108. On the Asymptotic Behaviors of Solutions of Difference Equations. II

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As for the applications of Lyapunov functions to the stability problems of difference equations with discrete variable, we can find some results in [2, 3, 5], and [4] concerning the criteria of Popov type for the absolute stability. In this paper, we shall show some other results including the construction of Lyapunov functions, that is, the so-called converse theorems, and the applications to perturbed systems.

The following is a result to show the existence of Lyapunov functions for linear systems, which will be often used to discuss the stability problems for perturbed systems.

Theorem 1. *Suppose that $A(t)$ be an $n\times n$ matrix defined for $t\in I_\infty$, and the trivial solution of*

(1) $$x(t+1)=A(t)x(t), \quad x(t_0)=x_0, \quad t\geqq t_0$$

is generalized exponentially asymptotically stable, where I_∞ is a set of nonnegative integers and $t_0\in I_\infty$. Then there exists a function $V(t,x)$ satisfying the following conditions:

(a) *$V(t,x)$ is defined for $t\in I_\infty$ and $|x|<\infty$, Lipschitzian in x for a function $K(t)$;*

(b) $$|x|\leqq V(t,x)\leqq K(t)|x|, \quad t\in I_\infty, \quad |x|<\infty;$$

(c) *for any solution $x(t)$ of* (1),

$$\Delta V(t,x(t))\leqq -(1-\exp(-\Delta p(t)))V(t,x(t)), \qquad t\geqq t_0.$$

This theorem will be proved by an analogous method as in differential equations, if we define a function $V(t,x)$ such that

$$V(t,x)=\sup_{\sigma\in I_\infty}|x(t+\sigma,t,x)|\,e^{p(t+\sigma)-p(t)}.$$

For the definition of the generalized exponentially asymptotic stability, see [1].

Theorem 2. *Suppose that*

(i) *$A(t)$ is defined for $t\in I_\infty$, and the trivial solution of* (1) *is generalized exponentially asymptotically stable;*

(ii) *$F(t,x)$ is defined for $t\in I_\infty$ and $|x|<\rho$, and $|F(t,x)|\leqq g(t,|x|)$, $t\in I_\infty$, $|x|<\rho$, where $g(t,r)$ is defined for $t\in I_\infty$ and $0\leqq r<\infty$, $g(t,0)\equiv 0$, and nondecreasing in r for any t.*

Then the stability or asymptotic stability of the trivial solution of

(2) $\Delta r(t) = -(1-\exp(-\Delta p(t)))r(t) + K(t+1)g(t, r(t)),\ r(t_0) = r_0 \geqq 0$
implies the stability or asymptotic stability of the perturbed system

$$(3) \qquad x(t+1) = A(t)x(t) + F(t, x(t)).$$

The proof of this theorem will be completed, if we obtain an inequality $V(t, x(t, t_0, x_0)) \leqq r(t, t_0, x_0)$, provided $V(t_0, x_0) \leqq r_0$, where $V(t, x)$ is a Lyapunov function satisfying the conditions in Theorem 1.

The following result has been proved in [3] by using the properties of fundamental matrices for linear systems. But, Lyapunov function obtained in Theorem 1 can be applied to prove it.

Theorem 3. *Suppose that*
(i) *the trivial solution of* (1) *is exponentially asymptotically stable*;
(ii) *the function* $F(t, x)$ *satisfies an inequality* $|F(t, x)| \leqq c|x|$, $t \in I_\infty$, $|x| < \rho$ *for a sufficiently small constant c.*

Then the trivial solution of (3) *is also exponentially asymptotically stable.*

As in differential equations, the following two results show the eventual stability of the trivial solution, and Lyapunov functions will be effectively applied to prove them.

Theorem 4. *Suppose that*
(i) *the condition* (i) *in Theorem* 3 *is satisfied*;
(ii) $F(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, *and for any given* $\varepsilon > 0$ *there exist* $\delta(\varepsilon)$ *and* $T(\varepsilon)$ *such that*

$$|F(t, x)| \leqq \varepsilon |x|, \quad t \geqq T(\varepsilon), \quad |x| < \delta(\varepsilon);$$

(iii) $G(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, $G(t, 0) \equiv 0$, *and there exists an* $\eta > 0$ *such that*

$$|G(t, x)| \leqq \gamma(t), \quad t \in I_\infty, \quad |x| < \eta,$$

where $\gamma(t) \to 0$ *as* $t \to \infty$.

Then there exists a T_0 *such that, for any* $t_0 \geqq T_0$, *the trivial solution of*

$$x(t+1) = A(t)x(t) + F(t, x(t)) + G(t, x(t)), \quad x(t_0) = x_0, \quad t \geqq t_0$$

is asymptotically stable.

Theorem 5. *Suppose that*
(i) $f(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, $f(t, 0) \equiv 0$, $f_x(t, x)$ *exists, and for any given* $\varepsilon > 0$ *there exist* $\delta(\varepsilon)$ *and* $T(\varepsilon)$ *such that* $|f(t, x) - f_x(t, 0)x| \leqq \varepsilon |x|$, *whenever* $|x| < \delta(\varepsilon)$ *and* $t \geqq T(\varepsilon)$;
(ii) *the trivial solution of* $\Delta x(t) = f_x(t, 0)x(t)$ *is exponentially asymptotically stable*;
(iii) $G(t, x)$ *is defined as in* (iii) *of Theorem* 4.

Then there exists a T_0 *such that, for any* $t_0 \geqq T_0$, *the trivial solution of*

$$\Delta x(t) = f(t, x(t)) + F(t, x(t)) + G(t, x(t)),\ x(t_0) = x_0, \qquad t \geqq t_0$$

is asymptotically stable.

