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On the asymptotic behaviour and smoothness properties of some positive linear operators for the approximation of continuous functions

R. J. Crawford
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ON THE ASYMPTOTIC BEHAVIOUR AND
SMOOTHNESS PROPERTIES OF SOME
POSITIVE LINEAR OPERATORS FOR THE
APPROXIMATION OF CONTINUOUS FUNCTIONS

R.J. CRAWFORD



M.Sc. THESIS
DECEMBER, 1972

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This thesis is submitted for the degree of Master of Science to the Wollongong University College in the University of New South Wales, and has not been submitted for a higher degree to any other University or institution.

Raymond J. Crawford
RAYMOND J. CRAWFORD

DECEMBER, 1972

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I am also grateful to Professor B. Mond for furnishing me with copies of two of his papers which enabled me to add a small but significant chapter on Degree of Approximation.

Finally, I would like to thank Mrs. L. Bott for typing the script so ably.

ABSTRACT

The uniform approximation of continuous functions of a real variable on intervals, through positive linear operators, is an active field of research at present. This study first surveys the various linear operators used in this connection, e.g. Operators of Bernstein, Szasz, Baskakov, Meyer-Konig and Zeller as well as those of Shah and Suryanarayana and the various results obtained from them.

The asymptotic form of most of these approximating operators has been studied. These results are analogous to those obtained by Sikkema with regards to the Meyer-Konig and Zeller operator.

Using the method of Shisha and Mond a quantitative estimation of the closeness of approximation of these operators is then obtained.

Finally it is examined whether the derivatives of these operators approximate uniformly the corresponding derivatives of the function. It has been shown that for the Meyer-Konig and Zeller, Baskakov and Szasz operators, they have the property that the v^{th} derivative of a function can be approximated uniformly by the v^{th} derivative of the operator.

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CHAPTER 1

INTRODUCTION

§

SURVEY OF VARIOUS RESULTS

CHAPTER 1

§1.1

The problem of uniform approximation of continuous functions of a real variable, on bounded closed intervals by polynomials, was originated by Weirstrass in 1885. The use of positive linear operators in connection with approximations has been a very active field of research since then, particularly in recent years.

S. Bernstein [1], O. Szasz [34], A. Lupas [23, 24, 25], M. Muller [25, 29], Meir & Sharma [26], Jakimovski & Leviatation [13, 14], Shah & Suryanarayana [30] are but a few of the names one associates with the work currently being carried out in this field. These papers introduce new positive linear operators and study their properties. We may study the following.

(i) If the operator approximates the function, whether, for continuously differentiable functions, the derivatives of the operator approximate the derivatives of the function.

(ii) Whether asymptotic expressions for these operators could be obtained, indicating the order of accuracy of these approximations.

(iii) Degree of convergence - How rapidly does the operator approach the function?

(iv) Whether these could be extended to include complex valued functions.

(v) Whether the operator is a member of some generalized family of positive linear operators.

(vi) Whether there is any connection of these operators with summability theory.

We give a brief survey of the work which has been done and then move onto further new results.

§1.2 Bernstein Polynomials

1.2.1 Definition

In 1912, Serge Bernstein introduced a polynomial "to demonstrate the Theorem of Weirstrass". This polynomial, the Bernstein polynomial, is defined as

$$\begin{aligned} B_n(f;x) &= \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad n=1,2, \\ &\equiv \sum_{k=0}^n b_{n,k}(x) f\left(\frac{k}{n}\right) \end{aligned} \tag{1}$$

$$\text{where } b_{n,k}(x) \equiv \binom{n}{k} x^k (1-x)^{n-k}.$$

It was shown that $\lim_{n \rightarrow \infty} B_n(f;x) = f(x)$ uniformly on $[0,1]$

Many properties of this polynomial have recently been studied in a book by G.G. Lorentz, "Bernstein Polynomials". [22]

1.2.2 Differentiability

The derivatives of an operator are a key to its shape preserving properties. Lupas [23, 24] in 1967 examined these properties for various operators and showed that for the positive linear operator

$$L_n(f; x) = \sum_{k=0}^{\infty} \frac{(-1)^k \phi_n^{(k)}(x) x^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0, a], \quad a > 0 \quad (1)$$

the $(\alpha+1)$ th derivative may be written as

$$\begin{aligned} L_n^{(\alpha+1)}(f; x) \\ = \frac{(\alpha+1)!}{n^{\alpha+1}} \sum_{k=0}^{\infty} (-1)^{k+\alpha+1} \frac{\phi_n^{(k+\alpha+1)}(x)}{k!} \left[\frac{k}{n}, \frac{k+1}{n}, \dots, \frac{k+\alpha+1}{n}; f \right] x^k \end{aligned} \quad (2)$$

where $\left[\frac{k}{n}, \frac{k+1}{n}, \dots, \frac{k+\alpha+1}{n}; f \right]$ is the $(\alpha+1)$ th divided difference of f .

When $\phi_n(x) \equiv (1-x)^n$, and $a=1$,

$$L_n^{(\alpha+1)}(f; x) \equiv B_n^{(\alpha+1)}(f; x).$$

The expression

$$B_n^{(\alpha)}(f; x) = \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{\alpha-1}{n}\right) \sum_{k=0}^{n-\alpha} f^{(\alpha)}\left(\frac{k}{n} + \theta_k \cdot \frac{\alpha}{n}\right) b_{n-\alpha, k}(x)$$

$$0 < \theta_k < 1,$$

was derived in Bernstein's initial paper of 1912, and used to show that whenever the derivative $f^{(\alpha)}$ exists and is continuous in $[0, 1]$, then

$$\lim_{n \rightarrow \infty} B_n^{(\alpha)}(f; x) = f^{(\alpha)}(x) \quad \text{uniformly on } [0, 1] \quad (3)$$

1.2.3 Asymptotic Relations

Voronowskaja [35] in 1932 considered approximations on the interval $[0,1]$;

"If $f(x)$ is bounded on $[0,1]$, then at every point x where the second derivative exists

$$\lim_{n \rightarrow \infty} n[B_n(f;x) - f(x)] = \frac{x(1-x)}{2} f^{(2)}(x). \quad (1)$$

Bernstein [1], in the same year wrote an addendum to this paper [35] when he considered

$$Q_n(f;x) = \sum_{k=0}^n \left[f\left(\frac{k}{n}\right) - \frac{x(1-x)}{2n} f^{(2)}\left(\frac{k}{n}\right) \right] b_{n,k}(x).$$

He showed that if $|f(x)| < M$ in the interval $[0,1]$ and the fourth derivative $f^{(4)}(x)$ exists at the point x , then

$$\begin{aligned} \lim_{n \rightarrow \infty} n^2 [Q_n(f;x) - f(x)] \\ = x(1-x)(1-2x) \frac{f^{(3)}(x)}{3!} - 3\{x(1-x)\}^2 \frac{f^{(4)}(x)}{4!} \end{aligned} \quad (2)$$

The result has since been generalized [22], to

$$\begin{aligned} \lim_{n \rightarrow \infty} n^k [B_n(f;x) - f(x) - \sum_{s=1}^{2k-1} \frac{n^{-s}}{s!} T_{n,s}(x) f^{(s)}(x)] \\ = \left\{ \frac{1}{2} x(1-x) \right\}^k \frac{f^{(2k)}(x)}{k!} \end{aligned}$$

where the $T_{n,s}(x)$ may be obtained from either

$$n^{-2k} T_{n,2k}(x) = \frac{2k!}{2^k k!} \{x(1-x)\}^k n^{-k} + o(n^{-k-1})$$

or the recursion formula

$$T_{n,r+1}(x) = x(1-x)[T_{n,r}^{(1)}(x) + nrT_{n,r-1}(x)].$$

P.L. Butzer [4] contributed greatly to this strand of work.

1.2.4 Degree of Convergence

Popoviciu in 1935 showed that

$$|B_n(f;x) - f(x)| \leq \frac{3}{2}\omega(n^{-\frac{1}{2}})$$

where ω is the modulus of continuity.

Cimoca and Lupas [8] verify this result using the results of a generalized polynomial.

1.2.5 Generalization

There are several generalizations of the Bernstein Polynomial some of which reduce to $B_n(f;x)$ as defined in 1.2.1 while others reduce to similar linear positive operators of Bernstein type.

In 1966 A. Jakimovski and D. Leviatan [13], defined the sequence

$$B_n(f;x,t) = \sum_{k=0}^n \binom{n+t}{n-k} (1-x)^{n-k} x^{k+t} f\left(\frac{k+t}{n+t}\right),$$

where $B_n(f;x,0) \equiv B_n(f;x)$.

A. Meir and A. Sharma in 1967, [26] developed a Bernstein polynomial using polynomial coefficients;

$$B_n^{(\alpha)}(f; x, t) = \frac{1}{L_n^{(\alpha)}(t)} \sum_{k=0}^n \binom{n+\alpha}{k+\alpha} L_k^{(\alpha)}\left(\frac{t}{x}\right) (1-x)^{n-k} x^k f\left(\frac{k}{n}\right)$$

$\alpha > -1, t \leq 0, \quad (1)$

where $L_n^{(\alpha)}(t)$ are Laguerre polynomials and

$$B_n^{(0)}(f; x, 0) \equiv B_n(f; x).$$

An operator which generalizes other common operators as well as the Bernstein, is that of Lupas [23, 24]

$$L_n(f; x) = \sum_{k=0}^{\infty} (-1)^k \phi_n^{(k)}(x) \frac{x^k}{k!} f\left(\frac{k}{n}\right),$$

the derivative of which is given in 1.2.2(2). Again the substitution, $\phi_n(x) \equiv (1-x)^n$, gives us $L_n(f; x) \equiv B_n(f; x)$.

Further generalizations of the Bernstein operator are examined by Jakimovski and Ramanujan [16], Cimoca and Lupas [8], Ibragimov and Gadziev [11] as well as Boehme and Powell [2].

1.2.6 Summability Matrices

Two matrices of importance in Summability Theory are those of Euler and Taylor. J.P. King [17] has shown that the $b_{n,k}(x)$, as defined in 1.2.1, generate these matrices. In particular

$$b_{n,k}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

generates the Euler matrix and

$$b_{n,k}(x) = \begin{cases} 0 & \text{if } k < n \\ \binom{n}{k} x^{k-n} (1-x)^{n+1} & \text{if } k \geq n \end{cases}$$

the Taylor matrix.

Similar work on Summability Theory may be found in Boehme and Powell [2], Meir and Sharma [26] and King et al. [18], [19].

1.2.7 Extension to the Complex Plane

Gergen, Dressel and Purcell [9] in 1962 extended 1.2.1 to hold in the complex plane; their theorem

"If $f(z)$, $z = x + iy$, is analytic in the interior E of the ellipse with foci at $z = 0$ and $z = 1$ then

$$B_n(f; z) \rightarrow f(z) \text{ as } n \rightarrow \infty \text{ on } E,$$

this convergence being uniform on each closed subset of E ." (1)

Nearly forty years elapsed before the first of the new operators was presented.

§1.3 Szasz Operators

1.3.1 Definition

Otto Szasz [34] in 1950 extended the range of polynomial approximations to the infinite interval $[0, \infty)$ by introducing the operator

$$S_n(f; x) \equiv e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \quad n > 0, x \in [0, a] \quad 0 < a < \infty$$

$$\equiv \sum_{k=1}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right)$$

$$\text{where } s_{n,k}(x) \equiv e^{-nx} \frac{(nx)^k}{k!}$$

He showed that this operator has properties analogous to those of the Bernstein polynomial, but for the infinite interval.

