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ON THE ASYMPTOTIC BEHAVIOR OF THE COEFFICIENTS OF ASYMPTOTIC POWER SERIES AND ITS RELEVANCE TO STOKES PHENOMENA*

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Abstract. This paper discusses the relevance of the asymptotic behavior of the coefficients of asymptotic power series for the study of Stokes phenomena. By way of illustration a connection problem is considered in the theory of linear difference equations.

Key words. asymptotic expansion, isomorphism of Malgrange, Cauchy-Heine transform, saddle-point method, Stokes phenomenon, linear analytic functional equation, difference equation

AMS(MOS) subject classifications. 30E15, 39

Introduction. In this paper we extend and apply ideas of Malgrange [10] and Ramis [12] concerning the connection between Stokes phenomena, in a wider sense, and formal power series. We start with an illustrative example.

Let y be an analytic function on the Riemann surface of $\log z$, with the following properties.

(i) y admits an asymptotic expansion of the form $\sum_{n=0}^{\infty} y_n z^{-n}$ as $z\to\infty$ in the sector S: $-\pi/2<\arg z<5\pi/2$.

(ii) $y(z)-y(z\,e^{2\pi i})=c\,e^{-z}$, $c\in\mathbb{C}^*$.

The second property implies that the asymptotic behavior of y changes abruptly as $\arg z$ becomes larger than $5\pi/2$ or less than $-\pi/2$. Such a change in asymptotic behavior will be called a Stokes phenomenon.

Now consider the function h defined by

$$h(z)=\frac{1}{2\pi i}\int_0^{\infty}\frac{e^{-t}}{t-z}\,dt,\qquad 0<\arg z<2\pi.$$

h is a Cauchy-Heine transform of e^{-z} (cf. [12]). By deformation of the path of integration it may be continued analytically to the Riemann surface of $\log z$. With the aid of residue calculus we readily verify that

$$h(z)-h(z\,e^{2\pi i})=e^{-z}. \tag{0.1}$$

Moreover, h admits the asymptotic expansion $\sum_{n=1}^{\infty} h_n z^{-n}$ as $z\to\infty$, $z\in S$, where

$$h_n=-\frac{1}{2\pi i}\int_0^{\infty} e^{-t}t^{n-1}\,dt,\qquad n\in\mathbb{N}. \tag{0.2}$$

From (ii) and (0.1) we conclude that

$$y(z\,e^{2\pi i})-ch(z\,e^{2\pi i})=y(z)-ch(z).$$

Thus it turns out that $y-ch$ is a single-valued analytic function, admitting an asymptotic expansion of the form $\sum_{n=0}^{\infty} a_n z^{-n}$ as $z\to\infty$, $z\in S$, where

$$a_n=y_n-ch_n. \tag{0.3}$$

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This implies that $y-ch$ is holomorphic at ∞ and, consequently, $\sum_{n=0}^{\infty} a_n z^{-n}$ is a convergent power series. From (0.2) and (0.3) it now follows that

$$c=-2\pi i \lim_{n\to\infty} \frac{y_n}{(n-1)!}.$$

Apparently, the constant c, which plays a central role in the Stokes phenomenon occurring in this example, is intimately related to the asymptotic behavior of the coefficients y_n. It is this relationship that forms the subject of this paper.

We shall consider the following situation. Suppose we are given a number of sectors S_ν, $\nu\in\{1,\cdots,N\}$, which cover a neighborhood of ∞ and a corresponding number of functions y_ν with the following properties: y_ν is analytic in S_ν and represented asymptotically by a series of the form $\sum_{n=0}^{\infty} \hat{y}_n z^{-n}$ (independent of ν) as $z\to\infty$, $z\in S_\nu$, $\nu\in\{1,\cdots,N\}$. Moreover, assume that

$$y_{\nu+1}(z)-y_\nu(z)=\sum_{j=1}^{m} c_j^\nu \varphi_j^\nu(z), \qquad z\in S_\nu\cap S_{\nu+1}, \quad \nu\in\{1,\cdots,N\}, \tag{0.4}$$

where $S_{N+1}=e^{2\pi i}S_1$, $y_{N+1}(z)\equiv y_1(z\,e^{-2\pi i})$, $c_j^\nu\in\mathbb{C}$, and the φ_j^ν belong to a certain class of analytic functions. We shall establish a relation between the complex numbers c_j^ν and the asymptotic behavior of $\hat{y}_n$ for $n\to\infty$. In some applications this relation may be exploited to "compute" at least part of the numbers c_j^ν from the coefficients $\hat{y}_n$ (cf. [9] and Remark 2 herein).

If the y_ν represent (sectorial models of) a resurgent function, our results could be derived from the work of Ecalle (cf. [4]). For the present purpose, however, this assumption is not needed and we shall establish the relation mentioned above in a more direct manner.

The argument is essentially the same as the one we used in [9]. It is based on the Propositions 1.1–1.3 herein. Proposition 1.1 concerns the properties of Cauchy-Heine transforms of functions like the φ_j^ν in (0.4). Proposition 1.2 enables us to construct, from the Cauchy-Heine transforms of the φ_j^ν, analytic functions H_ν with the same properties as the y_ν and only differing from the y_ν by a convergent power series in $1/z$. The coefficients of the asymptotic expansion $\hat{H}$ of the H_ν are given by the expression

$$\hat{H}_n=-\frac{1}{2\pi i}\sum_{\nu=1}^{N}\sum_{j=1}^{m} c_j^\nu \int_{\gamma_\nu} \varphi_j^\nu(t)t^{n-1}\,dt, \qquad \gamma_\nu\subset S_\nu.$$

Under certain conditions, like those mentioned in Proposition 1.3, the saddle-point method may be applied to the integral

$$\int_{\gamma_\nu} \varphi_j^\nu(t)t^{n-1}\,dt$$

to obtain its asymptotic behavior for $n\to\infty$. The main result is stated in Theorem 1.4. In § 2 this result is applied to a connection problem in the theory of homogeneous linear difference equations.

1. The general argument. Let $\mathbb{C}_\infty$ denote the Riemann surface of log z. Let $z_0\in\mathbb{C}_\infty$, α, $\beta\in\mathbb{R}$, $\alpha<\beta$. By $S(\alpha,\beta)$ we denote the sector

$$S(\alpha,\beta)=\{z\in\mathbb{C}_\infty\colon \alpha<\arg z<\beta\}$$

and by $S(z_0,\alpha,\beta)$ the set

$$S(z_0,\alpha,\beta)=\{z\in\mathbb{C}_\infty\colon \alpha<\arg(z-z_0)<\beta, |z|>|z_0|\}. \tag{1.1}$$

This will also be called a sector.

If S is a sector of the form $S=S(z_0, \alpha, \beta)$, then $\underline{S}$ will denote the sector $S(z_0, \alpha, \beta+2\pi)$.

Let $S=S(z_0, \alpha, \beta)$, $S'=S(z_1, \alpha', \beta')$ with $\alpha<\alpha'<\beta'<\beta$. We shall write

$$S' \Subset S$$

whenever $z_1 \in S$ and $S' \subset S(z_0, \alpha', \beta')$.

Let $\hat{h}=\sum_{n=0}^{\infty} h_n z^{-n}$ be a formal power series in z^{-1}, S a sector of the type (1.1), and h a function on S. We say that h is represented asymptotically by $\hat{h}$ as $z \to \infty$ in S, and write

$$h(z) \sim \sum_{n=0}^{\infty} h_n z^{-n}, \qquad z \to \infty \text{ in } S$$

if, for every $S' \Subset S$ and every $N \in \mathbb{N}$,

$$R_N(h; z) \equiv h(z) - \sum_{n=0}^{N-1} h_n z^{-n} = O(z^{-N}), \qquad z \to \infty, \quad z \in S'.$$

Any function φ which is analytic in a sector S and represented asymptotically by zero (i.e., the series with coefficients equal to zero) as $z \to \infty$ in S, may be written as the difference of two determinations of its Cauchy-Heine transform. The following proposition, due to Ramis, is concerned with the asymptotic properties of this Cauchy-Heine transform.

PROPOSITION 1.1 (cf. [12, Prop. 4.2]). *Let α and β be real numbers such that $\alpha<\beta$, $z_0 \in S(\alpha, \beta)$, and let φ be an analytic function on $S=S(z_0, \alpha, \beta)$. Suppose there exist positive numbers M_n, $n \in \mathbb{N}$, such that*

$$\sup_{z \in S} |z^n \varphi(z)| < M_n, \qquad n \in \mathbb{N}. \tag{1.2}$$

Then the function h defined by

$$h(z) = \frac{z}{2\pi i} \int_\gamma \frac{\varphi(\zeta)}{\zeta(\zeta - z)}\, d\zeta, \qquad z \in S(z_0, \Theta, \Theta+2\pi),$$

where γ is a half line in S from z_0 to ∞ with direction Θ, has the following properties:

(i) *h can be continued analytically to $\underline{S}$,*

(ii) *$h(z) - h(z\, e^{2\pi i}) = \varphi(z)$ for all $z \in S$,*

(iii) *h is represented asymptotically by*

$$\sum_{n=0}^{\infty} -\frac{1}{2\pi i}\left(\int_\gamma \varphi(\zeta)\zeta^{n-1}\, d\zeta\right) z^{-n},$$

as $z \to \infty$ in $\underline{S}$. Moreover, for every $S' \Subset \underline{S}$ there exists a positive constant $C_{S'}$ such that

$$\sup_{z \in S'} |z^n R_n(h; z)| \leqq C_{S'} M_{n+1}, \qquad n \in \mathbb{N}.$$

Proof. Let us suppose that S is a convex set, i.e., $\beta - \pi/2 < \arg z_0 < \alpha + \pi/2$. In that case every half line from z_0 to ∞ with direction $\Theta \in (\alpha, \beta)$ lies in S. If γ has direction Θ, h is obviously analytic in $S(z_0, \Theta, \Theta+2\pi)$. The analytic continuation to $\underline{S}$ is obtained by varying Θ. Part (ii) follows immediately from Cauchy's theorem.