The following three results show the direct applications of Lyapunov functions to perturbed systems.

Theorem 6. *Suppose that*

(i) *$f(t,x)$ and $F(t,x)$ are defined for $t\in I_\infty$ and $|x|<\infty$;*

(ii) *$V(t,x)$ is defined for $t\in I_\infty$ and $|x|<\infty$, Lipschitzian in x for a function $K(t)$, and*

$$a(|x|)\leqq V(t,x)\leqq b(|x|), \quad t\in I_\infty, \quad |x|<\infty,$$

where $a(r)$ and $b(r)$ are defined for $0\leqq r<\infty$, continuous, strictly monotone increasing, and $a(0)=b(0)=0$;

(iii) *$g(t,r)$ is defined for $t\in I_\infty$ and $0\leqq r<\infty$, nondecreasing in r, $g(t,0)\equiv 0$, and*

$$\Delta V(t,x(t))\leqq g(t,V(t,x(t))), \qquad t\geqq t_0,$$

where $x(t)$ is an arbitrary solution of

$$x(t+1)=f(t,x(t)), \quad x(t_0)=x_0, \quad t\geqq t_0;$$

(iv) *$w(t,r)$ is defined for $t\in I_\infty$ and $0\leqq r<\infty$, nondecreasing in r, $w(t,0)\equiv 0$, and*

$$|F(t,x)|\leqq w(t,|x|), \quad t\in I_\infty, \quad |x|<\infty.$$

Then the stability properties of the trivial solution of

$$\Delta r(t)=g(t,r(t))+K(t+1)w(t,a^{-1}(r(t)))$$

implies the corresponding stability properties of the trivial solution of

$$x(t+1)=f(t,x(t))+F(t,x(t)), \quad x(t_0)=x_0, \quad t\geqq t_0.$$

Theorem 7. *Suppose that*

(i) *$f(t,x)$ is defined for $t\in I_\infty$ and $|x|<\rho$, $f(t,0)\equiv 0$, $f_x(t,x)$ exists, and for any given $\varepsilon>0$, there exists a $\delta(\varepsilon)$ such that $|f(t,x)-f_x(t,0)x|\leqq\varepsilon|x|$ uniformly in $t\in I_\infty$, provided $|x|<\delta(\varepsilon)$;*

(ii) *$V(t,x)$ is defined for $t\in I_\infty$ and $|x|<\rho$, Lipschitzian in x for a constant $K>0$ and*

$$|x|\leqq V(t,x)\leqq K|x|, \quad t\in I_\infty, \quad |x|<\rho;$$

(iii) *for any solution $y(t)$ of $\Delta y(t)=f_x(t,0)y(t)$, an inequality*

$$\Delta V(t,y(t))\leqq\alpha(t)V(t,y(t)), \qquad t\geqq t_0$$

is satisfied, where

$$\overline{\lim_{t\to\infty}}\frac{1}{t-t_0}\sum_{s=t_0}^{t-1}\alpha(s)<0.$$

Then the trivial solution of $\Delta x(t)=f(t,x(t))$, $x(t_0)=x_0$, $t\geqq t_0$, is asymptotically stable.

Theorem 8. *Suppose that*

(i) *$V(t,x)$ is defined for $t\in I_\infty$ and $|x|<\rho$, and*

$$|x|\leqq V(t,x)\leqq K|x|, \quad t\in I_\infty, \quad |x|<\rho,$$

where K is a positive constant;

(ii) *for any solution of*

$$\Delta x(t)=f(t,x(t))+F(t,x(t)) \tag{4}$$

such that $|x(t)|<\rho$, where $F(t,x)$ is defined for $t\in I_\infty$ and $|x|<\rho$, there *holds an inequality*

$$\Delta V(t, x(t)) \leqq -c|x(t)|, \qquad t \geqq t_0,$$

where c is a positive constant such that $c < K$;

(iii) $w(t, r)$ *is defined for* $t \in I_\infty$ *and* $0 \leqq r < \infty$, $w(t, 0) \equiv 0$, *nondecreasing in r, and*

$$|F(t, x)| \leqq w(t, |x|), \quad t \in I_\infty, \quad |x| < \rho.$$

Then the stability properties of the trivial solution of

$$\Delta r(t) = -\frac{c}{K} r(t) - w(t, r(t))$$

implies the corresponding properties of the trivial solution of (4).

The following is a generalization of a result asserting the l^p-stability.

Theorem 9. *Suppose that there exists a function* $V(t, x)$ *satisfying the following conditions*:

(i) $V(t, x)$ *is defined and nonnegative for* $t \in I_\infty$ *and* $|x| < \infty$;

(ii) *for any solution* $x(t)$ *of* $x(t+1) = f(t, x(t))$, $x(t_0) = x_0$, *where* $f(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \infty$, *an inequality*

$$\Delta V(t, x(t)) + a(|x(t)|) \leqq g(t, V(t, x(t))), \qquad t \geqq t_0$$

is satisfied, where $a(r)$ *is the same function as before, and* $g(t, r)$ *is defined for* $t \in I_\infty$ *and* $0 \leqq r < \infty$, *and nondecreasing in r.*

Then an inequality

$$V(t, x(t)) + \sum_{s=t_0}^{t-1} a(|x(s)|) \leqq r(t), \qquad t \geqq t_0$$

is fulfilled, provided $V(t_0, x_0) \leqq r(t_0)$, *where* $r(t)$ *is a solution of* $\Delta r(t) = g(t, r(t))$, $r(t_0) = r_0$, $t \geqq t_0$.

In this result, the l^p-stability corresponds to the case where $a(r) = cr^p$ (c is a positive constant) and $g(t, r) \equiv 0$.

References

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