Lupas [23] in 1967 showed that

$$\lim_{n \rightarrow \infty} S_n(f; x) = S(f; x) \equiv \sum_{k=1}^{\infty} f^{(k)}(0) \frac{x^k}{k!}, \text{ if } f^{(k)}(0), k \in \omega \text{ exists,}$$

and so called it a "quasi" Maclaurin Operator.

1.3.2 Differentiability

The differential relationship 1.2.2(2), given by Lupas [4] also holds for the Szasz operator with the substitution

$$\phi_n(x) = e^{-nx}, \quad x \in [0, a], \quad 0 < a < \infty.$$

The analogue of 1.2.2(3), given by Szasz [34] is the Theorem

"If $f(x)$ is r -times differentiable, $f^{(r)}(x) = O(x^k)$ as $x \rightarrow \infty$ for some $k > 0$ and if $f^{(r)}(x)$ is continuous at $x = \xi$ then $S_n^{(r)}(f; x)$ converges to $f^{(r)}(x)$ at $x = \xi$ ". (1)

This result was obtained by Jakimovski and Leviatan [15] with the weaker restriction $|f(t)| \leq e^{At}$, $t \geq 0$ for some finite A .

1.3.3 Asymptotic Relations

Szasz analogue of 1.2.3(1) is Theorem 6 of his paper [34] "If $f(x)$ is bounded in every finite interval, if it is twice differentiable at a point $\xi > 0$, and if for some $k > 0$, $f(x) = O(x^k)$ $x \rightarrow \infty$ then

$$\lim_{n \rightarrow \infty} n \{S_n(f; \xi) - f(\xi)\} = \frac{1}{2} \xi f^{(2)}(\xi)" \quad (1)$$

1.3.4 Degree of Convergence

Two of the theorems in Szasz paper we may consider as giving us statements as to the order of accuracy of the Szasz operator. Firstly,

Theorem 1, [34];

$$\limsup |S_n(f; x) - f(x)| = O\{m(2\delta)\}$$

where $|x - \xi| < \delta$ and $\max |f(x) - f(\xi)| = m(\delta)$,

Theorem 5, [34];

This is an asymptotic result;

"If $S_n(f; x) \rightarrow f(x)$ uniformly as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \sqrt{n} \{S_n(f; \xi) - f(\xi)\} = 0,$$

if $f(x)$ is bounded in every finite interval and differentiable at a point $\xi > 0$, $f(x) = O(x^k)$."

(1)

It also means that the approximation by $S_n(f; x)$ is $O(\frac{1}{\sqrt{n}})$.

1.3.5 Generalization

The generalization, 1.2.2 (1) by Lupas, with

$$\phi_n(x) = e^{-nx}, \quad x \in [0, a] \quad 0 < a < \infty,$$

is one form for the Szasz operator.

In 1969 Jakimovski and Leviatan [15] defined

$$S_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad n > 0 \quad (1)$$

where the $f(t)$ are defined on $[0, \infty)$ and $p_k(x)$ are Appell polynomials defined by

$$\begin{aligned} g(n)e^{nx} &\equiv \sum_{k=0}^{\infty} p_k(x) n^k \\ \text{and } g(z) &= \sum_{n=0}^{\infty} a_n z^n, \\ p_k(x) &= \sum_{n=0}^k a_n \frac{x^{k-n}}{(k-n)!}. \end{aligned}$$

If $g(z) \equiv 1$; 1.3.5(1) reduces to the Szasz operator.

In 1967 Meir and Sharma [26] generalized both the Szasz operator and the asymptotic results 1.3.3(1) and 1.3.4(1).

Their operator

$$S_n(f; x, t) = e^{-nx} \operatorname{sech}(2t \sqrt{nx}) \sum_{k=0}^{\infty} (-1)^k H_{2k}(it) \frac{(nx)^k}{(2k)!} f\left(\frac{k}{n}\right),$$

where $H_k(t)$ are Hermite polynomials of degree k .

The relationship is

$$S_n(f; x, 0) \equiv S_n(f; x)$$

The two results

$$\lim_{n \rightarrow \infty} \sqrt{n} [S_n(f; \xi, \lambda) - f(\xi)] = \lambda \sqrt{\xi} f^{(1)}(\xi), \quad (2)$$

$$\lim_{n \rightarrow \infty} n [S_n(f; \xi, \lambda) - f(\xi) - \frac{\lambda \sqrt{\xi}}{\sqrt{n}} f^{(1)}(\xi)] = \frac{\lambda^2 + 1}{2} \xi \cdot f^{(2)}(\xi) \quad (3)$$

reduce to 1.3.4(1) and 1.3.3(1) resp. when $\lambda=0$.

A further new generalization is given by G.C. Jain [12] when he defines

$$P_n(f; x, \beta) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) n^x \frac{(nx+k\beta)^{k-1}}{k!} e^{-(nx+k\beta)}$$

The relationship here is

$$P_n(f; x, 0) \equiv S_n(f; x)$$

Other generalizations are in Lupas [24].

1.3.6 Extension to the Complex Plane

Cheney and Sharma [7] in 1963, analysed the behaviour of $S_n(f; z)$ and found it analogous to the $B_n(f; z)$, 1.2.7(1);

"If $f(z)$, $z=x+iy$ is analytic in the interior E of the ellipse with foci at $z=0$ and $z=1$ then

$$S_n(f; z) \rightarrow f(z) \text{ as } n \rightarrow \infty \text{ on } E,$$

this convergence being uniform on each closed subset of E ."

§1.4 Baskakov Operator

1.4.1 Definition

A.V. Baskakov in 1957 introduced another operator on the infinite interval, with similar properties to Szasz. The operator

$$K_n(f; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right) \quad x \in [0, a] \\ 0 < a < \infty$$

$$\equiv \sum_{k=0}^{\infty} h_{n,k}(x) f\left(\frac{k}{n}\right)$$

$$\text{where } h_{n,k}(x) \equiv \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \text{ and } \sum_{k=0}^{\infty} h_{n,k}(x) = 1.$$

The substitution of

$$\phi_n(x) = (1+x)^{-n}$$

in 1.2.2(1) and 1.2.2(2) gives the operator and its $(\alpha+1)$ th derivative resp.

Ibragimov and Gadziev [11] as well as Lupas [24] have considered generalizations of this operator.

§1.5 Meyer-Konig and Zeller Operator

1.5.1 Definition

A new "family" of Bernstein type approximation for the infinite interval was introduced in 1960 by Meyer-Konig and Zeller [27]. They defined

$$M_n(f; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{k} x^k f\left(\frac{k}{k+n}\right)$$

$$\equiv \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{k+n}\right)$$

$$\text{where } m_{n,k}(x) \equiv (1-x)^{n+1} \binom{k+n}{k} x^k$$

$$\text{and } \sum_{k=0}^{\infty} m_{n,k}(x) = 1$$

$M_n(f; x)$ approaches $f(x)$ uniformly on $[0,1]$.

The essential change is away from $f(\frac{k}{n})$ to $f(\frac{k}{k+n})$.

1.5.2 Differentiability

In a study of shape preserving properties of this operator, Lupas [23], has derived the first, second and third derivatives. Manfred Muller [28], starting with Lupas' first derivative showed;

"If the function f in $[0,1]$ is uniformly differentiable then $\lim_{n \rightarrow \infty} M_n^{(1)}(f; x) = f^{(1)}(x)$ for each point $[0,1]$ ".

1.5.3 Asymptotic Relations

Cheney and Sharma [7] have shown that if at a point $\xi \in [0,1)$, $f(t)$ possesses a finite second derivative,

$$|M_n(f; \xi) - f(\xi)| \leq \frac{1-\xi}{3n} |f^{(2)}(\xi)| + o(n+1)^{-1}; \quad n \rightarrow \infty$$

Muller [29] improved this result to

$$|M_n(f; \xi) - f(\xi)| \leq \frac{\xi(1-\xi)}{2(n+1)} |f^{(2)}(\xi)| + o(n+1)^{-1},$$

only to combine with A. Lupas in 1970 [25] to obtain

$$M_n(f; \xi) - f(\xi) = \frac{\xi(1-\xi)^2}{2n} f^{(2)}(\xi) + o(n^{-1}).$$

In a recent paper P. Sikkema [33] has extended this result to:

"If $f(t) \in [0,1]$, and if $f(t)$ is continuous to the left at $t=1$ and if at $\xi \in [0,1]$, $f(t)$ possesses a finite fourth derivative, $f^{(4)}(\xi)$, we have

$$\begin{aligned} M_n(f; \xi) - f(\xi) &= \frac{\xi(1-\xi)^2}{2n} f^{(2)}(\xi) + \\ &+ \frac{1}{n^2} \left\{ \frac{1}{2} \xi(1-\xi)^2(2\xi-1) f^{(2)}(\xi) + \frac{1}{6} \xi(1-\xi)^3(1-5\xi) f^{(3)}(\xi) \right. \\ &\quad \left. + \frac{1}{8} \xi^2(1-\xi)^4 f^{(4)}(\xi) \right\} \\ &+ o(n^{-2}). \end{aligned}$$

1.5.4 Degree of Convergence

Lupas and Muller [25] showed

$$|M_n(f; x) - f(x)| \leq \frac{31}{27} \omega(n^{-\frac{1}{2}}), \quad n=2,3,\dots$$

where ω is the modulus of continuity.

Sikkema, using the results of Shisha and Mond [31], has improved on this result; if $x \in [0,1]$,

$$|M_n(f;x) - f(x)| \leq \left\{1 + \frac{4}{27} \left(1 - \frac{n^2-5}{4(n^2-1)^2}\right)\right\} \omega(n^{-\frac{1}{2}})$$

$$\leq \frac{31}{27} \omega(n^{-\frac{1}{2}}) \quad n=1,2,\dots$$

and added two more inequalities;

$$|M_n(f;x) - f(x)| \leq \left\{1 + x(1-x)^2 + \frac{x^2(1-x)(2-x)}{n+1}\right\} \omega[(n+1)^{-\frac{1}{2}}],$$

$$n=1,2,\dots$$

and

$$|M_n(f;x) - f(x)| \leq 2\omega \left(\sqrt{\frac{x(1-x)^2}{n+1} + \frac{x^2(1-x)(2-x)}{(n+1)^2}} \right) \quad n=1,2,\dots$$

Other results for both $M_n(f;x)$ and its first derivative $M_n^{(1)}(f;x)$ are given in Muller [29] and Cimoca & Lupas [8].

1.5.5 Generalization

One generalization, analogous to that for the Bernstein polynomial 1.2.5(1), was given in 1963 by Cheney and Sharma [7]

$$M_n(f;x,t) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} L_k^{(n)}(t) f\left(\frac{k}{k+n}\right) x^k$$

where $L_k^{(n)}(t)$ are the Laguerre polynomials of degree k and

$$M_n(f;x,0) \equiv M_n(f;x) .$$

A. Jakimovski and M. Ramanujan [16] showed that if $f(x)$ is continuous in $[\delta,1]$, $\delta > 0$, and $t > 0$

$$M_n(f; x, t) = \sum_{k=0}^{\infty} \binom{k+t}{k-n} (1-x)^{k-n} x^{n+t+1} f\left(\frac{n+t+1}{k+t+1}\right)$$

converges uniformly to $f(x)$ for all x in $[\delta, 1]$, $0 < \delta < 1$.

Further generalizations are investigated in Jakimovski and Leviatan [14], Cimoca and Lupas [8].