Now let $S'=S(z_1, \alpha', \beta') \Subset \underline{S}$. Then there is a number $\varepsilon \in (0, \pi/2)$ such that $\alpha+\varepsilon < \arg(z-z_0) < \beta+2\pi-\varepsilon$ for all $z \in S'$. Let $z \in S'$ and choose $\Theta \in (\alpha, \beta)$ in such a way that $\Theta+\varepsilon<\arg(z-z_0)<\Theta+2\pi-\varepsilon$. Let γ_Θ be the half line from z_0 to ∞ with direction Θ. For all $\zeta\in\gamma_\Theta$ the following inequality holds:

$$|\zeta-z|>|z-z_0|\sin\varepsilon>|z|\left(1-\left|\frac{z_0}{z_1}\right|\right)\sin\varepsilon. \tag{1.3}$$

It is easily seen that

$$z^nR_n(h;z)=\frac{z}{2\pi i}\int_{\gamma_\Theta}\frac{\varphi(\zeta)}{\zeta-z}\zeta^{n-1}\,d\zeta,\qquad n\in\mathbb{N}.$$

With (1.2) and (1.3) it follows that

$$|z^nR_n(h;z)|<\frac{1}{2\pi\sin\varepsilon}\left(1-\left|\frac{z_0}{z_1}\right|\right)^{-1}\int_{\gamma_\Theta}\left|\frac{d\zeta}{\zeta^2}\right|M_{n+1}.$$

Hence

$$\sup_{z\in S'}|z^nR_n(h;z)|<\frac{1}{2\pi\sin\varepsilon}\left(1-\left|\frac{z_0}{z_1}\right|\right)^{-1}\sup_{\Theta\in(\alpha,\beta)}\int_{\gamma_\Theta}\left|\frac{d\zeta}{\zeta^2}\right|M_{n+1}$$

and this proves (iii).

If S is not convex the above argument must be adapted in an obvious manner.

PROPOSITION 1.2 (cf. [10], [12]). *Let $N\in\mathbb{N}$. Let α_ν, β_ν, $\nu\in\{1,\cdots,N\}$, be real numbers such that $\alpha_\nu\leqq\alpha_{\nu+1}<\beta_\nu\leqq\beta_{\nu+1}$ if $\nu<N$ and $\alpha_N\leqq\alpha_{N+1}\equiv\alpha_1+2\pi<\beta_N\leqq\beta_{n+1}\equiv\beta_1+2\pi$. Let $z_\nu\in S(\alpha_{\nu+1},\beta_\nu)$ and $S^\nu=S(z_\nu,\alpha_{\nu+1},\beta_\nu)$, $\nu=1,\cdots,N$.*

Suppose that, for every $\nu\in\{1,\cdots,N\}$, we are given an analytic function φ_ν on S^ν with the property that $\varphi_\nu(z)\sim0$ as $z\to\infty$ in S^ν. Let

$$h_\nu(z)=\frac{z}{2\pi i}\int_{\gamma_\nu}\frac{\varphi_\nu(\zeta)}{\zeta(\zeta-z)}\,d\zeta,\qquad z\in S(z_\nu,\Theta_\nu,\Theta_\nu+2\pi),\quad \nu\in\{1,\cdots,N\},$$

where γ_ν is a half line in S^ν from z_ν to ∞ with direction Θ_ν and let

$$H_\nu(z)=\sum_{\mu=1}^{\nu-1}h_\mu(z)+\sum_{\mu=\nu}^{N}h_\mu(z\,e^{2\pi i})\quad \textit{if } \nu\in\{2,\cdots,N\},$$

$$H_1(z)=\sum_{\mu=1}^{N}h_\mu(z\,e^{2\pi i})\qquad \textit{and}\quad H_{N+1}(z)=\sum_{\mu=1}^{N}h_\mu(z).$$

The functions H_ν have the following properties:

(i) *For every $\nu\in\{1,\cdots,N+1\}$ there exists a $\tilde z_\nu\in S(\alpha_\nu,\beta_\nu)$ such that $\tilde z_{N+1}=\tilde z_1\,e^{2\pi i}$ and H_ν is analytic on $S_\nu=S(\tilde z_\nu,\alpha_\nu,\beta_\nu)$.*

(ii) *$H_{\nu+1}(z)-H_\nu(z)=\varphi_\nu(z)$ for all $z\in S_\nu\cap S_{\nu+1}$, $\nu\in\{1,\cdots,N\}$, and $H_{N+1}(z)=H_1(z\,e^{-2\pi i})$ for all $z\in e^{2\pi i}S_1$.*

(iii) *H_ν admits an asymptotic power series expansion $\hat H$ independent of ν, as $z\to\infty$ in S_ν.*

Moreover, if $\tilde H_\nu$, $\nu=1,\cdots,N+1$, are functions with the same properties, then there exists a function h, holomorphic at ∞, such that

$$\tilde H_\nu-H_\nu=h\quad \textit{for all } \nu\in\{1,\cdots,N\}.$$

Proof. From Proposition 1.1(i) we deduce that H_ν is analytic in

$$\bigcap_{\mu=1}^{\nu-1}S(z_\mu,\alpha_{\mu+1},\beta_\mu+2\pi)\bigcap_{\mu=\nu}^{N}S(z_\mu,\alpha_{\mu+1}-2\pi,\beta_\mu)\quad \text{if } \nu\in\{2,\cdots,N\},$$

in

$$\bigcap_{\mu=1}^{N} S(z_\mu, \alpha_{\mu+1}-2\pi, \beta_\mu) \quad \text{if } \nu=1,$$

and in

$$\bigcap_{\mu=1}^{N} S(z_\mu, \alpha_{\mu+1}, \beta_\mu+2\pi) \quad \text{if } \nu=N+1,$$

and this set contains a sector of the form $S(\tilde{z}_\nu, \alpha_\nu, \beta_\nu)$ for a suitable choice of $\tilde{z}_\nu$. Part (ii) follows immediately from Proposition 1.1(ii) by observing that

$$H_{\nu+1}(z)-H_\nu(z)=h_\nu(z)-h_\nu(z\,e^{2\pi i}) \quad \text{for all } \nu\in\{1,\cdots,N\}.$$

Furthermore, Proposition 1.1(iii) implies that $H_\nu(z)\sim\sum_{n=0}^{\infty} H_n z^{-n}$, as $z\to\infty$ in S_ν, where

$$(1.4) \qquad H_n=-\frac{1}{2\pi i}\sum_{\nu=1}^{N}\int_{\gamma_\nu}\varphi_\nu(\zeta)\zeta^{n-1}\,d\zeta, \qquad n\in\mathbb{N}.$$

Now suppose that $\tilde{H}_\nu$, $\nu=1,\cdots,N+1$, are functions with the properties (i)–(iii) mentioned in Proposition 1.2. Then there exist $z'_\nu\in S(\alpha_\nu,\beta_\nu)$ such that both H_ν and $\tilde{H}_\nu$ are analytic on $\tilde{S}_\nu=S(z'_\nu,\alpha_\nu,\beta_\nu)$ and we have

$$\tilde{H}_{\nu+1}(z)-\tilde{H}_\nu(z)=H_{\nu+1}(z)-H_\nu(z), \quad z\in\tilde{S}_\nu\cap\tilde{S}_{\nu+1}, \quad \nu\in\{1,\cdots,N\}$$

and

$$\tilde{H}_{N+1}(z)-\tilde{H}_1(z\,e^{-2\pi i})=H_{N+1}(z)-H_1(z\,e^{-2\pi i}), \qquad z\in\tilde{S}_{N+1}.$$

It follows that

$$\tilde{H}_{\nu+1}-H_{\nu+1}=\tilde{H}_\nu-H_\nu \quad \text{for all } \nu\in\{1,\cdots,N\}$$

and, moreover,

$$\tilde{H}_{N+1}(z)-H_{N+1}(z)=\tilde{H}_1(z\,e^{-2\pi i})-H_1(z\,e^{-2\pi i}).$$

Hence the function $h=\tilde{H}_1-H_1$ can be continued analytically to a reduced neighborhood of ∞. Furthermore, property (iii) implies that h admits an asymptotic power series expansion in z^{-1} as $z\to\infty$ in a neighborhood of ∞ and, consequently, h is analytic in a full neighborhood of ∞.

