3. *Let $x(t)$ be an arbitrary solution of* (1.2). *Suppose that there exist $\alpha(t)$, $\Delta(t)$, and $\omega(t,s)$ satisfying Assumption* A,

$$\|\Delta(t)\Phi(t)\|\leqslant\beta(t),\qquad \|\Phi^{-1}(t)f(t,\,x)\|\leqslant\omega(t,\,\|\Delta(t)x(t)\|\,\beta^{-1}(t)),$$

and

$$\lim_{t\to\infty}\int_t^\infty \omega(s,\,1)\,ds=0.$$

Then there exists a solution $y(t)$ of (1.1) *such that* (2.1) *holds.*

Remark 2.3. In Corollary 2.3 we do not require the initial condition of a given solution $x(t)$ to be sufficiently small as stated in [1], Theorem 1, and we use the condition, $\lim_{t\to\infty}\int_t^\infty \omega(s,\,1)\,ds=0$ which is weaker than part (i) of Lemma 2.1 as stated in [1], Theorem 1. Moreover, $\beta(t)$ depends on the given solution $x(t)$ in Corollary 2.3 while $\beta(t)$ depends on the fundamental matrix, $\Phi(t)$, of (2.1) in [1], Theorem 1.

Corresponding to Corollary 2.2, if we let the coefficient $A(t)$ in (1.1) be constant, $\alpha(t)=1$, and $\Delta(t)=I$, then Theorem 2.2 implies the following corollary.

COROLLARY 2.4. *Suppose that $A(t)$ is a constant $n\times n$ matrix and $\beta(t)$ is a positive continuous function on J. Let $x(t)$ be an arbitrary nontrivial solution of* (1.2). *Suppose also that there exists $\omega(t,s)$ satisfying Assumption* A,

$$\|x(t)\|\leqslant\beta(t),\qquad \|f(t,\,x)\|\exp(\|A\|\,t)\leqslant\omega(t,\,\|x\|\,\beta^{-1}(t))$$

and

$$\lim_{t\to\infty} \int_t^{\infty} \omega(s, 1)\, ds = 0.$$

Then there exists a solution $y(t)$ *of* (1.1) *such that* (2.1) *holds.*

III. Example. Consider the following differential equation

$$\theta''(t)+2\theta(t)+f(t)\theta^r(t) = 0, \qquad t \geqslant 0, \tag{3.1}$$

where $\alpha > 0$, $r \geqslant 1$, and $f(t)$ is a real continuous function for $t \geqslant 0$. It is clear that (3.1) can be rewritten as

$$\frac{dx(t)}{dt} = Ax(t)+F\big(t, x(t)\big), \tag{3.2}$$

where

$$x(t) = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix} \quad \text{and} \quad F\big(t, x(t)\big) = f(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\theta^r(t).$$

Thus we could consider (3.2) as a perturbed linear system of

$$\frac{dy(t)}{dt} = Ay(t).$$

If we apply Theorem A to (3.2), we obtain that if

$$\int^{\infty} |f(s)|\, e^{(1+r)s}\, ds < \infty, \tag{3.3}$$

then there exists a nontrivial solution $x(t)$ of (3.2) for which

$$\|x(t)-e^{-\alpha t}\| = o(e^{-\alpha t}) \quad \text{as } t \to \infty. \tag{3.4}$$

However, from Corollary 2.2 we obtain that if

$$\int^{\infty} |f(s)|\, e^{(1-r)s}\, ds < \infty, \tag{3.5}$$

then there exists a nontrivial solution $x(t)$ of (3.2) for which (3.4) holds. This later criterion is an improvement of the early criterion from Theorem A because of (3.3) and (3.5).

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Reçu par la Rédaction le 19. 7. 1978