§1.6 The Operator Class of Shah and Suryanarayana

1.6.1 Definition

Another form of positive linear operator was introduced in 1965 by Shah and Suryanarayana [30];

$$P_n(f; x) = g(n, x) \sum_{k=0}^{\infty} f\left(\frac{kA_k}{n}\right) \phi_k(n) \frac{x^k}{k!} \quad (1)$$

$$\text{where } \sum_{k=0}^{\infty} g(n, x) \phi_k(n) \frac{x^k}{k!} = 1 \quad (2)$$

and $\{\phi_n(x)\}$ is a sequence of real polynomials satisfying

- (i) $\phi_n(x)$ is a polynomial of degree n
- (ii) It possesses a recurrence relation of the type

$$A_n \phi_n(x) = x \phi_{n-1}(x) + C_n \phi_{n-2}(x)$$
- (iii) All the zeros of $\{\phi_n(x)\}$ lie in $(-\infty, \alpha]$, $0 < \alpha < \infty$ and $\phi_n(x) > 0$ for all n and $x > \alpha$.
- (iv) $\phi_n(x) < (bx)^n$, $x > \alpha$ and $b > 0$.

In particular if

1.6.2

(a) $\phi_n(x) \equiv T_n(x)$ the Chebychev polynomials with recurrence relation

$$\frac{1}{2}T_n(x) = xT_{n-1}(x) - \frac{1}{2}T_{n-2}(x) \quad (1)$$

and $T_n(x) < (2x)^n \quad (2)$

1.6.3

(b) $\phi_n(x) \equiv P_n(x)$ the Legendre polynomials with recurrence relation

$$\frac{n}{2n-1} P_n(x) = xP_{n-1}(x) - \frac{n-1}{2n-1} P_{n-2}(x) \quad (1)$$

and $P_n(x) < (2x)^n \quad (2)$

§1.7

From this survey we can isolate the operators and areas which have undergone most intensive study. Obviously the oldest and most familiar Bernstein operator has been in the vanguard of any new area of study. So the Bernstein operator is the "reference" operator, and the aim is to emulate its results with other operators.

We notice the asymptotic Bernstein results in 1.2.3, the similar results obtained for the Szasz operator 1.3.3, and more recently the analogous results for the Meyer-Konig and Zeller operator in the paper by Sikkema [33].

This is the area of study of our first set of three theorems in Chapter 3; to extend the asymptotic Szasz results to a greater order of accuracy and to include a wider range of functions; to obtain a similar set of asymptotic Baskakov results and to obtain the first asymptotic result, or Voronowskaja type of result for two of the members in the class of Shah and Suryanarayana. They are:

Theorem 1

1.7.1 If $f(x)$, $|f(x)| < Ae^{\beta x}$, is defined on $[0, \infty)$ and has all finite derivatives up to, and including the eighth at a point $x = \xi$, where A , ξ and β are all >0 , then

$$\begin{aligned} S_n(f; \xi) - f(\xi) = & \frac{\xi}{2n} f^{(2)}(\xi) + \frac{1}{n^2} \left\{ \xi \frac{f^{(2)}(\xi)}{3!} + 3\xi^2 \frac{f^{(4)}(\xi)}{4!} \right. \\ & + \frac{1}{n^3} \left\{ \xi \frac{f^{(3)}(\xi)}{3!} + 10\xi^2 \frac{f^{(5)}(\xi)}{5!} + 15\xi^3 \frac{f^{(6)}(\xi)}{6!} \right\} \\ & + \frac{1}{n^4} \left\{ \xi \frac{f^{(5)}(\xi)}{5!} + 25\xi^2 \frac{f^{(6)}(\xi)}{6!} + 15\xi^3 \frac{f^{(7)}(\xi)}{7!} \right. \\ & \left. \left. + 105\xi^4 \frac{f^{(8)}(\xi)}{8!} \right\} + o(n^{-4}) \right\} \end{aligned}$$

where $S_n(f; \xi)$ is as defined in 1.3.1.

The existing theorem in 1.3.3 only admits functions $f(x)$, $|f(x)| = O(x^k)$, and ceases at $o(n^{-2})$.

1.7.2 The Baskakov operator, defined in 1.4.1 would, like the Szasz operator, have asymptotic formulae to any desired order. The asymptotic result for $o(n^{-3})$ is,

Theorem 2

If $f(x), |f(x)| < Ae^{\beta x}$, is defined in $[0, \infty)$ and has finite derivatives up to and including the fourth at a point ξ , where A, ξ and β are all >0 , then

$$\begin{aligned} K_n(f; \xi) - f(\xi) &= \frac{\xi(1+\xi)}{n} \frac{f^{(2)}(\xi)}{2!} + \\ &\frac{\xi(1+\xi)}{n^2} \left\{ (1+2\xi) \frac{f^{(3)}(\xi)}{3!} + 3\xi(1+\xi) \frac{f^{(4)}(\xi)}{4!} \right\} \\ &+ \frac{1}{n^3} \left\{ \xi(1+6(\xi+1)) \frac{f^{(4)}(\xi)}{4!} + 10\xi^2(1+\xi)^2(1+2\xi) \frac{f^{(5)}(\xi)}{5!} \right. \\ &\left. + 15\xi^3(1+\xi)^3 \frac{f^{(6)}(\xi)}{6!} \right\} + o(n^{-3}) \end{aligned}$$

where $K_n(f; \xi)$ is as defined in 1.4.1.

For the operator class of Shah and Suryanarayana;

Theorem 3a

If $f(x), |f(x)| < Ae^{\beta x}$, is defined in the interval $[0, \infty)$ and has finite first and second derivatives at a point ξ ; A, β and ξ all being >0 , then,

1.7.3

$$P_n(f; \xi) - f(\xi) = \frac{\xi}{4n} f^{(2)}(\xi) + o(n^{-1})$$

when the coefficients are the Chebychev polynomials 1.6.2,

and Theorem 3b

1.7.4

$$P_n(f; \xi) - f(\xi) = \frac{\xi}{8n} f^{(2)}(\xi) + o(n^{-1})$$

when the coefficients are the Legendre polynomials 1.6.3.

Closely allied with this asymptotic work is of course the idea of degree of convergence or rapidity of convergence of the actual operators. This next chapter, chapter 4, then, makes use of some lemmas derived for the previous theorems, together with the theorem of Shisha and Mond to prove the following set of three theorems,

Theorem 4a

1.8.1

If $S_n(f; x)$ is the Szasz operator, then at a point $x = \xi$, $\xi \in [a, b]$, $0 < a < b < \infty$, f is continuous in $[a, b]$,

$$|S_n(f; \xi) - f(\xi)| \leq 2\omega\left(\frac{\xi}{n}\right)^{\frac{1}{2}}$$

where ω is the modulus of continuity.

Theorem 4b

1.8.2

If $K_n(f; x)$ is the Baskakov operator, then at a point $x = \xi$, $\xi \in [a, b]$, $0 < a < b < \infty$, f is continuous in $[a, b]$,

$$|K_n(f;\xi) - f(\xi)| \leq 2\omega \left(\frac{\xi(1+\xi)}{n} \right)^{\frac{1}{2}}$$

where ω is the modulus of continuity.

Theorem 4c

If $P_n(f;x)$ is the operator class of Shah and Suryanarayana then at a point $x=\xi$, $\xi \in [a,b]$, $0 < a < b < \infty$, f is continuous in $[a,b]$,

1.8.3

$$|P_n(f;\xi) - f(\xi)| \leq 2\omega \left(\frac{\xi}{2n} \right)^{\frac{1}{2}}$$

when the coefficients are Chebychev polynomials and

Theorem 4d

1.8.4

$$|P_n(f;\xi) - f(\xi)| \leq 2\omega \left(\frac{\xi}{4n} + \frac{\xi^2}{16n^2} \right)^{\frac{1}{2}}$$

when the coefficients are Legendre polynomials.

We notice in §1.2.2, Bernstein considered the v^{th} derivative of his operator, $B_n^{(v)}(f;x)$ and showed that in the limit as $n \rightarrow \infty$, it did approach the v^{th} derivative of $f(x)$, $f^{(v)}(x)$. Szasz has enunciated a similar theorem for the Szasz operator $S_n(f;x)$, but it appears that it has only been shown to be true for the first derivative of the Meyer-Konig and Zeller operator. No studies appear to have been done

regarding the derivatives of the Baskakov operator.

The next set of three theorems in chapter 5 will then;

- (a) improve the Szasz result to include functions in an arbitrary bounded closed interval $[a,b]$,
- (b) extend the M-K and Z result to all derivatives, and
- (c) derive an analogous result for the Baskakov operator.

1.9.1 Theorem 5

If $K_n(f;x)$ is the Baskakov operator, defined in 1.4.1, and $f \in C_0^v[a,b]$ functions which are v times continuously differentiable having compact support, then,

$$\lim_{n \rightarrow \infty} K_n^{(v)}(f;x) = f^{(v)}(x), \quad v = 1, 2, \dots$$

uniformly in any bounded closed interval $[a,b]$, $0 < a < b < \infty$.

Corollary.

If $F = f ; \quad x \in [a-2\varepsilon, b+2\varepsilon], \quad f \in C^v[a,b]$

$= 0$ outside this interval

and \tilde{F} is the regularization of F such that

$$F = f ; \quad x \in [a,b]$$

$$= 0 \quad \text{outside } (a-\varepsilon, b+\varepsilon)$$

$$\tilde{F} \in C_0^\infty(-\infty, \infty)$$

then $\lim_{n \rightarrow \infty} D^{(v)}[K_n(\tilde{F}; x)] = f^{(v)}(x) \quad \text{for all } f \in C^v[a,b]$

$$v = 1, 2, \dots$$

1.9.2 Theorem 6

If $S_n(f;x)$ is the Szasz operator and $F \in C^v[a,b]$
then

$$\lim_{n \rightarrow \infty} S_n^{(v)}(f;x) = f^{(v)}(x), \quad v = 1, 2, \dots$$

uniformly in any bounded closed interval $[a,b]$, $0 < a < b < \infty$

Corollary.

If $F = f$; $x \in [a-2\varepsilon, b+2\varepsilon]$, $f \in C^v[a,b]$

$= 0$ outside this interval

and \tilde{F} is the regularization of F such that

$$\tilde{F} = f \quad x \in [a,b]$$

$= 0$ outside $(a-\varepsilon, b+\varepsilon)$

$$\tilde{F} \in C_0^\infty(-\infty, \infty)$$

then $\lim_{n \rightarrow \infty} D^{(v)}[S_n(\tilde{F}; x)] = f^{(v)}(x)$ for all $f \in C^v[a,b]$

$v = 1, 2, \dots$

1.9.3 Theorem 7

If $M_n(f;x)$ is the Meyer-Konig and Zeller operator,
 $f(x)$ is v times differentiable then

$$\lim_{n \rightarrow \infty} M_n^{(v)}(f;x) = f^{(v)}(x) \quad v = 1, 2, \dots$$

uniformly in $[0,1)$.

CHAPTER 2

PRELIMINARIES

CHAPTER 2

§2.0

In this chapter we collect the essential theorems to be used in the following chapters. Shorter results are classified as lemmas and given prior to the proving of the relevant new theorem.

§2.1 Theorem 2.1

This is Theorem 137 in Hardy [10].

Suppose that $x > 0$ and

$$\begin{aligned} U_m &= U_m(x) \\ &= e^{-x} \frac{x^m}{m!} \quad (m = 0, 1, 2, \dots) \end{aligned}$$

so that $\sum U_m = 1$.