The next proposition concerns the asymptotic behavior of integrals of the type

$$\int_\gamma \varphi(z)z^n\,dz,$$

where γ is a half line and φ is an analytic function with the property that $\varphi(z)\sim 0$ as $z\to\infty$ in some sector S containing γ. The conditions (iii)–(v) below are purely technical and have been chosen in such a way that the result follows by a straightforward application of the saddle-point method. They might be relaxed or replaced by other conditions. We have merely tried to define a class of functions for which this method works.

PROPOSITION 1.3 (cf. [2, Thm. 7, Remark 6]). *Let α and β be real numbers such that $\alpha<\beta$, $z_0\in S(\alpha,\beta)$, and let ψ be an analytic function on $S=S(z_0,\alpha,\beta)$ with the property that*

(i) $\exp\psi(z)\sim 0$ *as $z\to\infty$ in S.*

Let $g: S\times\mathbb{N}\to\mathbb{C}$ be defined by

$$g(z,n)=\psi(z)+n\log z.$$

Suppose there exists $n_0 \in \mathbb{N}$ such that for all $n \geqq n_0$ the following conditions hold:

(ii) *The equation $\partial g/\partial z = 0$ has a solution $s_n \in S$ such that the half line γ_n from z_0 to ∞ through s_n is contained in S. Moreover, $s_n \to \infty$ as $n \to \infty$.*

(iii) *There exists a number $\Theta \in (0, \pi/2)$ such that*

$$\left|\arg - s_n^2 \frac{\partial^2 g}{\partial z^2}(s_n, n)\right| \leqq \Theta$$

and, furthermore, $s_n^2(\partial^2 g/\partial z^2)(s_n, n) \to \infty$ as $n \to \infty$.

(iv) *There exist positive numbers ε_0 and K such that*

$$\left| z \frac{\partial^3}{\partial z^3} g(z, n)\left\{\frac{\partial^2}{\partial z^2} g(z, n)\right\}^{-1}\right| \leqq K$$

if $|z - s_n| < \varepsilon_0 |s_n|$.

(v) *Let $\alpha_n = \arg(s_n - z_0) - \arg s_n$. There exists a positive number ε_1, a function $n_1 : (0, \varepsilon_1) \to \mathbb{N}$, a bounded function $g_1 : (0, \varepsilon_1) \times (-1, 0) \to \mathbb{R}$, and a function $g_2 : (0, \varepsilon_1) \times (0, \infty) \to \mathbb{R}$ such that, for all $\varepsilon \in (0, \varepsilon_1)$, $\exp g_2(\varepsilon, \cdot) \in \mathcal{L}(0, \infty)$, and, for all $n \geqq n_1(\varepsilon)$,*

$$\operatorname{Re}\{g(s_n(1 + \tau e^{i\alpha_n}), n) - g(s_n(1 - \varepsilon e^{i\alpha_n}), n)\} \leqq g_1(\varepsilon, \tau)$$

if $\tau \in (-|1 - z_0/s_n|, -\varepsilon)$, whereas

$$\operatorname{Re}\{g(s_n(1 + \tau e^{i\alpha_n}), n) - g(s_n(1 + \varepsilon e^{i\alpha_n}), n)\} \leqq g_2(\varepsilon, \tau)$$

if $\tau \in (\varepsilon, \infty)$.

Furthermore, let f be a bounded analytic function on S and suppose there exists a positive number ε such that

(vi) $\sup_{z \in I_n(\varepsilon)} |f(z) - 1| \to 0$ *if $n \to \infty$, where $I_n(\varepsilon)$ denotes the segment between $s_n(1 - \varepsilon e^{i\alpha_n})$ and $s_n(1 + \varepsilon e^{i\alpha_n})$.*

Let

$$\varphi(z) = f(z) \exp \psi(z),$$

and

$$J_n = \frac{1}{2\pi i}\int_\gamma \varphi(z) z^n \, dz$$

where γ is a half line in S from z_0 to ∞. Then we have

$$J_n = \left\{2\pi s_n^2 \frac{\partial^2 g}{\partial z^2}(s_n, n)\right\}^{-1/2} s_n \exp g(s_n, n)(1 + o(1)), \quad n \to \infty,$$

where $\arg\{s_n^2(\partial^2 g/\partial z^2)(s_n, n)\}^{-1/2} \in (-\pi, 0)$.

Proof. We shall closely follow the proof of Theorem 7 in [2]. Let $n \geqq n_0$. Due to (i), (ii) and the properties of f, we may replace γ by γ_n. Let $\varepsilon > 0$. We begin by considering the integrand on the segment $I_n(\varepsilon)$. We put

$$\left|\frac{\partial^2 g}{\partial z^2}(z, n) - \frac{\partial^2 g}{\partial z^2}(s_n, n)\right| = h(z).$$

From (iv) we deduce that

$$h(z) \leqq K \left|\frac{\partial^2 g}{\partial z^2}(s_n, n)\right| \int_{s_n}^{z} \left|\frac{d\zeta}{\zeta}\right| + K \int_{s_n}^{z} h(\zeta)\left|\frac{d\zeta}{\zeta}\right|$$

provided $|z-s_n|<\varepsilon_0|s_n|$. With the aid of Gronwall's generalized inequality (cf. [5, p. 36]) we find

$$h(z)\leqq \tilde{K}\left|\frac{z-s_n}{s_n}\right|\left|\frac{\partial^2 g}{\partial z^2}(s_n,n)\right|,$$

where $\tilde{K}$ is a positive constant, provided $|z-s_n|<\varepsilon_0|s_n|$. Hence it follows that, for all $\varepsilon\in(0,\varepsilon_0)$,

$$\frac{\partial g}{\partial z}(z,n)=(z-s_n)\frac{\partial^2 g}{\partial z^2}(s_n,n)(1+\varepsilon O(1)),\qquad n\to\infty \tag{1.5}$$

$$g(z,n)-g(s_n,n)=\frac{1}{2}(z-s_n)^2\frac{\partial^2 g}{\partial z^2}(s_n,n)(1+\varepsilon O(1)),\qquad n\to\infty, \tag{1.6}$$

uniformly on $I_n(\varepsilon)$. Here $O(1)$ is uniformly bounded in ε.

We introduce a new variable w by means of

$$\tfrac{1}{2}w^2=g(s_n,n)-g(s_n(1+\tau e^{i\alpha_n}),n),\qquad |\tau|\leqq\varepsilon. \tag{1.7}$$

Due to (1.6) we have

$$w^2=-\tau^2 e^{2i\alpha_n}s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)(1+\varepsilon O(1))$$

and we remove the ambiguity in the definition of w by demanding that

$$w=\tau e^{i\alpha_n}\left(-s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{1/2}(1+\varepsilon O(1)), \tag{1.8}$$

where the square root has its principal value. Equations (1.7) and (1.5) imply that

$$w\frac{dw}{d\tau}=-s_n e^{i\alpha_n}\frac{\partial g}{\partial z}(s_n(1+\tau e^{i\alpha_n}),n)=-s_n^2\tau e^{2i\alpha_n}\frac{\partial^2 g}{\partial z^2}(s_n,n)(1+\varepsilon O(1)).$$

Consequently,

$$\frac{dw}{d\tau}=e^{i\alpha_n}\left(-s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{1/2}(1+\varepsilon O(1)). \tag{1.9}$$

Let $w_\pm$ correspond to $\tau=\pm\varepsilon$. From (1.8) it follows that

$$w_\pm=\pm\varepsilon e^{i\alpha_n}\left(-s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{1/2}(1+\varepsilon O(1)). \tag{1.10}$$

From (1.7), (1.9), (1.10), and condition (vi) of Proposition 1.3 we deduce that

$$\begin{aligned}&\int_{s_n(1-\varepsilon e^{i\alpha_n})}^{s_n(1+\varepsilon e^{i\alpha_n})}\varphi(z)z^n\,dz\\ &\quad=s_n e^{i\alpha_n}\exp g(s_n,n)\int_{w_-}^{w_+}e^{-w^2/2}\left(\frac{dw}{d\tau}\right)^{-1}dw(1+o(1))\\ &\quad=\left(-s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{-1/2}s_n\exp g(s_n,n)\int_{w_-}^{w_+}e^{-w^2/2}\,dw(1+o(1))(1+\varepsilon O(1)).\end{aligned}$$

Using (1.10) and condition (iii) and noting that $\lim_{n\to\infty}\alpha_n=0$, we conclude that, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\int_{w_-}^{w_+}e^{-w^2/2}\,dw=\sqrt{2\pi}.$$

Hence

$$\frac{1}{2\pi i}\int_{s_n(1-\varepsilon e^{i\alpha_n})}^{s_n(1+\varepsilon e^{i\alpha_n})}\varphi(z)z^n\,dz=\left(2\pi s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{-1/2}s_n$$
$$\cdot(\exp g(s_n,n)(1+o(1))(1+\varepsilon O(1)),$$

where $\arg\{s_n^2(\partial^2 g/\partial z^2)(s_n,n)\}^{-1/2}\in(-\pi,0)$.

Next we consider the integral

$$J_n^+(\varepsilon)=\int_{s_n(1+\varepsilon e^{i\alpha_n})}^{\infty}\varphi(z)z^n\,dz.$$

From (1.7), condition (v), and the properties of f we deduce that, for $n\geqq n_1(\varepsilon)$,

$$|J_n^+(\varepsilon)|\leqq C\left|s_n\exp\left\{g(s_n,n)-\frac{1}{2}w_+^2\right\}\right|\int_\varepsilon^\infty\exp g_2(\varepsilon,\tau)\,d\tau$$
$$\leqq C_1\left|s_n\exp\left\{g(s_n,n)-\frac{1}{2}w_+^2\right\}\right|,$$

where C and C_1 are positive constants. In view of (1.10) and condition (iii) this implies that

$$J_n^+(\varepsilon)=\left(s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{-1/2}s_n\exp g(s_n,n)o(1),\qquad n\to\infty$$

and the same property holds for the integral over the segment between z_0 and $s_n(1-\varepsilon e^{i\alpha_n})$. Combining the above estimates we find

$$J_n=\left(2\pi s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{-1/2}s_n\exp g(s_n,n)(1+\varepsilon O(1))(1+o(1)),\qquad n\to\infty.$$

Since this is true for every sufficiently small ε the result follows.

THEOREM 1.4. *Let $N\in\mathbb{N}$. Let α_ν, β_ν, $\nu\in\{1,\cdots,N\}$, be real numbers such that $\alpha_\nu\leqq\alpha_{\nu+1}<\beta_\nu\leqq\beta_{\nu+1}$ if $\nu<N$ and $\alpha_N\leqq\alpha_{N+1}\equiv\alpha_1+2\pi<\beta_N\leqq\beta_{N+1}\equiv\beta_1+2\pi$. Let $\tilde z_\nu\in S(\alpha_\nu,\beta_\nu)$ and $S_\nu=S(\tilde z_\nu,\alpha_\nu,\beta_\nu)$, $\nu=1,\cdots,N$, $S_{N+1}=e^{2\pi i}S_1$. Suppose that, for each $\nu\in\{1,\cdots,N\}$, we are given an analytic function y_ν on S_ν, admitting an asymptotic expansion $\sum_{n=0}^\infty\hat y_n z^{-n}$ as $z\to\infty$ in S_ν, independent of ν. Let*

$$y_{N+1}(z)=y_1(z\,e^{-2\pi i}),\qquad z\in S_{N+1},$$

and

$$\varphi_\nu(z)=y_{\nu+1}(z)-y_\nu(z),\qquad z\in S_\nu\cap S_{\nu+1},\quad \nu\in\{1,\cdots,N\}.$$

Suppose that for every $\nu\in\{1,\cdots,N\}$ there exists a sector $\tilde S^\nu\subset S_\nu\cap S_{\nu+1}$, a positive integer $m(\nu)$, and, for every $j\in\{1,\cdots,m(\nu)\}$, analytic functions ψ_j^ν and f_j^ν on $\tilde S^\nu$, satisfying the conditions of Proposition 1.3, and a complex number c_j^ν such that

$$\varphi_\nu(z)=\sum_{j=1}^{m(\nu)}c_j^\nu f_j^\nu(z)\exp\psi_j^\nu(z),\qquad z\in\tilde S^\nu.$$

Let $g_j^\nu(z,n)=\psi_j^\nu(z)+n\log z$, let $s_n^{\nu,j}$ denote its saddle point, and let

$$M_j^\nu(n)=\left\{2\pi(s_n^{\nu,j})^2\frac{\partial^2 g_j^\nu}{\partial z^2}(s_n^{\nu,j},n)\right\}^{-1/2}s_n^{\nu,j}\exp g_\nu(s_n^{\nu,j},n),$$
$$j\in\{1,\cdots,m(\nu)\},\quad \nu\in\{1,\cdots,N\},$$

where $\arg\{(s_n^{\nu,j})^2(\partial^2 g_j^\nu/\partial z^2)(s_n^{\nu,j}, n)\}^{-1/2} \in (-\pi, 0)$. *Then there exists a convergent power series* $\sum_{n=0}^{\infty} h_n z^{-n}$ *such that*

$$\hat{y}_n = h_n - \sum_{\nu=1}^{N} \sum_{j=1}^{m(\nu)} c_j^\nu \{M_j^\nu(n-1)(1+o(1))\}, \qquad n \to \infty. \tag{1.11}$$

Proof. There exists $z_\nu \in S_\nu \cap S_{\nu+1}$ such that $S_\nu \cap S_{\nu+1}$ contains the sector $S^\nu = S(z_\nu, \alpha_{\nu+1}, \beta_\nu)$. As y_ν and $y_{\nu+1}$ admit the same asymptotic expansion, it follows that

$$\varphi_\nu(z) = y_{\nu+1}(z) - y_\nu(z) \sim 0 \quad \text{as } z \to \infty \text{ in } S^\nu, \quad \nu \in \{1, \cdots, N\}.$$

Obviously, the functions y_ν possess the properties (i)–(iii) mentioned in Proposition 1.2.

According to Proposition 1.2 there exists a function h, holomorphic at ∞, such that $y_\nu = h + H_\nu$ for all $\nu \in \{1, \cdots, N\}$. Let $\sum_{n=0}^{\infty} h_n z^{-n}$ be the power series expansion of h. With (1.4) we find

$$\begin{aligned}\hat{y}_n &= h_n - \sum_{\nu=1}^{N} \frac{1}{2\pi i} \int_{\gamma_\nu} \varphi_\nu(z) z^{n-1}\, dz \\ &= h_n - \sum_{\nu=1}^{N} \sum_{j=1}^{m(\nu)} \frac{c_j^\nu}{2\pi i} \int_{\gamma_\nu} f_j^\nu(z) \exp \psi_j^\nu(z) z^{n-1}\, dz, \qquad n \in \mathbb{N},\end{aligned}$$

where γ_ν is a half line $\tilde{S}^\nu$, $\nu \in \{1, \cdots, N\}$.

The proof is completed by application of Proposition 1.3 to each term of the sum in the right-hand side of the above identity.

Remark 1. If the y_ν as well as the functions $f_j^\nu \exp \psi_j^\nu$ are solutions of some homogeneous linear functional equation, the numbers c_j^ν play a role similar to the Stokes multipliers in the theory of linear differential equations.

Remark 2. If one of the functions M_j^ν in (1.11) dominates the rest for $n \to \infty$, the corresponding coefficient c_j^ν may be determined from the asymptotic behavior of $\hat{y}_n$.

Remark 3. Propositions 1.1 and 1.2 may also be used to obtain estimates of the growth of the remainder terms $R_n(y_\nu; z)$ as $n \to \infty$. This will be illustrated by the application to linear difference equations in the next section.

Example. The nonlinear differential equation

$$\frac{dy}{dz} = \frac{a}{z^2} + y + \frac{b}{z^2} y^3, \qquad a, b \in \mathbb{C}^* \tag{1.12}$$

possesses three formal solutions of the form $\sum_{n=-1}^{\infty} \hat{y}_n z^{-n}$. The coefficients $\hat{y}_n$ can be determined from the recursive relations

$$\begin{aligned}&-2\hat{y}_{n+2} + (n+4)\hat{y}_{n+1} + b \sum_{\substack{m_i \leqq n \\ m_1+m_2+m_3=n}} \hat{y}_{m_1}\hat{y}_{m_2}\hat{y}_{m_3} = 0, \qquad n \geqq -1, \\ &(\hat{y}_{-1})^2 = -\frac{1}{b}, \qquad \hat{y}_0 = -\frac{1}{2}\hat{y}_{-1}\end{aligned} \tag{1.13}$$

and

$$\begin{aligned}&\hat{y}_{n+2} + (n+1)\hat{y}_{n+1} + b \sum_{m_1+m_2+m_3=n} \hat{y}_{m_1}\hat{y}_{m_2}\hat{y}_{m_3} = 0, \qquad n \geqq 1, \\ &\hat{y}_{-1} = \hat{y}_0 = \hat{y}_1 = 0, \qquad \hat{y}_2 = -a.\end{aligned} \tag{1.14}$$

Let $\hat{y}$ denote one of the formal solutions and let S be a sector of aperture less than π. It is a well-known fact that there exists a solution of (1.12), analytic in S and represented asymptotically by $\hat{y}$ as $z \to \infty$ in S, uniformly on S (cf. [13]). Suppose that y_1 and y_2 are two solutions with these properties. Obviously,

$$\frac{d}{dz}(y_1-y_2)=y_1-y_2+\frac{b}{z^2}(y_1^2+y_1y_2+y_2^2)(y_1-y_2). \tag{1.15}$$

Let $\hat{y}=\sum_{n=-1}^{\infty}\hat{y}_n^- z^{-n}$ and suppose the coefficients $\hat{y}_n^-$ satisfy (1.13). Then we have

$$y_1^2+y_1y_2+y_2^2=-\frac{3}{b}(z^2-z)+h(z), \tag{1.16}$$

where h is a bounded analytic function on S, admitting an asymptotic expansion as $z \to \infty$ in S. Inserting (1.16) into (1.15) we obtain

$$\frac{d}{dz}(y_1-y_2)=\left\{-2+\frac{3}{z}+\frac{b}{z^2}h(z)\right\}(y_1-y_2)$$

and this implies that

$$y_1-y_2=c\,e^{-2z}z^3\left(1+O\left(\frac{1}{z}\right)\right) \quad z\to\infty \text{ in } S,$$

where c is a complex number. Hence it follows that (1.12) has a unique solution y^-, analytic in a left half plane and represented asymptotically by the series $\sum_{n=-1}^{\infty}\hat{y}_n^- z^{-n}$ as $z \to \infty$ in this half plane. Moreover, it is easily seen that y^- may be continued analytically to a sector of the form $S(z_1, -3\pi/2, 3\pi/2)$, with $z_1 \in \mathbb{C}_\infty$, without a change in asymptotic behavior.