Then (i) the largest U_m is U_M , where

$$M = [x],$$

two terms U_{M-1} and U_M , being equal if M is an integer;

(ii) if $m = M + h$ and

$0 < \delta < 1$, then

$$\sum_{|h| > \delta x} U_m = O(e^{-\gamma x})$$

$$\text{where } \gamma = \frac{1}{3} \delta^2$$

§2.2 Theorem 2.2

This is Theorem 139 in Hardy [10].

Suppose $0 < k < 1$ and

$$\begin{aligned} U_m &= U_m(n) \\ &= k^{n+1} \binom{m}{n} (1-k)^{m-n} \quad (m \geq n) \end{aligned}$$

so that

$$\begin{aligned} \sum U_m &= k^{n+1} \left\{ 1 + (n+1)(1-k) + \frac{(n+1)(n+2)}{2!} (1-k)^2 + \dots \right\} \\ &= 1 \end{aligned}$$

Then (i) the largest U_m is U_M where

$$M = \left[\frac{n}{k} \right],$$

two terms U_{M-1} and U_M being equal if $\frac{n}{k}$ is an integer

(ii) and $0 < \delta < 1$, then

$$\sum_{|h| > \delta n} U_m = O(e^{-\gamma n})$$

where $\gamma = \gamma(k, \delta) > 0$.

§2.3 Theorem 2.3

This is Theorem 1 of Shisha and Mond [31], which is a quantitative form of Korovkin's result.

Let $-\infty < a < b < \infty$, and let L_1, L_2, \dots be linear positive operators, all having the same domain D which contains the

restrictions of $1, t, t^2$ to $[a, b]$. For $n = 1, 2, \dots$, suppose $L_n(1; x)$ is bounded. Let $f \in D$ be continuous in $[a, b]$, with modulus of continuity ω . Then for $n = 1, 2, \dots$,

2.3.1

$$||f - L_n(f)|| \leq ||f|| \cdot ||L_n(1) - 1|| + ||L_n(1) + 1|| \omega(\mu_n),$$

$$\text{where } \mu_n = ||L_n[(t-x)^2; x]||^{\frac{1}{2}},$$

and $|| \cdot ||$ stands for the sup norm over $[a, b]$. In particular, if $L_n(1; x) = 1$, as is often the case, 2.3.1 reduces to

$$||f - L_n(f)|| \leq 2\omega(\mu_n).$$

§2.4 Approximation of a Function by its Regularisation

These are lemmas from §3.2 in the book by Hans Bremermann [3].

2.4.1 Lemma 1.

Let $f(t)$ be continuous on E^n . Then for $r \rightarrow 0$ the regularized function $f * \rho_r$ converges to f uniformly on every compact set in E^n .

The regularization of f , $f * \rho_r$ is defined as

$$(f * \rho_r)(x) = \int_{E^n} f(t) \rho_r(x-t) dt$$

where $\rho_r(t) = \begin{cases} 0 & \text{for } |t| > r \\ \frac{k}{r} \exp[-1/(1-t^2)] & \text{for } |t| \leq r, \end{cases}$ k is a constant chosen so that

$$\int_{E_n} \rho_r(t) dt = 1.$$

2.4.2

Lemma 2

Let f be a (C^m) function on E^n . The $f * \rho_r$ converges to f uniformly up to m^{th} order on every compact subset of E^n .

2.4.3

Regularization of Characteristic Functions

The characteristic function f_s of $[a, b]$ is defined as

$$f_s(x) = \begin{cases} 1 & \text{for } x[a, b] \\ 0 & \text{outside this interval} \end{cases}$$

The regularized function $f_s * \rho_r$ is (C^∞) and equals f_s everywhere except in $(a - \frac{1}{r}, a + \frac{1}{r})$, $(b - \frac{1}{r}, b + \frac{1}{r})$.

CHAPTER 3

NEW ASYMPTOTIC THEOREMS

CHAPTER 3

Asymptotic Relations

§3.1

The methods of proof used to obtain the asymptotic results, expressed as theorems 1, 2, 3a and 3b will all follow the same pattern as outlined below.

If a function $f(x)$ is α times continuously differentiable, then it has a Taylor Series expansion,

3.1.1

$$f(t) = f(x) + \sum_{j=1}^{\alpha} f^{(j)}(x) \frac{(t-x)^j}{j!} + (t-x)^{\alpha} \phi(t)$$
$$\alpha=1,2,\dots$$

where $|\phi(t)| < \epsilon$ for $|t-x| < \delta$ and x in the interval.

Applying P_n to both sides yields

3.1.2

$$P_n(f;x) = f(x) + \sum_{j=1}^{\alpha} \frac{f^{(j)}(x)}{j!} P_n[(t-x)^j;x] + P_n[(t-x)^{\alpha} \phi(t);x]$$
$$\alpha=1,2,\dots$$

If we can show that $O(P_n[(t-x)^{\alpha} \phi(t);x])$ is small compared to

$$P_n(f;x) - f(x) - \sum_{j=1}^{\alpha} \frac{f^{(j)}(x)}{j!} P_n[(t-x)^j;x]$$

then

3.1.3

$$P_n(f;x) - f(x) \sim \sum_{j=1}^{\alpha} \frac{f^{(j)}(x)}{j!} P_n[(t-x)^j;x]$$

becomes an asymptotic result.

§3.2

Before we proceed with the proof of the first asymptotic result, Theorem 1, we will require the following definitions and lemmas.

Definition 1

For the Szasz operator

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}$$

$$\begin{aligned} \text{let } D_m &\equiv S_n(t^m; x) \\ &= \sum_{k=0}^{\infty} \left(\frac{k}{n}\right)^m s_{n,k}(x) \end{aligned}$$

In particular notice

$$D_0 = \sum_{k=0}^{\infty} s_{n,k}(x) = 1$$

3.2.1 Lemma 1

$$\begin{aligned} D_1 &\equiv \sum_{k=0}^{\infty} \left(\frac{k}{n}\right) s_{n,k}(x) \\ &= x. \end{aligned}$$

3.2.2 Lemma 2

$$D_m = x^{2^{m-2}} \sum_{\omega=0}^{m-2} \binom{m-2}{\omega} \left(\frac{2}{n}\right)^{\omega} D_{m-(2+\omega)} + \frac{D_{m-1}}{n}, \quad m=2,3,\dots$$

Proof of Lemma 2

$$D_m \equiv e^{nx} \sum_{k=0}^{\infty} \left(\frac{k}{n}\right)^m \frac{(nx)^k}{k!} \quad (1)$$

$$= e^{-nx} \sum_{k=0}^{\infty} \frac{k^{m-1}}{n^m} (k-1) \frac{(nx)^k}{k!} + \frac{e^{-nx}}{n} \sum_{k=0}^{\infty} \frac{k^{m-1}}{n^{m-1}} \frac{(nx)^k}{k!}$$

$$= \frac{x^2}{n^{m-2}} e^{-nx} \sum_{k=2}^{\infty} k^{m-2} \frac{(nx)^{k-2}}{(k-2)!} + \frac{D_{m-1}}{n}$$

$$= \frac{x^2}{n^{m-2}} e^{-nx} \sum_{k=0}^{\infty} (k+2)^{m-2} \frac{(nx)^k}{k!} + \frac{D_{m-1}}{n}$$

$$= x^2 e^{-nx} \sum_k \frac{(nx)^k}{k!} \sum_{\omega} \binom{m-2}{\omega} \left(\frac{2}{n}\right)^{\omega} \left(\frac{k}{n}\right)^{m-(2+\omega)} + \frac{D_{m-1}}{n}$$

$$= x^2 \sum_{\omega=0}^{m-2} \binom{m-2}{\omega} \left(\frac{2}{n}\right)^{\omega} D_{m-(2+\omega)} + \frac{D_{m-1}}{n} ; \quad m = 2, 3, \dots$$

With D_0 , a consequence of the definition and D_1 as obtained in Lemma 1, the recurrence relation above is used to obtain:

(2)

$$D_2 = x^2 + \frac{x}{n}$$

$$D_3 = x^3 + \frac{3x^2}{n} + \frac{x}{n^2}$$

$$D_4 = x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3}$$

$$D_5 = x^5 + \frac{10x^4}{n} + \frac{25x^3}{n^2} + \frac{15x^2}{n^3} + \frac{x}{n^4}$$

$$D_6 = x^6 + \frac{15x^5}{n} + \frac{65x^4}{n^2} + \frac{90x^3}{n^3} + \frac{31x^2}{n^4} + \frac{x}{n^5}$$

$$D_7 = x^7 + \frac{21x^6}{n} + \frac{140x^5}{n^2} + \frac{350x^4}{n^3} + \frac{301x^3}{n^4} + \frac{63x^2}{n^5} + \frac{x}{n^6}$$

$$D_8 = x^8 + \frac{28x^7}{n} + \frac{266x^6}{n^2} + \frac{1050x^5}{n^3} + \frac{1701x^4}{n^4} + \frac{966x^3}{n^5} + \frac{97x^2}{n^6} + \frac{x}{n^7}$$

Definition 2

For the Szasz operator let

$$S_n[(t-x)^\alpha; x] \equiv S_{n,\alpha}(x).$$

3.2.3 Lemma 3

$$S_n[(t-x)^\alpha; x] = \sum_{v=0}^{\infty} (-1)^v x^v \binom{\alpha}{v} D_{\alpha-v} \quad \alpha=1,2,\dots$$

Proof of Lemma 3

$$S_n[(t-x)^\alpha; x] \equiv \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^\alpha s_{n,k}(x) \quad (1)$$

$$= e^{-nx} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^\alpha \frac{(nx)^k}{k!}$$

$$\begin{aligned}
 &= e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \sum_{v=0}^{\alpha} \binom{\alpha}{v} (-1)^v x^v \left(\frac{k}{n}\right)^{\alpha-v} \\
 &= \sum_{v=0}^{\alpha} (-1)^v x^v \binom{\alpha}{v} \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^{\alpha-v} \\
 &= \sum_{v=0}^{\alpha} (-1)^v x^v \binom{\alpha}{v} D_{\alpha-v} \quad \alpha=1,2,\dots
 \end{aligned}$$

By substituting the required values of α in lemma 3, the following are obtained:

$$S_n[(t-x); x] = 0$$

$$S_n[(t-x)^2; x] = \frac{x}{n}$$

$$S_n[(t-x)^3; x] = \frac{x^2}{n^2}$$

$$S_n[(t-x)^4; x] = \frac{3x^2}{n^2} + \frac{x}{n^3}$$

$$S_n[(t-x)^5; x] = \frac{10x^2}{n^3} + \frac{x}{n^4}$$

$$S_n[(t-x)^6; x] = \frac{15x^3}{n^3} + \frac{25x^2}{n^4} + \frac{x}{n^5}$$

$$S_n[(t-x)^7; x] = \frac{15x^3}{n^4} + \frac{63x^2}{n^5} + \frac{x}{n^6}$$

$$S_n[(t-x)^8; x] = \frac{105x^4}{n^4} + \frac{490x^3}{n^5} + \frac{89x^2}{n^6} + \frac{x}{n^7}$$

These $S_{n,\alpha}(x)$ are a linear combination of the D_v computed in Lemma 2. Each D_v is a linear combination of

the D_i ($i=0, \dots, v-1$), so $S_{n,\alpha}(x)$ is consequently a linear combination of the $S_{n,i}(x)$, ($i=0, \dots, \alpha-1$). The explicit relationship is given by