Furthermore, we have

$$y^-(z)-y^-(z\,e^{2\pi i})=c^-\,e^{-2z}z^3\left(1+O\left(\frac{1}{z}\right)\right), \qquad c^- \in \mathbb{C},$$

as $z\to\infty$ in $S(-3\pi/2+\varepsilon, -\pi/2-\varepsilon)$ for any $\varepsilon \in (0, \pi/2)$. Applying Theorem 1.4 we find

$$c^- = -2\pi i \lim_{n\to\infty}\frac{2^{n+3}\hat{y}_n^-}{(n+2)!}.$$

In a similar manner it is shown that (1.12) possesses a unique solution y^+ analytic in $S(z_2, -\pi/2, 5\pi/2)$ for some $z_2 \in \mathbb{C}_\infty$ and represented asymptotically by the series $\sum_{n=-1}^{\infty}\hat{y}_n^+ z^{-n}$ determined by (1.14), as $z\to\infty$ in this sector. Moreover, it turns out that

$$y^+(z)-y^+(z\,e^{2\pi i})=c^+\,e^{z}\left(1+O\left(\frac{1}{z}\right)\right), \qquad c^+ \in \mathbb{C},$$

as $z\to\infty$ in $S(-\pi/2+\varepsilon, \pi/2-\varepsilon)$ for any $\varepsilon \in (0, \pi/2)$. Application of Theorem 1.4 now yields the relation

$$c^+ = 2\pi i \lim_{n\to\infty}\frac{(-1)^{n-1}\hat{y}_n^+}{(n-1)!}.$$

2. An application to linear difference equations. We consider the mth-order homogeneous linear difference equation

$$\sum_{j=0}^{m} a_j(z)y(z+j)=0, \tag{2.1}$$

where $a_j \in \mathbb{C}\{z^{-1}\}$, $j=1,\cdots,m$ (or, equivalently, a system of m first-order difference equations). The "generic" case is when the characteristic equation of (2.1) has m distinct roots. This case has been treated in [8]. Here we shall deal with a more singular class of equations.

Under certain conditions, (2.1) possesses m linearly independent formal solutions of the form

$$\hat{y}_j(z)=\hat{h}_j(z)z^{\rho_j}\exp(d_j z\log z+\mu_j z),\qquad j=1,\cdots,m, \tag{2.2}$$

where $\hat{h}_j(z)=\sum_{n=0}^{\infty}\hat{h}_{jn}z^{-n}$ with $\hat{h}_{j0}=1$, $\rho_j\in\mathbb{C}$, $d_j\in\mathbb{Q}$ and $\mu_j\in\mathbb{C}$ for all $j\in\{1,\cdots,m\}$ (cf. [3], [11]).

We put

$$\rho_i-\rho_j=\rho_{ij},\quad d_i-d_j=d_{ij},\quad\text{and}\quad \mu_i-\mu_j=\mu_{ij},\qquad i,j\in\{1,\cdots,m\}$$

and we assume that, for all $i,j\in\{1,\cdots,m\}$ such that $i\neq j$ and $d_{ij}=0$,

$$\operatorname{Re}\mu_{ij}\neq 0. \tag{2.3}$$

For merely technical reasons we further assume that

$$\operatorname{Im}\mu_{ij}\notin\{0,-d_{ij}\pi\}\bmod 2\pi\quad\text{if } i\neq j,\quad i,j\in\{1,\cdots,m\} \tag{2.4}$$

but this condition can easily be removed. For all $i,j\in\{1,\cdots,m\}$ such that $i\neq j$ we shall denote by n_{ij} the integer determined by

$$\begin{aligned}&0<\operatorname{Im}\mu_{ij}+2n_{ij}\pi<2\pi &&\text{if } d_{ij}\leqq 0,\\ &0<\operatorname{Im}\mu_{ij}+(2n_{ij}+d_{ij})\pi<2\pi &&\text{if } d_{ij}>0.\end{aligned} \tag{2.5}$$

Let $S_1,\cdots,S_7$ be sectors of the following form:

$$S_1=S(R\,e^{-i(\pi/2)},-\pi,0),\quad S_2=e^{i(\pi/2)}S_1,\quad S_3=S_4=e^{i\pi}S_1,$$
$$S_5=e^{i(3\pi/2)}S_1\quad\text{and}\quad S_6=S_7=e^{2\pi i}S_1,$$

where $R>0$. If R is chosen sufficiently large, equation (2.1) possesses, for each $j\in\{1,\cdots,m\}$ and $\nu\in\{1,3,4,6,7\}$, a unique solution y_j^ν, represented asymptotically by $\hat{y}_j$ as $z\to\infty$, uniformly on

$$\begin{aligned}&\left(\frac{\nu-1}{3}-1\right)\pi+\delta<\arg(z-R\,e^{(\nu/3-5/6)\pi i})\leqq\frac{\nu-1}{3}\pi &&\text{if }\nu\in\{1,4,7\},\\ &\left(\frac{\nu}{3}-1\right)\pi\leqq\arg(z-R\,e^{(\nu/3-1/2)\pi i})<\frac{\nu}{3}\pi-\delta &&\text{if }\nu\in\{3,6\}\end{aligned} \tag{2.6}$$

for every $\delta\in(0,\pi/2)$ (cf. [6, Thm. 2.4.5]; note that this is a stronger statement than $y_j^\nu\sim\hat{y}_j$ as $z\to\infty$ in S_ν). Moreover, we have

$$y_j^4-y_j^3=p_{jj}^3y_j^3,\qquad y_j^7-y_j^6=p_{jj}^6y_j^6, \tag{2.7}$$

where p_{jj}^3 and p_{jj}^6 are periodic functions of period 1 with the property that

$$\lim_{\operatorname{Im}z\to\infty}p_{jj}^3(z)=\lim_{\operatorname{Im}z\to-\infty}p_{jj}^6(z)=0,\qquad j\in\{1,\cdots,m\}. \tag{2.8}$$

Furthermore, for each $j\in\{1,\cdots,m\}$, equation (2.1) possesses a unique solution y_j^2, analytic in S_2 and represented asymptotically by $\hat{y}_j$ as $z\to\infty$ in S_2, such that

$$y_j^2-y_j^1=\sum_{i=1}^m p_{ij}^1y_i^1,\qquad y_j^3-y_j^2=\sum_{i=1}^m p_{ij}^2y_i^2,$$

where p_{ij}^1 and p_{ij}^2 are periodic functions of period 1 with the following properties:

$$\begin{gathered}p_{ij}^1=p_{ij}^2\equiv 0\quad\text{if } d_{ij}>0\text{ or } d_{ij}=0\text{ and }\operatorname{Re}\mu_{ij}\geqq 0,\\ \lim_{\operatorname{Im}z\to-\infty}p_{ij}^1(z)\exp\{-2(n_{ij}-1)\pi iz\}\text{ and }\lim_{\operatorname{Im}z\to\infty}p_{ij}^2(z)\exp\{-2n_{ij}\pi iz\}\\ \text{exist for all } i,j\in\{1,\cdots,m\}\text{ such that } i\neq j.\end{gathered} \tag{2.9}$$

Similarly, for each $j\in\{1,\cdots,m\}$, there exists a unique solution y_j^5, analytic in S_5 and represented asymptotically by $\hat{y}_j$ as $z\to\infty$ in S_5, such that

$$y_j^5-y_j^4=\sum_{i=1}^m p_{ij}^4 y_i^4,\qquad y_j^6-y_j^5=\sum_{i=1}^m p_{ij}^5 y_i^5,$$

where p_{ij}^4 and p_{ij}^5 are periodic functions of period 1 with the following properties:

$$p_{ij}^4=p_{ij}^5\equiv 0\quad \text{if } d_{ij}<0 \text{ or } d_{ij}=0 \text{ and } \operatorname{Re}\mu_{ij}\leqq 0,$$

(2.10) $\lim_{\operatorname{Im} z\to\infty} p_{ij}^4(z)\exp\{-2n_{ij}\pi iz\}$ and $\lim_{\operatorname{Im} z\to-\infty} p_{ij}^5(z)\exp\{-2(n_{ij}-1)\pi iz\}$ exist for all $i,j\in\{1,\cdots,m\}$ such that $i\neq j$.