3.2.4 Lemma 4

$$S_{n,\alpha+1}(x) = S_{n,\alpha}^{(1)}(x) + \alpha \frac{x}{n} S_{n,\alpha-1}(x); \quad \alpha=2,3,\dots$$

Proof of Lemma 4

(1)

From definition 2;

$$S_{n,\alpha}(x) \equiv e^{-nx} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^{\alpha} \frac{(nx)^k}{k!}$$

differentiate both sides to obtain

$$\begin{aligned} S_{n,\alpha}^{(1)}(x) &= -nS_{n,\alpha}(x) - \alpha S_{n,\alpha-1}(x) + e^{-nx} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^{\alpha} nk \frac{(nx)^{k-1}}{k!} \\ &= -\alpha S_{n,\alpha-1}(x) + e^{-nx} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^{\alpha} \frac{(nx)^k}{k!} \left\{\frac{k}{x} - n\right\} \\ &= -\alpha S_{n,\alpha-1}(x) + \frac{n}{x} S_{n,\alpha+1}(x). \end{aligned}$$

Therefore if we rearrange these terms

$$S_{n,\alpha+1}(x) = \frac{x}{n} S_{n,\alpha}^{(1)}(x) + \alpha \frac{x}{n} S_{n,\alpha-1}(x)$$

3.2.5 Lemma 5

$$S_n[(t-x)^{\delta} \phi(t); x] = o(n^{-4}) \quad \text{for } |\phi(t)| < \varepsilon$$

when $|t-x| < \delta, \delta > 0$

Proof of Lemma 5

(1)

$$S_n[(t-x)^8 \phi(t); x] = \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^8 \phi\left(\frac{k}{n}\right) s_{n,k}(x)$$

$$= e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left(\frac{k}{n} - x\right)^8 \phi\left(\frac{k}{n}\right)$$

$$= \sum_{\left|\frac{k}{n} - x\right| < \delta} \left(\frac{k}{n} - x\right)^8 \phi\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!}$$

$$+ \sum_{\left|\frac{k}{n} - x\right| > \delta} \left(\frac{k}{n} - x\right)^8 \phi\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!}$$

$$= \sum_1$$

$$+ \sum_2$$

$$\sum_1 \leq \varepsilon \left(\frac{105x^4}{n^4} + \frac{490x^3}{n^5} + \frac{89x^2}{n^6} + \frac{x}{n^7} \right)$$

$$= \varepsilon o(n^{-4})$$

(2)

Also if $\phi\left(\frac{k}{n}\right) < Ae^{\frac{\beta k}{n}}$ for $\left|\frac{k}{n} - x\right| > \delta$, $A > 0$, $\beta > 0$

then

$$\sum_2 < A \sum_{\left|\frac{k}{n} - x\right| > \delta} e^{\frac{\beta k}{n}} e^{-nx} \frac{(nx)^k}{k!}$$

$$= A \sum_{\left|\frac{k}{n} - x\right| > \delta} e^{-nx} \frac{(e^{\beta/n} nx)^k}{k!}$$

Let $e^{\beta/n} < 1+\eta$

$$\begin{aligned} \text{so that } \sum_2 &< A e^{-nx} \sum_{\left| \frac{k}{n} - x \right| > \delta} \frac{\{nx(1+\eta)\}^k}{k!} \\ &= A e^{nx\eta} e^{-nx(1+\eta)} \sum_{|k-nx(1+\eta)| > nx(1+\eta)\delta'} \frac{\{nx(1+\eta)\}^k}{k!} \end{aligned}$$

where $\delta' = \frac{\delta+x\eta}{x(1+\eta)}$

and for sufficiently small η

$$0 < \delta' < 1.$$

This now has the form of Hardy's theorem, 2.1 so that

$$\begin{aligned} \sum_2 &= e^{nx\eta} O\{\exp(-\frac{1}{3} \delta'^2 nx(1+\eta))\} \\ &= O\{\exp(-\frac{1}{3} nx(\delta'^2(1+\eta)-3\eta))\} \end{aligned}$$

For sufficiently small η , let

$$\delta'^2(1+\eta) - 3\eta \equiv \eta_1 > 0$$

so that

$$\begin{aligned} \sum_2 &= O\{\exp(-\frac{1}{3} n \delta \eta_1)\} \\ &= o(n^{-4}) \end{aligned} \tag{3}$$

Now then if the results 3.2.5(2) and 3.2.5(3) are combined,

$$S_n[(t-x)^8 \phi(t); x] = o(n^{-4})$$

3.2.6 Proof of Theorem 1

Let the positive linear operator in 3.1.2 be the Szasz operator with $\alpha = 8$;

$$S_n(f; x) = f(x) + \sum_{j=1}^8 \frac{f^{(j)}(x)}{j!} S_n[(t-x)^j; x] + S_n[(t-x)^8 \phi(t); x]$$

Using Lemma 5 we have

$$S_n(f; x) = f(x) + \sum_{j=1}^8 \frac{f^{(j)}(x)}{j!} S_n[(t-x)^j; x] + o(n^{-4})$$

and on substituting from Lemma 3

$$\begin{aligned} S_n(f; x) = & f(x) + \frac{x}{2!n} f^{(2)}(x) + \frac{x}{3!n^2} f^{(3)}(x) + \frac{f^{(4)}(x)}{4!} \left(\frac{3x^2}{n^2} + \frac{x}{n^3} \right) \\ & + \frac{f^{(5)}(x)}{5!} \left(\frac{10x^2}{n^3} + \frac{x}{n^4} \right) + \frac{f^{(6)}(x)}{6!} \left(\frac{15x^3}{n^3} + \frac{25x^2}{n^4} + \frac{x}{n^5} \right) \\ & + \frac{f^{(7)}(x)}{7!} \left(\frac{15x^3}{n^4} + \frac{63x^2}{n^5} + \frac{x}{n^6} \right) + \frac{f^{(8)}(x)}{8!} \left(\frac{105x^4}{n^4} + \frac{490x^3}{n^5} + \frac{89x^2}{n^6} + \frac{x}{n^7} \right) \\ & + o(n^{-4}) \end{aligned}$$

Thus for $f(x)$ defined on $[0, \alpha)$ and possessing all the derivatives up to and including the eighth at a point $x = \xi$, $\xi > 0$, we have;

$$S_n(f; \xi) - f(\xi) = \frac{\xi}{2n} f^{(2)}(\xi) + \frac{1}{n^2} \left\{ \xi \frac{f^{(3)}(\xi)}{3!} + 3\xi^2 \frac{f^{(4)}(\xi)}{4!} \right\}$$

$$\begin{aligned}
 & + \frac{1}{n^3} \left\{ \xi \frac{f^{(3)}(\xi)}{3!} + 10\xi^2 \frac{f^{(5)}(\xi)}{5!} + 15\xi^3 \frac{f^{(6)}(\xi)}{6!} \right\} \\
 & + \frac{1}{n^4} \left\{ \xi \frac{f^{(5)}(\xi)}{5!} + 25\xi^2 \frac{f^{(6)}(\xi)}{6!} + 15\xi^3 \frac{f^{(7)}(x)}{7!} + 15\xi^4 \frac{f^{(8)}(\xi)}{8!} \right\} \\
 & + o(n^{-4})
 \end{aligned}$$

§3.3

The proof of the next asymptotic theorem, Theorem 2 will need similar preliminary definitions and lemmas. Again an iterative process becomes obvious in the computations. We will obtain a result with an accuracy $o(n^{-3})$; to obtain a higher order result would involve tedious detail.

Definition 3

For the Baskakov operator

$$K_n(f; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} f\left(\frac{k}{n}\right) \frac{x^k}{(1+x)^{n+k}}$$

$$\text{Let } K_{n,\alpha}(x) \equiv K_n\left[\left(\frac{k}{n} - x\right)^\alpha; x\right] \quad x \in [0, a], \quad 0 < a < \infty$$

$$= \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^\alpha \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$$

3.3.1 Lemma 6

$$(i) \quad \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n}\right) = x \quad (1)$$

$$(ii) \quad \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n}\right)^2 = x^2 + \frac{x(1+x)}{n} \quad (2)$$

Proof of Lemma 6

(i) The identity for the Baskakov operator is

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k.$$

Differentiation of this identity yields

$$n(1+x)^{n-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} k \left(\frac{x}{1+x}\right)^{k-1} \frac{1}{(1+x)^2},$$

which, after multiplying both sides by x and redistributing terms becomes

$$x(1+x)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{k}{n}\right) \left(\frac{x}{1+x}\right)^k \quad (3)$$

$$\begin{aligned} \text{or} \quad x &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{k}{n}\right) \frac{x^k}{(1+x)^{n+k}} \\ &= \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n}\right) \end{aligned}$$

(ii) Now differentiate 3.3.1(3) to obtain

$$x(1+x)^{n-1} + \frac{(1+x)^n}{n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{k}{n}\right)^2 \frac{x^{k-1}}{(1+x)^{k+1}}$$

which, on multiplying both sides by $x(1+x)^{1-n}$ becomes

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{k}{n}\right)^2 \frac{x^k}{(1+x)^{n+k}} = x^2 + \frac{x(1+x)}{n}$$

$$\text{or } \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n}\right)^2 = x^2 + \frac{x(1+x)}{n}$$

3.3.2 Lemma 7

$$(i) \quad K_n[(t-x); x] \equiv K_{n,1}(x)$$

$$= 0$$

$$(ii) \quad K_n[(t-x)^2; x] \equiv K_{n,2}(x)$$

$$= \frac{x(1+x)}{n}$$

Proof of Lemma 7

$$K_n[(t-x); x] = \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n} - x\right) \quad (1)$$

$$= \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n}\right) - x$$

$$= x - x \quad \text{from 3.3.1(1)}$$

$$= 0$$

$$K_n[(t-x)^2; x] = \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n} - x\right)^2 \quad (2)$$

$$= \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n}\right)^2 - 2x \sum_{k=0}^{\infty} h_{n,k}(x) \left(\frac{k}{n}\right) + x^2$$

$$= x^2 + \frac{x(1+x)}{n} - 2x^2 + x^2 \quad \text{from 3.3.1(2)}$$

$$= \frac{x(1+x)}{n}$$

3.3.3 Lemma 8

$$K_{n,\alpha+1}(x) = \frac{x(1+x)}{n} \{K_{n,\alpha}^{(1)}(x) + \alpha K_{n,\alpha-1}(x)\}$$

where $K_{n,\alpha}^{(1)}(x) \equiv \frac{d}{dx} K_{n,\alpha}(x) \quad \alpha = 2, 3, \dots$

Proof of Lemma 8

(1)

$$\begin{aligned} K_{n,\alpha}(x) &\equiv K_n\left[\left(\frac{k}{n} - x\right)^\alpha; x\right] \\ &= \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^\alpha \frac{x^k}{(1+x)^{n+k}} \binom{n+k-1}{k} \end{aligned}$$

Therefore on differentiating both sides of this we have

$$\begin{aligned} K_{n,\alpha}^{(1)}(x) &= -\alpha K_{n,\alpha-1}(x) + \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^\alpha \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \left\{ \frac{k}{x} - \frac{(n+k)}{1+x} \right\} \\ &= -\alpha K_{n,\alpha-1}(x) + \frac{1}{x(1+x)} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^\alpha \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \{k - nx\} \\ &= -\alpha K_{n,\alpha-1}(x) + \frac{n}{x(1+x)} K_{n,\alpha+1}(x). \end{aligned}$$

To rearrange the terms of this will yield

$$K_{n,\alpha+1}(x) = \frac{x(1+x)}{n} \{K_{n,\alpha}^{(1)}(x) + \alpha K_{n,\alpha-1}(x)\}$$

$$\alpha = 2, 3, \dots$$

3.3.4 Lemma 9

Let $X \equiv x(1+x)$ and $Y = 1+2x$

Then

(i) $K_{n,3}(x) = \frac{XY}{n^2}$

$$(ii) \quad K_{n,4}(x) = \frac{X}{n^2} \left\{ 3X + \frac{1+6X}{n} \right\}$$

$$(iii) \quad K_{n,5}(x) = \frac{XY}{n^3} \left\{ 10X + \frac{1+12X}{n} \right\}$$

$$(iv) \quad K_{n,6}(x) = \frac{X}{n^3} \left\{ 15X^2 + \frac{10X(13X+2)}{n} + \frac{30Y^2+1}{n^2} \right\}$$

$$(v) \quad K_{n,7}(x) = \frac{XY}{n^4} \left\{ 105X^2 + \frac{46X+462X^2}{n} + \frac{30Y^2+120X+1}{n^2} \right\}$$

Proof of Lemma 9

For $\alpha=2$ in 3.3.3(1) (1)

$$\begin{aligned} K_{n,3}(x) &= \frac{x(1+x)}{n} \{ K_{n,2}^{(1)}(x) + 2K_{n,1}(x) \} \\ &= \frac{x(1+x)}{n} \cdot \frac{(1+2x)}{n} \\ &= \frac{XY}{n^2} \end{aligned}$$

For $\alpha=3$ in 3.3.3(1) (2)

$$\begin{aligned} K_{n,4}(x) &= \frac{x(1+x)}{n} \{ K_{n,3}^{(1)}(x) + 3K_{n,2}(x) \} \\ &= \frac{X}{n} \left\{ \frac{2X + Y^2}{n^2} + \frac{3X}{n} \right\} \\ &= \frac{X}{n^2} \left\{ 3X + \frac{1+6X}{n} \right\} \end{aligned}$$

For $\alpha=4$ in 3.3.3(1) (3)

$$\begin{aligned} K_{n,5}(x) &= x(1+x) \{ K_{n,4}^{(1)}(x) + 4K_{n,3}(x) \} \\ &= \frac{X}{n} \left\{ \frac{Y}{n^2} \left(3X + \frac{1+6X}{n} \right) + \frac{X}{n^2} \left(3Y + \frac{6Y}{n} \right) + \frac{4XY}{n^2} \right\} \end{aligned}$$

$$= \frac{XY}{n^3} \left\{ 10X + \frac{1+12X}{n} \right\}$$

For $\alpha=5$ in 3.3.3(1) (4)

$$\begin{aligned} K_{n,6}(x) &= x(1+x) \{K_{n,5}^{(1)}(x) + 5K_{n,4}(x)\} \\ &= \frac{X}{n} \left\{ \frac{(2X+Y^2)}{n^3} (10X + \frac{1+12X}{n}) + \frac{XY}{n^3} (10Y + \frac{12Y}{n}) + \frac{5X}{n^2} (3X + \frac{1+6X}{n}) \right\} \\ &= \frac{X}{n^3} \left\{ 15X^2 + \frac{20X^2(1+5X)+5X(1+6X)}{n} + \frac{30X+120X^2+1}{n^2} \right\} \\ &= \frac{X}{n^3} \left\{ 15X^2 + \frac{10X(13X+2)}{n} + \frac{30Y^2+1}{n^2} \right\} \end{aligned}$$

3.3.5 Lemma 10

$$\sum_{\left| \frac{k}{n} - x \right| > \delta} \left(\frac{k}{n} - x \right)^6 \phi\left(\frac{k}{n}\right) h_{n,k}(x) = o(n^{-3})$$

Proof of Lemma 10

Recall Hardy's theorem, §2.2 (1)

$$\sum_{\left| \frac{k}{n} - \frac{1-y}{y} \right| > \delta} \binom{k+n}{k} y^{n+1} (1-y)^k \approx o(e^{-\gamma n})$$

where $\gamma = \frac{1}{3}\delta^2$ and $0 < \delta < 1$, for this proof.

In the sum

$$\sum_{\left| \frac{k}{n} - x \right| > \delta} \left(\frac{k}{n} - x \right)^6 \phi\left(\frac{k}{n}\right) \binom{k+n-1}{k} \frac{x^k}{(1+x)^{n+k}}$$

$$\text{let } \left| \left(\frac{k}{n} - x \right)^6 \phi\left(\frac{k}{n}\right) \right| < A e^{\frac{\beta k}{n}} ;$$

such that

$$\sum_{\left|\frac{k}{n} - x\right| > \delta} \left(\frac{k}{n} - x\right)^6 \phi\left(\frac{k}{n}\right) \binom{k+n-1}{k} \left(\frac{x}{1+x}\right)^k (1+x)^{-n}$$

$$\leq A \sum_{\left|\frac{k}{n} - x\right| > \delta} \binom{k+n}{k} \left(\frac{x}{1+x} \cdot e^{\frac{\beta}{n}}\right)^k (1+x)^{-n}$$

Further let $\frac{x}{1+x} \cdot e^{\frac{\beta}{n}} \equiv 1-Y$

which means

$$\frac{1+x - x e^{\frac{\beta}{n}}}{1+x} \equiv Y$$

As $\lim_{n \rightarrow \infty} \left(\frac{1+x - x e^{\frac{\beta}{n}}}{(1+x)^{1-\frac{\beta}{n}}} \right) \rightarrow \frac{1}{(1+x)e^{\beta x}}$

$$A \left| \sum_{\left|\frac{k}{n} - x\right| > \delta} \binom{k+n}{k} \left(\frac{x}{1+x} \cdot e^{\frac{\beta}{n}}\right)^k (1+x)^{-n} \right|$$

$$\leq A \left| \sum_{\left|\frac{k}{n} - x\right| > \delta} \binom{k+n}{k} (1+x) e^{\beta x} \left(\frac{x}{1+x} \cdot e^{\frac{\beta}{n}}\right)^k \left(\frac{1+x - x e^{\frac{\beta}{n}}}{1+x}\right)^{n+1} \right|$$

$$= A(1+x)e^{\beta x} \left| \sum_{\left|\frac{k}{n} - \frac{x e^{\frac{\beta}{n}}}{1+x - x e^{\frac{\beta}{n}}}\right| > \delta'} \binom{k+n}{k} \left(\frac{x}{1+x} \cdot e^{\frac{\beta}{n}}\right)^k \left(\frac{1+x - x e^{\frac{\beta}{n}}}{1+x}\right)^{n+1} \right| \quad (2)$$

where

$$\delta' = \delta - \frac{x(1-e^{\frac{\beta}{n}})(x-1)}{1+x(1-e^{\frac{\beta}{n}})}$$

$$\equiv \delta - \psi(n)$$

$$\text{and } \psi(n) = \frac{x(1-e^{\beta/n})(x-1)}{1+x(1-e^{\beta/n})}$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty$$

Therefore $\delta' \rightarrow \delta$ and $0 < \delta' < 1$

Now 3.3.5(2) is in the form of Hardy's theorem §2.2 so

$$= O(e^{-\frac{1}{3}\delta'^2 n})$$

$$= O(e^{-\frac{1}{3}\delta^2 n})$$

$$= O(n^{-4})$$

$$= o(n^{-3})$$

3.3.6 Lemma 11

$$K_n[(t-x)^6 \phi(t); x] = o(n^{-3})$$

for $|\phi(t)| < \epsilon$ and $|t-x| < \delta$

Proof of Lemma 11

$$K_n[(t-x)^6 \phi(t); x] = \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^6 \phi\left(\frac{k}{n}\right) h_{n,k}(x) \quad (1)$$

$$= \sum_{\left|\frac{k}{n} - x\right| < \delta} \left(\frac{k}{n} - x\right)^6 \phi\left(\frac{k}{n}\right) h_{n,k}(x) + \sum_{\left|\frac{k}{n} - x\right| > \delta} \left(\frac{k}{n} - x\right)^6 \phi\left(\frac{k}{n}\right) h_{n,k}(x)$$

$$= \sum_1 + \sum_2$$

$$\sum_1 < \epsilon \cdot \frac{X}{n^3} \left\{ 15X^2 + \frac{10X(13X+2)}{n} + \frac{30Y^2+1}{n^2} \right\} \quad \text{From 3.3.4(4)}$$

$$= o(n^{-3})$$

$$\text{and } \sum_2 = o(n^{-3}) \quad \text{from 3.3.5}$$

Thus the Lemma.

3.3.7 Proof of Theorem 2

Let the positive linear operator in 3.1.2 be the Baskakov operator with $\alpha=6$.

$$K_n(f; x) = f(x) + \sum_{j=1}^6 \frac{f^{(j)}(x)}{j!} K_n[(t-x)^j; x] + K_n[(t-x)^6 \phi(t); x]$$

which from Lemma 11 becomes

$$f(x) + \sum_{j=1}^6 \frac{f^{(j)}(x)}{j!} K_n[(t-x)^j; x] + o(n^{-3})$$

The equalities found in Lemmas 7 and 9 give

$$\begin{aligned} K_n(f; x) &= f(x) + \frac{X}{n} \frac{f^{(2)}(x)}{2!} + \frac{f^{(3)}(x)}{3!} \frac{XY}{n^2} + \frac{f^{(4)}(x)}{4!} \frac{X}{n^2} \left\{ 3X + \frac{1+6X}{n} \right\} + \\ &+ \frac{f^{(5)}(x)}{5!} \frac{XY}{n^3} \left\{ 10X + \frac{1+12X}{n} \right\} \\ &+ \frac{f^{(6)}(x)}{6!} \cdot \frac{X}{n^3} \left\{ 15X^2 + \frac{10X(13X+2)}{n} + \frac{30Y^2+1}{n^2} \right\} \\ &+ o(n^{-3}) \\ &= f(x) + \frac{X}{n} \frac{f^{(2)}(x)}{2!} + \frac{1}{n^2} \left\{ \frac{f^{(3)}(x)}{3!} XY + \frac{f^{(4)}(x)}{4!} 3X^2 \right\} \\ &+ \frac{1}{n^3} \left\{ \frac{f^{(4)}(x)}{4!} X(1+6X) + \frac{f^{(5)}(x)}{5!} 10X^2Y + \frac{f^{(6)}(x)}{6!} 15X^3 \right\} + o(n^{-3}) \end{aligned}$$

In particular at the point ξ where $f(x)$ has all the derivatives up to and including the sixth, we replace X and Y by their respective values to obtain

$$\begin{aligned} K_n(f; \xi) - f(\xi) &= \frac{\xi(1+\xi)}{n} \frac{f^{(2)}(\xi)}{2} + \\ &+ \frac{1}{n^2} \left\{ \xi(1+\xi)(1+2\xi) \frac{f^{(3)}(\xi)}{3!} + 3\xi^2(1+\xi)^2 \frac{f^{(4)}(\xi)}{4!} \right\} \\ &+ \frac{1}{n^3} \left\{ \xi(1+6(\xi+1)) \frac{f^{(4)}(\xi)}{4!} + 10\xi^2(1+\xi)^2(1+2\xi) \frac{f^{(5)}(\xi)}{5!} \right. \\ &\left. + 15\xi^3(1+\xi)^3 \frac{f^{(6)}(\xi)}{6!} \right\} + o(n^{-3}). \end{aligned}$$

§3.4

We now consider the asymptotic results, with degree of approximation of $o(n^{-1})$, for two members of the operator class of Shah and Suryanarayana. This class of operator has polynomial coefficients and we firstly construct a set of definitions and lemmas when the coefficients are the Chebychev polynomials, $T_n(x)$. These will form the data for theorem 3a.