Now let

$$h_j^\nu(z)=y_j^\nu(z)z^{-\rho_j}\exp(-d_jz\log z-\mu_jz),\qquad j\in\{1,\cdots,m\},\quad \nu\in\{1,\cdots,7\}.$$

Obviously, h_j^ν is represented asymptotically by $\hat{h}_j$ as $z\to\infty$ in S_ν for all $j\in\{1,\cdots,m\}$ and all $\nu\in\{1,\cdots,7\}$. Moreover, if $\nu\in\{1,3,4,6,7\}$, the asymptotic expansion is uniformly valid on (2.6) for every $\delta\in(0,\pi/2)$. The uniqueness of h_j^ν implies that

$$h_j^7(z)=h_j^1(z\,e^{-2\pi i})\quad \text{for all } j\in\{1,\cdots,m\}. \tag{2.11}$$

Furthermore, we have, for all $j\in\{1,\cdots,m\}$ and $\nu\in\{1,\cdots,6\}$,

$$h_j^{\nu+1}(z)-h_j^\nu(z)=\sum_{i=1}^m p_{ij}^\nu(z)h_i^\nu(z)z^{\rho_{ij}}\exp(d_{ij}z\log z+\mu_{ij}z). \tag{2.12}$$

For all $i,j\in\{1,\cdots,m\}$ and all $\nu\in\{1,\cdots,6\}$ we define an integer n_{ij}^ν and complex numbers c_{ij}^ν and μ_{ij}^ν as follows:

(2.13)

$$n_{ij}^\nu=\begin{cases}\max\left\{n\in\mathbb{Z}:\ \lim\limits_{\operatorname{Im} z\to\infty} p_{ij}^\nu(z)\exp(-2n\pi iz)\text{ exists}\right\} & \text{if } \nu\in\{2,3,4\} \text{ and } p_{ij}^\nu\not\equiv 0,\\ \min\left\{n\in\mathbb{Z}:\ \lim\limits_{\operatorname{Im} z\to-\infty} p_{ij}^\nu(z)\exp(-2n\pi iz)\text{ exists}\right\} & \text{if } \nu\in\{1,5,6\} \text{ and } p_{ij}^\nu\not\equiv 0,\\ 0\quad \text{otherwise},\end{cases}$$

$$c_{ij}^\nu=\begin{cases}0 & \text{if } p_{ij}^\nu\equiv 0,\\ \lim_{\operatorname{Im} z\to\pm\infty} p_{ij}^\nu(z)\exp(-2n_{ij}^\nu\pi iz) & \text{otherwise},\end{cases} \tag{2.14}$$

$$\mu_{ij}^\nu=\mu_{ij}+2n_{ij}^\nu\pi iz. \tag{2.15}$$

Furthermore, we define analytic functions f_{ij}^ν and φ_{ij}^ν by

$$f_{ij}^\nu(z)=\begin{cases}0 & \text{if } c_{ij}^\nu=0,\\ (c_{ij}^\nu)^{-1}p_{ij}^\nu(z)\exp(-2n_{ij}^\nu\pi iz)h_i^\nu(z) & \text{otherwise},\end{cases} \tag{2.16}$$

$$\varphi_{ij}^\nu(z)=p_{ij}^\nu(z)h_i^\nu(z)z^{\rho_{ij}}\exp(d_{ij}z\log z+\mu_{ij}z). \tag{2.17}$$

Obviously,

$$\varphi_{ij}^\nu(z)=c_{ij}^\nu f_{ij}^\nu(z)z^{\rho_{ij}}\exp(d_{ij}z\log z+\mu_{ij}^\nu z),\qquad i,j\in\{1,\cdots,m\},\quad \nu\in\{1,\cdots,6\}. \tag{2.18}$$

In order to check whether the conditions of Theorem 1.4 are satisfied, we will first study the properties of the function $g:S\times\mathbb{N}\to\mathbb{C}$ defined by

$$g(z,n)=dz\log z+\mu z=(n+\rho)\log z,$$

where $d \in \mathbb{Q}$, $\mu \in \mathbb{C}$, $\rho \in \mathbb{C}$, and S is one of the sectors $S_\nu \cap S_{\nu+1}$, $\nu \in \{1, \cdots, 6\}$. From (2.18), (2.3), (2.7)-(2.10), and the definitions (2.13)-(2.15) we conclude that the following cases need to be considered:

1. $d=0$, $\rho=0$, $\mu=2m\pi i$, $m \in \mathbb{N}$, $S=S_3$,
2. $d=0$, $\rho=0$, $\mu=-2m\pi i$, $m \in \mathbb{N}$, $S=S_6$,
3. $d=0$, $\operatorname{Re}\mu<0$, $\operatorname{Im}\mu<0$, $S=S_1\cap S_2$,
4. $d=0$, $\operatorname{Re}\mu<0$, $\operatorname{Im}\mu>0$, $S=S_2\cap S_3$,
5. $d=0$, $\operatorname{Re}\mu>0$, $\operatorname{Im}\mu>0$, $S=S_4\cap S_5$,
6. $d=0$, $\operatorname{Re}\mu>0$, $\operatorname{Im}\mu<0$, $S=S_5\cap S_6$,
7. $d<0$, $\operatorname{Im}\mu<0$, $S=S_1\cap S_2$,
8. $d<0$, $\operatorname{Im}\mu>0$, $S=S_2\cap S_3$,
9. $d>0$, $\operatorname{Im}\mu+d\pi>0$, $S=S_4\cap S_5$,
10. $d>0$, $\operatorname{Im}\mu+d\pi<0$, $S=S_5\cap S_6$.

In the first six cases, $\partial g/\partial z=0$ has a unique solution s_n given by

$$s_n=-\frac{n+\rho}{\mu}. \tag{2.19}$$

Hence

$$\arg s_n=\arg\left(-\frac{1}{\mu}\right)(1+o(1)), \qquad n\to\infty. \tag{2.20}$$

Furthermore, we have

$$\frac{\partial^2 g}{\partial z^2}(s_n, n)=-\frac{\mu^2}{n+\rho}, \qquad s_n^2\frac{\partial^2 g}{\partial z^2}(s_n, n)=-n-\rho, \tag{2.21}$$

$$z\frac{\partial^3 g}{\partial z^3}(z, n)\left\{\frac{\partial^2 g}{\partial z^2}(z, n)\right\}^{-1}=-2. \tag{2.22}$$

Let $S' \Subset S$. In each of the cases 1-6 there exists a positive number δ such that

$$\cos(\arg z+\arg\mu)<-\frac{\delta}{|\mu|} \quad \text{for all } z\in S'.$$

This implies that, for all $z \in S'$,

$$\operatorname{Re} g(z, n)\leqq -\delta|z|+(n+\operatorname{Re}\rho)\log|z|-\operatorname{Im}\rho\arg z.$$

Hence we easily deduce the existence of positive constants $A_{S'}$ and $C_{S'}$ such that

$$\sup_{z\in S'}|\exp g(z, n)|<C_{S'}A_{S'}^n n^n. \tag{2.23}$$

Now consider the cases 7-10. There $d\neq 0$ and the saddle point s_n is a solution of the equation

$$s_n\left(\log s_n+\frac{\mu}{d}+1\right)=-\frac{n+\rho}{d}. \tag{2.24}$$

Let h be the inverse of the function $z\to z\log z$ (cf. [9, Ex. III], [4, § 3.6]). It has the following asymptotic behavior:

$$h(z)=\frac{z}{\log z}(1+o(1)), \qquad z\to\infty. \tag{2.25}$$

From (2.24) we deduce

$$
\begin{aligned}
s_n &= \exp\left(-\frac{\mu}{d}-1\right) h\left(-\frac{n+\rho}{d}\exp\left(\frac{\mu}{d}+1\right)\right) \\
&= -\frac{n+\rho}{d}\left\{\log h\left(-\frac{n+\rho}{d}\exp\left(\frac{\mu}{d}+1\right)\right)\right\}^{-1}.
\end{aligned} \tag{2.26}
$$

With (2.25) it follows that

$$
s_n = -\frac{n}{d\log n}(1+o(1)), \qquad n\to\infty. \tag{2.27}
$$

Equating the imaginary parts on both sides of (2.24), we get

$$
\operatorname{Im} s_n\left(\log|s_n|+\frac{\operatorname{Re}\mu}{d}+1\right)+\operatorname{Re} s_n\left(\arg s_n+\frac{\operatorname{Im}\mu}{d}\right)=\frac{\operatorname{Im}\rho}{d}.
$$

With (2.27) we find

$$
\operatorname{Im} s_n = \frac{n}{d(\log n)^2}\left(\arg s_n+\frac{\operatorname{Im}\mu}{d}\right)(1+o(1)), \qquad n\to\infty.
$$

Hence

$$
\operatorname{Im} s_n = \begin{cases} \dfrac{n}{d^2(\log n)^2}\operatorname{Im}\mu(1+o(1)), & n\to\infty \quad \text{if } d<0, \\[2ex] \dfrac{n}{d^2(\log n)^2}(\operatorname{Im}\mu+d\pi)(1+o(1)), & n\to\infty \quad \text{if } d>0. \end{cases} \tag{2.28}
$$

Furthermore, we have

$$
\frac{\partial^2 g}{\partial z^2}(s_n,n)=\frac{d}{s_n}-\frac{n+\rho}{s_n^2}=-\frac{d^2}{n+\rho}\log h\left(-\frac{n+\rho}{d}\exp\left(\frac{\mu}{d}+1\right)\right)(1+o(1)), \quad n\to\infty \tag{2.29}
$$

and hence

$$
s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)=-n(1+o(1)), \qquad n\to\infty. \tag{2.30}
$$

We easily verify that

$$
z\frac{\partial^3 g}{\partial z^3}(z,n)\left\{\frac{\partial^2 g}{\partial z^2}(z,n)\right\}^{-1}=-\frac{2(n+\rho)-dz}{n+\rho-dz} \tag{2.31}
$$

and the expression on the right-hand side is obviously uniformly bounded on the half plane $-d\operatorname{Re} z>0$ and thus on S, provided $n\geqq n_0$, where n_0 is some sufficiently large number.