Definition 4

From the identity relationship for Chebychev polynomials

$$e^{nx} \cosh(x\sqrt{n^2-1}) = \sum_{k=0}^{\infty} T_k(n) \frac{x^k}{k!}$$

let

$$\alpha_{n,k}(x) \equiv e^{-nx} \operatorname{sech}(x\sqrt{n^2-1}) T_k(n) \frac{x^k}{k!}$$

such that $\sum_{k=0}^{\infty} \alpha_{n,k}(x) = 1$

3.4.1 Lemma 12

$$\alpha_{n,k}(x) \leq 2Ke^{-2nx} \frac{(nx)^k}{k!} \quad x \geq 1$$

$$\text{where } K = \max [e^{nx(1-\sqrt{1-\frac{1}{n^2}})}] \quad (1)$$

Proof of Lemma 12

$$\text{Since } T_k(n) < (2n)^k$$

$$\alpha_{n,k}(x) < e^{-nx}(2n)^k \frac{x^k}{k!} \operatorname{sech}(x\sqrt{n^2-1})$$

$$\begin{aligned} \text{and this} \quad &= e^{-nx} \frac{(2nx)^k}{k!} \frac{2}{e^{x\sqrt{n^2-1}} + e^{-x\sqrt{n^2-1}}} \\ &= 2e^{-nx} \frac{(2nx)^k}{k!} \cdot \frac{e^{-x\sqrt{n^2-1}}}{1+e^{-2x\sqrt{n^2-1}}} \\ &< 2Ke^{-2nx} \frac{(2nx)^k}{k!} \end{aligned}$$

$$\text{where } K = \max [e^{nx(1-\sqrt{1-\frac{1}{n^2}})}] \equiv \max [\exp(\frac{x}{n(\sqrt{1-\frac{1}{n^2}}+1)})]$$

Definition 5

$$\text{Let } G_\alpha \equiv \sum_{k=0}^{\infty} k^\alpha T_k(n) \frac{x^k}{k!}$$

$$\text{and } G_\alpha^* = G_\alpha g(n,x)$$

$$\text{where } \frac{1}{g(n,x)} \equiv e^{nx} \cosh(x\sqrt{n^2-1})$$

Corollary

$$G_0 = e^{-nx} \operatorname{sech}(x\sqrt{n^2-1}) \sum_{k=0}^{\infty} T_k(n) \frac{x^k}{k!}$$

= 1 from definition 4.

3.4.2 Lemma 13

$$G_1^* = nx + x\sqrt{n^2-1} \cdot \tanh(x\sqrt{n^2-1}).$$

Proof of Lemma 13

(1)

If we differentiate both sides of the identity

$$\sum_{k=0}^{\infty} T_k(n) \frac{x^k}{k!} = e^{nx} \cosh(x\sqrt{n^2-1})$$

such that

$$\sum_{k=0}^{\infty} k T_k(n) \frac{x^{k-1}}{(k-1)!} = ne^{nx} \cosh(x\sqrt{n^2-1}) + e^{nx} \cdot \sqrt{n^2-1} \sinh(x\sqrt{n^2-1})$$

Now multiplying both sides throughout by $x.g(n,x)$ will yield the result:

$$\begin{aligned} G_1^* &= g(n,x) \sum_{k=0}^{\infty} k T_k(n) \frac{x^k}{k!} \\ &= nx + x\sqrt{n^2-1} \cdot \tanh(x\sqrt{n^2-1}). \end{aligned}$$

3.4.3 Lemma 14

$$G_{\alpha}^* = (2nx + 1)G_{\alpha-1}^* - x^2 2^{\alpha-2} + \sum_{v=0}^{\alpha-3} G_{\alpha-v-2}^* \{2nx \binom{\alpha-2}{v+1} - x^2 2^v \binom{\alpha-2}{v}\}$$

$$\alpha = 2, 3, \dots$$

Proof of Lemma 14

(1)

$$\begin{aligned} G_{\alpha} &\equiv \sum_0^{\infty} k^{\alpha} T_k(n) \frac{x^k}{k!} \\ &= \sum_0^{\infty} k^{\alpha-1} (k-1) \frac{x^k}{k!} T_k(n) + \sum_0^{\infty} k^{\alpha-1} \frac{x^k}{k!} T_k(n) \end{aligned}$$

Use the recurrence relationship for $T_k(n)$ to obtain

$$\begin{aligned} G_{\alpha} &= \sum_0^{\infty} k^{\alpha-1} (k-1) \frac{x^k}{k!} \{2nT_{k-1}(n) - T_{k-2}(n)\} + G_{\alpha-1} \\ &= 2nx \sum_1^{\infty} k^{\alpha-2} \frac{(k-1)}{(k-1)!} x^{k-1} T_{k-1}(n) - x^2 \sum_2^{\infty} k^{\alpha-2} T_{k-2}(n) \frac{x^{k-2}}{(k-2)!} + G_{\alpha-1} \\ &= 2nx \sum_0^{\infty} (k+1)^{\alpha-2} k \frac{x^k}{k!} T_k(n) - x^2 \sum_0^{\infty} (k+2)^{\alpha-2} \frac{x^k}{k!} T_k(n) + G_{\alpha-1} \\ &= 2nx \sum_0^{\infty} k \frac{x^k}{k!} T_k(n) \left(\sum_{v=0}^{\alpha-2} \binom{\alpha-2}{v} k^{\alpha-(2+v)} \right) \\ &\quad - x^2 \sum_0^{\infty} \frac{x^k}{k!} T_k(n) \left(\sum_{v=0}^{\alpha-2} \binom{\alpha-2}{v} k^{\alpha-(2+v)} 2^v \right) + G_{\alpha-1} \\ &= 2nx \sum_0^{\infty} \frac{x^k}{k!} T_k(n) \{k^{\alpha-1} + \sum_{v=1}^{\alpha-2} \binom{\alpha-2}{v} k^{\alpha-(v+2)}\} \\ &\quad - x^2 \sum_0^{\infty} \frac{x^k}{k!} T_k(n) \{2^{\alpha-2} + \sum_{v=0}^{\alpha-3} \binom{\alpha-2}{v} k^{\alpha-(2+v)} 2^v\} + G_{\alpha-1} \\ &= 2nx G_{\alpha-1} + \sum_{v=0}^{\alpha-3} G_{\alpha-(2+v)} \{2nx \binom{\alpha-2}{v+1} - x^2 2^v \binom{\alpha-2}{v}\} - 2^{\alpha-2} x^2 G_0 + G_{\alpha-1} \\ &= (2nx+1) G_{\alpha-1} - 2^{\alpha-2} x^2 G_0 + \sum_{v=0}^{\alpha-3} G_{\alpha-(2+v)} \{2nx \binom{\alpha-2}{v+1} - x^2 2^v \binom{\alpha-2}{v}\}. \end{aligned}$$

Therefore

$$G_{\alpha}^* = (2nx+1) G_{\alpha-1}^* - 2^{\alpha-2} x^2 + \sum_{v=0}^{\alpha-3} G_{\alpha-(2+v)}^* \{2nx \binom{\alpha-2}{v+1} - x^2 2^v \binom{\alpha-2}{v}\},$$

$$\alpha = 2, 3, \dots$$

Corollary

(2)

In particular when $\alpha=2$ in lemma 14,

$$\begin{aligned} G_2^* &= (2nx+1)G_1^* - 2^0 x^2 + 0 \\ &= (2nx+1)\{nx + x\sqrt{n^2-1} \tanh(x\sqrt{n^2-1})\} - x^2 \\ &= (2nx+1)nx - x^2 + (2nx+1) x\sqrt{n^2-1} \tanh(x\sqrt{n^2-1}) \end{aligned}$$

3.4.4 Lemma 15

$$P_n\{(t-x)^\alpha; x\} = \sum_{v=0}^{\alpha} \binom{\alpha}{v} G_{\alpha-v}^* \frac{(-1)^v x^v}{(2n)^{\alpha-v}} \quad (1)$$

Corollary

In particular when $\alpha=1$ in lemma 15

$$\begin{aligned} P_n\{t-x; x\} &= -\frac{x}{2} \left\{1 - \sqrt{1 - \frac{1}{n^2}} \cdot \tanh(x\sqrt{n^2-1})\right\} \\ &= -\frac{x}{4n^2} - \frac{x}{16n^4} + o(n^{-6}) \end{aligned} \quad (2)$$

and when $\alpha=2$,

$$\begin{aligned} P_n\{(t-x)^2; x\} &= x^2 - \frac{x^2}{4n^2} - \sqrt{1 - \frac{1}{n^2}} \cdot \tanh(x\sqrt{n^2-1}) + 1 \left(\frac{x^2}{2} - \frac{x}{4n}\right) \\ &= \frac{x}{2n} - \frac{x}{8n^3} + o(n^{-5}) \end{aligned} \quad (3)$$

Proof of Lemma 15

$$P_n\{(t-x)^\alpha; x\} = g(n, x) \sum_k \left(\frac{k}{2n} - x\right)^\alpha T_k(n) \frac{x^k}{k!}$$

$$\begin{aligned}
 &= g(n, x) \sum_k \frac{x^k}{k!} T_k(n) \sum_{v=0}^{\alpha} \binom{\alpha}{v} \frac{k^{\alpha-v}}{(2n)^{\alpha-v}} (-1)^v x^v \\
 &= g(n, x) \sum_{v=0}^{\alpha} \binom{\alpha}{v} \frac{(-1)^v x^v}{(2n)^{\alpha-v}} \sum_k k^{\alpha-v} T_k(n) \frac{x^k}{k!} \\
 &= g(n, x) \sum_{v=0}^{\alpha} \binom{\alpha}{v} \frac{(-1)^v x^v}{(2n)^{\alpha-v}} G_{\alpha-v} \\
 &= \sum_{v=0}^{\alpha} \binom{\alpha}{v} \frac{(-1)^v x^v}{(2n)^{\alpha-v}} G_{\alpha-v}^*
 \end{aligned}$$

When $\alpha=1$

$$\begin{aligned}
 P_n\{(t-x); x\} &= \sum_{v=0}^1 \frac{(-1)^v x^v}{(2n)^{1-v}} G_{1-v}^* \quad (4) \\
 &= \frac{G_1^*}{2n} - x \\
 &= -\frac{x}{2} \left\{ 1 - \sqrt{1 - \frac{1}{n^2}} \tanh(x\sqrt{n^2-1}) \right\} \\
 &\approx -\frac{x}{2} \left\{ 1 - \left(1 - \frac{1}{2n^2} - \frac{1}{8n^4} \dots \right) \tanh(x\sqrt{n^2-1}) \right\} \\
 &= -\frac{x}{4n^2} - \frac{x}{16n^4} + o(n^{-6})
 \end{aligned}$$

When $\alpha=2$

$$\begin{aligned}
 P_n\{(t-x)^2; x\} &= \sum_{v=0}^2 \binom{2}{v} \frac{(-1)^v x^v}{(2n)^{2-v}} G_{2-v}^* \quad (5) \\
 &= \frac{G_2^*}{4n^2} - \frac{x}{n} G_1^* + x^2 \\
 &= x^2 - \frac{x^2}{4n^2} - \left(\sqrt{1 - \frac{1}{n^2}} \cdot \tanh(x\sqrt{n^2-1}) + 1 \right) \left(\frac{x^2}{2} - \frac{x}{4n} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= x^2 - \frac{x^2}{4n^2} - \left\{ \left(1 - \frac{1}{2n^2} - \frac{1}{8n^4} \dots\right) \tanh(x\sqrt{n^2-1}) + 1 \right\} \left(\frac{x^2}{2} - \frac{x}{4n} \right) \\
 &= \frac{x}{2n} - \frac{x}{8n^3} + o(n^{-5})
 \end{aligned}$$