Let $S'\Subset S$. In each of the cases considered this implies the existence of a positive number δ such that

$$
d\cos\arg z<-\delta \quad \text{for all } z\in S'.
$$

Let $0<\varepsilon<\delta$. Then there exists a positive constant C such that

$$
|\exp g(z,n)|<C\exp(-\varepsilon|z|\log|z|)|z|^n, \qquad z\in S'.
$$

The expression to the right of the inequality sign attains its maximum as $|z| = h(ne/\varepsilon)/e$ and the maximum value is equal to

$$\exp\left(-2n+\frac{\varepsilon}{e}h\left(\frac{ne}{\varepsilon}\right)\right)h\left(\frac{ne}{\varepsilon}\right)^n.$$

In view of (2.25) it follows that there exist positive constants $A_{S'}$ and $C_{S'}$ such that

$$\sup_{z\in S'}|\exp g(z,n)|<C_{S'}A_{S'}^n\left(\frac{n}{\log n}\right)^n. \tag{2.32}$$

With the aid of (2.19), (2.21), (2.26), and (2.29) we can derive an explicit expression for the function $M:\mathbb{N}\to\mathbb{C}$ given by

$$M(n)=\left(2\pi s_n^2\frac{\partial^2 g}{\partial z^2}(s_n,n)\right)^{-1/2}s_n\exp g(s_n,n), \tag{2.33}$$

where $\arg(s_n^2(\partial^2 g/\partial z^2)(s_n,n))^{-1/2}\in(-\pi,0)$, in each of the cases considered above. With (2.30) we find

$$M(n)=\begin{cases}\{-2\pi(n+\rho)\}^{-1/2}\exp(-n-\rho)\left(\dfrac{n+\rho}{\mu}\right)^{n+\rho+1} & \text{if } d=0,\\ (-2\pi n)^{-1/2}\exp\{(n+\rho)\chi(n)^{-1}-1\}\left(\dfrac{n+\rho}{d\chi(n)}\right)^{n+\rho+1}(1+o(1)), & n\to\infty, \text{if } d\neq 0,\end{cases}$$

where $\chi(n)=\log h((n+\rho)/d\exp(\mu/d+1))$. Let us define a function $M_{d,\mu}:\mathbb{C}\to\mathbb{C}$ by

(2.34)

$$-2\pi i M_{d,\mu}(s)$$
$$=\begin{cases}\Gamma(s)(-\mu)^{-s} & \text{if } d=0,\\ \Gamma(s)\exp\left\{\dfrac{s}{\log h(-s/d\exp(\mu/d+1))}\right\}\left\{-d\log h\left(-\dfrac{s}{d}\exp\left(\dfrac{\mu}{d}+1\right)\right)\right\}^{-s}, & \\ & \text{if } d\neq 0.\end{cases}$$

Using Stirling's formula and the properties of the function h, we readily verify that

$$-M(n-1)=M_{d,\mu}(n+\rho)(1+o(1)),\qquad n\to\infty. \tag{2.35}$$

Now let $\nu\in\{1,\cdots,6\}$, $\tilde z_\nu\in S_\nu\cap S_{\nu+1}$, and let $\tilde S^\nu$ be a sector of the following form:

$$\tilde S^\nu=S\left(\tilde z_\nu,\left(\frac{\nu}{3}-\frac{5}{6}\right)\pi+\delta,\frac{\nu-1}{3}\pi\right)\quad \text{if } \nu\in\{1,4\},$$

$$\tilde S^\nu=S\left(\tilde z_\nu,\frac{\nu-2}{3}\pi,\left(\frac{\nu}{3}-\frac{1}{6}\right)\pi\right)\quad \text{if } \nu\in\{2,5\},$$

$$\tilde S^\nu=S\left(\tilde z_\nu,\left(\frac{\nu}{3}-1\right)\pi+\delta,\frac{\nu}{3}\pi-\delta\right)\quad \text{if } \nu\in\{3,6\},$$

where $\delta\in(0,\pi/2)$. Let $i,j\in\{1,\cdots,m\}$ such that $c_{ij}^\nu\neq 0$, and let

$$g_{ij}^\nu(z,n)=d_{ij}z\log z+\mu_{ij}^\nu z+(n+\rho_{ij})\log z,\qquad z\in\tilde S^\nu,\quad n\in\mathbb{N}.$$

From (2.19)-(2.22) and (2.27)-(2.31) we deduce that conditions (ii)-(iv) of Proposition 1.3 are satisfied, provided δ is chosen sufficiently small. We readily verify that condition (v) holds as well (with $z_0=\tilde z_\nu$).

Next, we consider the function f_{ij}^ν defined by (2.16). The asymptotic properties of h_i^ν imply that

$$\lim_{z\to\infty} h_i^\nu(z)=1 \quad \text{uniformly on } \tilde{S}^\nu. \tag{2.36}$$

Furthermore, from (2.14) and the fact that p_{ij}^ν is analytic on either a lower or an upper half plane it follows that

$$|p_{ij}^\nu(z)\exp(-2n_{ij}^\nu\pi iz)-c_{ij}^\nu|\leqq K\exp(-2\pi|\mathrm{Im}\, z|), \qquad z\in\tilde{S}^\nu, \tag{2.37}$$

where K is a positive constant. From (2.36) and (2.37) it is obvious that f_{ij}^ν is bounded on $\tilde{S}^\nu$. Moreover, with the aid of (2.20) it is easily seen that, in the case that $d_{ij}=0$, f_{ij}^ν satisfies condition (vi) of Proposition 1.3. Now suppose that $\nu\in\{1,2,4,5\}$ and $d_{ij}\neq 0$. Formulas (2.4) and (2.28) imply that $|\mathrm{Im}\, s_n|\to\infty$ as $n\to\infty$, where s_n denotes the saddle point of $g_{ij}^\nu(z,n)$. With (2.36) and (2.37) it follows that, also in this case, condition (vi) of Proposition 1.3 is fulfilled.

Apparently, all conditions of Theorem 1.4 are satisfied. Applying this theorem and using (2.33) and (2.35), we obtain the following result.

THEOREM 2.1. *For each $j\in\{1,\cdots,m\}$ there exists a convergent power series $\sum_{n=0}^\infty h_{jn}z^{-n}$ such that*

$$\hat{h}_{jn}=h_{jn}+\sum_{i=1}^m\sum_{\nu=1}^6 c_{ij}^\nu\{M_{d_{ij},\mu_{ij}^\nu}(n+\rho_{ij})(1+o(1))\}, \qquad n\to\infty,$$

where c_{ij}^ν and M_{d_{ij},μ_{ij}^ν} are defined by (2.14) *and* (2.34), *respectively.*

With the aid of Propositions 1.1 and 1.2 we are able to estimate the growth of the remainder terms $R_n(h_j^\nu;z)$ for $n\to\infty$, $j\in\{1,\cdots,m\}$. Let $\nu\in\{1,\cdots,6\}$. $S_\nu\cap S_{\nu+1}$ is a sector of the form $S(z_\nu,\alpha_\nu,\beta_\nu)$. We begin by considering the functions h_{ij}^ν defined by

$$h_{ij}^\nu(z)=\frac{z}{2\pi i}\int_{\gamma_\nu}\frac{\varphi_{ij}^\nu(\zeta)}{\zeta(\zeta-z)}\,d\zeta, \qquad i,j\in\{1,\cdots,m\}, \quad \nu\in\{1,\cdots,6\}, \tag{2.38}$$

where γ is a half line in $S_\nu\cap S_{\nu+1}$ from z_ν to ∞ and φ_{ij}^ν is defined by (2.17).

PROPOSITION 2.2. *Let $i,j\in\{1,\cdots,m\}$, $\nu\in\{1,\cdots,6\}$. The function h_{ij}^ν defined by* (2.38) *is analytic in $\underline{S_\nu\cap S_{\nu+1}}$ and represented asymptotically by*

$$\sum_{n=0}^\infty -\frac{1}{2\pi i}\left(\int_{\gamma_\nu}\varphi_{ij}^\nu(\zeta)\zeta^{n-1}\,d\zeta\right)z^{-n}$$

as $z\to\infty$ in $\underline{S_\nu\cap S_{\nu+1}}$. Moreover, for every $S'\Subset\underline{S_\nu\cap S_{\nu+1}}$ there exist positive constants $A_{S'}$ and $C_{S'}$ such that, for all $n\in\mathbb{N}$,

$$\sup_{z\in S'}|z^nR_n(h_{ij}^\nu;z)|\leqq\begin{cases}C_{S'}A_{S'}^n n! & \text{if } d_{ij}=0,\\ C_{S'}A_{S'}^n(n/\log n)^n & \text{if } d_{ij}\neq 0.\end{cases} \tag{2.39}$$

Proof. The first two statements follow immediately from Proposition 1.1 and the properties of φ_{ij}^ν. Now let $S'\Subset\underline{S_\nu\cap S_{\nu+1}}$. We can choose a sector $S''\Subset\underline{S_\nu\cap S_{\nu+1}}$ of the form $S''=S(\tilde{z}_\nu,\tilde{\alpha}_\nu,\tilde{\beta}_\nu)$ such that $S'\Subset\underline{S''}$. Let $\tilde{\gamma}_\nu$ be a half line in S'' from $\tilde{z}_\nu$ to ∞ and

$$\tilde{h}_{ij}^\nu(z)=\frac{z}{2\pi i}\int_{\tilde{\gamma}_\nu}\frac{\varphi_{ij}^\nu(\zeta)}{\zeta(\zeta-z)}\,d\zeta.$$

As $h_{ij}^\nu-\tilde{h}_{ij}^\nu$ is holomorphic at ∞, it is obviously sufficient to prove (2.39) for $\tilde{h}_{ij}^\nu$ instead of h_{ij}^ν. Using (2.18), (2.23), and (2.32) and noting that, due to (2.16), (2.36), and (2.37),

f_{ij}^{ν} is bounded on S'', we conclude that there exist positive numbers $A_{S''}$ and $C_{S''}$ such that, for all $n \in \mathbb{N}$,

$$\sup_{z\in S''} |z^n \varphi_{ij}^{\nu}(z)| \leqq \begin{cases} C_{S''}A_{S''}^n n! & \text{if } d_{ij}=0, \\ C_{S''}A_{S''}^n (n/(\log n))^n & \text{if } d_{ij}\neq 0. \end{cases}$$

The result now follows by application of Proposition 1.1.

THEOREM 2.3 (cf. also [7]). *Let* $j \in \{1, \cdots, m\}$, $\nu \in \{1, \cdots, 6\}$. *For every* $S' \Subset S_\nu$ *there exist positive constants* $A_{S'}$ *and* $C_{S'}$ *such that*

$$\sup_{z\in S'} |z^n R_n(h_j^\nu; z)| < C_{S'}A_{S'}^n n!, \qquad n \in \mathbb{N}$$

Moreover, if the numbers c_{ij}^{μ} *defined by* (2.14) *vanish for all* $i \in \{1, \cdots, m\}$ *such that* $d_{ij}=0$ *and all* $\mu \in \{1, \cdots, 6\}$, *then there exist positive constants* $\tilde{C}_{S'}$ *and* $\tilde{A}_{S'}$ *such that*

$$\sup_{z\in S'} |z^n R_n(h_j^\nu; z)| < \tilde{C}_{S'}\tilde{A}_{S'}^n \left(\frac{n}{\log n}\right)^n.$$

Proof. Using (2.11), (2.12), and the definitions (2.17) and (2.38), and applying Proposition 1.2, we conclude that there exists a function h_j, holomorphic at ∞, such that

$$h_j^\nu(z) = h_j(z) + \sum_{i=1}^m \left\{ \sum_{\mu=1}^{\nu-1} h_{ij}^\nu(z) + \sum_{\mu=\nu}^{6} h_{ij}^\mu(z\, e^{2\pi i}) \right\}.$$

Thus the statements of the theorem are seen to be an immediate corollary of Proposition 2.2.

To conclude this section we shall apply the above results to the second-order difference equation

$$\{(z+2)^2+\alpha(z+2)+\beta\}y(z+2)-\{(z+1)^2+\gamma(z+1)^2+\gamma(z+1)+\delta\}y(z+1)+\sigma y(z)=0,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\sigma \in \mathbb{C}^*$ (this is a particular case of the class of equations considered in [1]). This equation possesses two formal solutions $\hat{y}_1$ and $\hat{y}_2$ of the form

$$\hat{y}_1(z) = \hat{h}_1(z) z^{\gamma-\alpha-2},$$
$$\hat{y}_2(z) = \hat{h}_2(z) z^{-\gamma-2} \exp\{-2z \log z + (2+\log\sigma)z\},$$

where $\hat{h}_j(z) = \sum_{n=0}^{\infty} \hat{h}_{jn} z^{-n}$ with $\hat{h}_{j0}=1$, $j=1, 2$. Thus we have

$$\rho_{12} = 2\gamma-\alpha = -\rho_{21}, \quad d_{12}=2=-d_{21}, \quad \mu_{12} = -(2+\log\sigma) = -\mu_{21}.$$

Assumption (2.4) is equivalent to

$$\arg\sigma \neq 0 \bmod 2\pi.$$

We shall choose $\arg\sigma \in (0, 2\pi)$. With (2.5) it follows that $n_{12}=n_{21}=0$. Hence, by (2.9) the following limits exist:

$$\lim_{\operatorname{Im} z\to-\infty} p_{21}^1(z)\exp 2\pi i z \quad \text{and} \quad \lim_{\operatorname{Im} z\to\infty} p_{21}^2(z).$$

From these and other considerations, based on the particular form of the equation, it can be deduced that the periodic functions $p_{21}^1, p_{21}^2, p_{11}^3$ and p_{11}^6 must be of the following form:

$$p_{11}^3(z) = \frac{c_{11}^3 \exp 2\pi i z + (\exp 2\pi i y - \exp 2\pi i \alpha)\exp 4\pi i z}{(1-\exp 2\pi i(z-a))(1-\exp 2\pi i(z-b))}, \tag{2.40}$$

$$p_{11}^6(z) = (1+p_{11}^3(z))^{-1} \exp 2\pi i(\gamma-\alpha) - 1, \tag{2.41}$$

(2.42)

$$p_{21}^1(z)=-p_{21}^2(z)$$
$$=\frac{-c_{21}^2+c_{21}^1\exp 2\pi i(z-\gamma)}{1+\{c_{11}^3-\exp(-2\pi ia)-\exp(-2\pi ib)\}\exp 2\pi iz+\exp 2\pi i(\gamma+2z)},$$

where a and b denote the roots of the polynomial $z^2+\alpha z+\beta$, and c_{21}^1, c_{21}^2, and c_{11}^3 are defined by (2.14). From (2.7), (2.9), and (2.10) it is seen that $c_{11}^\nu=0$ for $\nu\in\{1,2,4,5\}$ and $c_{21}^\nu=0$ for $\nu\in\{3,4,5,6\}$. According to Theorem 2.1 there exists a convergent power series $\sum_{n=0}^\infty h_{1n}z^{-n}$ such that

$$\begin{aligned}\hat{h}_{1n}&=h_{1n}+c_{11}^3M_{0,\mu_{11}^3}(n)(1+o(1))+c_{11}^6M_{0,\mu_{11}^6}(n)(1+o(1))\\ &\quad+c_{21}^1M_{-2,\mu_{21}^1}(n+\alpha-2\gamma)(1+o(1)\\ &\quad+c_{21}^2M_{-2,\mu_{21}^2}(n+\alpha-2\gamma)(1+o(1)),\qquad n\to\infty.\end{aligned}\tag{2.43}$$

From (2.40)-(2.42) we deduce, with (2.13), that $n_{11}^3=-n_{11}^6=1$, $n_{21}^1=-1$, $n_{21}^2=0$ and hence, with (2.15), that

$$\mu_{11}^3=-\mu_{11}^6=2\pi i,\quad \mu_{21}^1=2+\log\sigma-2\pi i,\quad \mu_{21}^2=2+\log\sigma.$$

Using (2.34), we find

$$M_{0,\mu_{11}^3}(n)=(-1)^nM_{0,\mu_{11}^6}(n)=\Gamma(n)(-2\pi i)^{-n-1}.$$

As the dominating terms in (2.43) are the ones with coefficients c_{11}^3 and c_{11}^6 we conclude that

$$c_{11}^3+c_{11}^6=-\lim_{n\to\infty}\frac{\hat{h}_{12n}(2\pi i)^{2n+1}}{(2n-1)!},$$

$$c_{11}^3-c_{11}^6=\lim_{n\to\infty}\frac{\hat{h}_{12n+1}(2\pi i)^{2n+2}}{(2n)!}.$$

If $c_{11}^3=0$, then, by (2.41), $p_{11}^6\equiv\exp 2\pi i(\gamma-\alpha)-1$ and, in view of (2.8), this implies $c_{11}^6=0$ and $\gamma-\alpha\in\mathbb{Z}$. In that case (2.42) becomes

$$p_{21}^1(z)=-p_{21}^2(z)=\frac{-c_{21}^2+c_{21}^1\exp 2\pi i(z-\alpha)}{(1-\exp 2\pi i(z-a))(1-\exp 2\pi i(z-b))}$$

(where we have used the identity $a+b=-\alpha$), and the coefficient c_{21}^ν, $\nu\in\{1,2\}$, of the dominating term in (2.43) may be determined from the asymptotic behavior of $\hat{h}_{1n}$ for $n\to\infty$.

On the other hand, if $c_{11}^3\neq 0$, then the coefficients c_{21}^1 and c_{21}^2 cannot be determined by this method.

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