3.4.5 Lemma 16

For $x \geq 0$ and for any fixed $\delta > 0$ (1)

$$\begin{aligned}
 \sigma_n(x) &\equiv \sum_{\left| \frac{k}{2n} - x \right| > \delta} \alpha_{n,k}(x) e^{\frac{\beta k}{2n}} \\
 &= o(n^{-1}) \quad \text{uniformly in } 0 \leq x \leq b < \infty.
 \end{aligned}$$

Proof of Lemma 16 (2)

Part 1: If $x \geq \delta$, let η be a positive number to be specified later in the proof. From lemma 12 it follows that

$$\sigma_n(x) < \sum_{\left| \frac{k}{2n} - x \right| > \delta} 2Ke^{-2nx} 2^k \frac{(nx)^k}{k!} e^{\frac{\beta k}{2n}}$$

which

$$\begin{aligned}
 &= \sum 2Ke^{-2nx} \frac{\left(\frac{\beta}{2n} e^{2nx} \right)^k}{k!} \\
 &= 2Ke^{-2nx} \sum \frac{\left(\frac{\beta}{2n} + \log 2 \right)^k}{k!}
 \end{aligned}$$

Now let $\exp\left(\frac{\beta}{2n} + \log 2\right) < 2+\eta$ so that

$$\sigma_n(x) < 2Ke^{-2nx} \sum_{\left| \frac{k}{2n} - x \right| > \delta} \frac{(nx(2+\eta))^k}{k!}$$

which

$$= 2Ke^{nx\eta} e^{-nx(2+\eta)} \sum_{|k-nx(2+\eta)| > nx(2+\eta)\delta'} \frac{(nx(2+\eta))^k}{k!}$$

where $\delta' = \frac{2\delta+x\eta}{x(2+\eta)}$ and for sufficiently small η

$$0 < \delta' < 1$$

Thus from Hardy's theorem, §2.2

$$\begin{aligned} \sigma_n(x) &= 2Ke^{xn\eta} \cdot O\{\exp(-\frac{1}{3}\delta'^2 nx(2+\eta))\} \\ &= O\{\exp(-\frac{1}{3}nx(\delta'^2(2+\eta)-3\eta))\} \end{aligned}$$

For sufficiently small η let

$$\delta'^2(2+\eta)-3\eta \equiv \eta_1 > 0$$

so that

$$\begin{aligned} \sigma_n(x) &\leq O\{\exp(-\frac{1}{3}n\delta'\eta_1)\} \\ &= o(n^{-1}) \end{aligned}$$

Part 2: If $0 \leq x < \delta$, $x = \theta\delta$, $0 \leq \theta < 1$, then for sufficiently small n

$$\sigma_n(x) = \sum_{k > 2(nx+n\delta)} a_{n,k}(x) e^{\frac{\beta k}{2n}} \quad (3)$$

which

$$\leq e^{-2nx} 2K \sum_{k > 2(nx+n\delta)} \frac{(nx(2+\eta))^k}{k!}$$

$$< 2Ke^{-2nx} \left(\frac{nx(2+\eta)}{M!} \right)^{M+1} \cdot \frac{1}{M-nx(2+\eta)}$$

$$\text{where } M = [2(nx+n\delta)]$$

$$< C \frac{e^{-2n\theta\delta}}{\sqrt{M}} \left(\frac{en\theta\delta(2+\eta)}{M} \right)^M \cdot \frac{1}{n\delta(2-\theta\eta)}$$

$$\text{since } M! \approx \sqrt{2\pi M} (Me^{-1})^M$$

Thus

$$\sigma_n(x) < \frac{C}{n\sqrt{n}} e^{2n\delta} \left(\frac{\theta(2+\eta)}{2(1+\theta)} \right)^{2n\delta(1+\theta)}$$

$$\text{Now as } 0 \leq \theta < 1, \left(\frac{\theta}{1+\theta} \right)^{1+\theta} < \frac{1}{4},$$

for sufficiently small η

$$\sigma_n(x) < \frac{C}{n^{3/2}} \cdot \left(\frac{e}{3} \right)^{2n\delta}$$

$$= o(n^{-1})$$

Thus the lemma is valid for all $\delta > 0$.

3.4.6 Proof of Theorem 3a

Let $\alpha=2$ in 3.1.2 to obtain;

$$P_n(f;x) = f(x) + \sum_{j=1}^2 \frac{f^{(j)}(x)}{j!} P_n\{(t-x)^j; x\} + P_n\{(t-x)^2 \phi(t); x\} \quad (1)$$

If given $\epsilon > 0$, let $0 < \delta < x$ be such that $|\phi(t)| < \epsilon$ for
for all t in $|t-x| < \delta$, and for all t and some $A > 0$

$$|\phi(t)| < Ae^{\beta t}$$

then

$$P_n\{(t-x)^2 \phi(t); x\} = \sum_{|\frac{k}{2n}-x| \leq \delta} + \sum_{|\frac{k}{2n}-x| > \delta} \alpha_{n,k}(x) \left(\frac{k}{2n}-x\right)^2 \phi\left(\frac{k}{2n}\right)$$

$$= \sigma_1 + \sigma_2$$

$$\sigma_1 \leq \varepsilon \sum_{|\frac{k}{2n}-x| \leq \delta} \alpha_{n,k}(x) \left(\frac{k}{2n}-x\right)^2$$

$$= \varepsilon o(n^{-1}) \quad \text{from 3.4.4(5)}$$

$$\sigma_2 \leq A \sum_{|\frac{k}{2n}-x| > \delta} e^{\frac{\beta k}{2n}} \alpha_{n,k}(x)$$

$$= o(n^{-1}) \quad \text{from 3.4.5(2)}$$

Therefore

$$P_n\{(t-x)^2 \phi(t); x\} = o(n^{-1}) \quad (2)$$

So if $f(x)$ is twice differentiable at a point x ,
3.4.6(1) may be written as

$$P_n(f; x) - f(x) = f^{(1)}(x) P_n(t-x; x) + \frac{f^{(2)}(x)}{2} P_n\{(t-x)^2; x\} + o(n^{-1}) \quad (3)$$

We may now substitute the results of lemma 15 into
this to get

$$P_n(f; x) - f(x) = f^{(1)}(x) \left\{ \frac{x}{4n} + \frac{x}{16n^4} + \dots \right\} + \frac{f^{(2)}(x)}{2} \left\{ \frac{x}{2n} - \frac{x}{8n^3} + \dots \right\}$$

$$+ o(n^{-1})$$

Thus if $f(x)$ is twice differentiable at a point $x=\xi$, $\xi \in [a,b]$, then

$$\lim_{n \rightarrow \infty} n[P_n(f; \xi) - f(\xi)] = \frac{\xi f^{(2)}(\xi)}{4}$$

or in the alternate form

$$P_n(f; \xi) - f(\xi) = \frac{\xi f^{(2)}(\xi)}{4n} + o(n^{-1})$$

§ 3.5

To prove the asymptotic result for the other member in the class of Shah & Suryanarayana we will need a similar set of definitions and lemmas. The coefficients of the positive linear operators in this case are Legendre polynomials.

Definition 6

From the identity relationship for Legendre polynomials

$$e^{nx} J_0(x\sqrt{1-n^2}) = \sum_{k=0}^{\infty} P_k(n) \frac{x^k}{k!}$$

let

$$\alpha_{n,k}(x) \equiv P_k(n) \frac{x^k}{k!}$$

so that $\sum_{k=0}^{\infty} \alpha_{n,k}(x) = 1$.

3.5.1 Lemma 17

$$\alpha_{n,k}(x) \leq 2Ke^{-2nx} \frac{(2nx)^k}{k!}$$

where

$$2K = \max \left(\sqrt{2\pi n x} \sqrt{1 - \frac{1}{n^2}} \exp(n x (1 - \sqrt{1 - \frac{1}{n^2}})) \right)$$

Proof of Lemma 17

As $P_k(n) < (2n)^k$ and $I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}$, we may rewrite

the relationship

$$\begin{aligned} \alpha_{n,k}(x) &= P_k(n) \frac{x^k}{k!} \frac{e^{-nx}}{J_0(x\sqrt{1-n^2})} \\ &< (2n)^k \frac{x^k}{k!} \frac{e^{-nx}}{J_0(ix\sqrt{n^2-1})} \\ &= \frac{(2nx)^k}{k!} \frac{e^{-nx\sqrt{2\pi x\sqrt{n^2-1}}}}{e^{x(\sqrt{n^2-1})}} \\ &< 2K \frac{(2nx)^k}{k!} e^{-2nx} \sqrt{n} \end{aligned}$$

$$\text{where } 2K = \max \sqrt{2\pi x} \sqrt{1 - \frac{1}{n^2}} \exp(n x (1 - \sqrt{1 - \frac{1}{n^2}}))$$

Definition 7

$$\text{Let } L_\alpha \equiv \sum_{k=0}^{\infty} k^\alpha P_k(n) \frac{x^k}{k!}$$

$$\text{and } L_\alpha^* = L_\alpha \cdot g(n, x)$$

$$\text{where } \frac{1}{g(n, x)} = e^{nx} J_0(x\sqrt{1-n^2})$$

Corollary

$$\begin{aligned} L_0^* &= \frac{e^{-nx}}{J_0(x\sqrt{1-n^2})} \sum_{k=0}^{\infty} P_k(n) \frac{x^k}{k!} \\ &= 1 \text{ from definition 6.} \end{aligned}$$

3.5.2 Lemma 18

$$L_1^* = nx + nx\sqrt{1-\frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1-\frac{1}{n^2}})}{I_0(nx\sqrt{1-\frac{1}{n^2}})} \right\} \quad (1)$$

and

$$L_2^* = (2n^2x^2 + nx - x^2) + 2n^2x^2 \sqrt{1-\frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1-\frac{1}{n^2}})}{I_0(nx\sqrt{1-\frac{1}{n^2}})} \right\} \quad (2)$$

Proof of Lemma 18

Consider the identity

$$\sum_{k=0}^{\infty} P_k(n) \frac{x^k}{k!} = e^{nx} J_0(x\sqrt{1-n^2})$$

differentiate both sides to obtain

$$\sum_k k P_k(n) \frac{x^{k-1}}{k!} = ne^{nx} J_0(x\sqrt{1-n^2}) - e^{nx} \sqrt{1-n^2} J_1(x\sqrt{1-n^2})$$

Multiply this throughout by $x.g(n,x)$ to get

$$\begin{aligned} L_1^* &= nx - x\sqrt{1-n^2} \cdot \frac{J_1(x\sqrt{1-n^2})}{J_0(x\sqrt{1-n^2})} \\ &= nx + nx \sqrt{1-\frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1-\frac{1}{n^2}})}{I_0(nx\sqrt{1-\frac{1}{n^2}})} \right\} \end{aligned}$$

Now consider

$$\begin{aligned} L_2 &= \sum_k k^2 P_k(n) \frac{x^k}{k!} \\ &= \sum_k k P_k(n) \frac{x^k}{(k-1)!} \end{aligned}$$

$$= \sum_k \frac{x^k}{(k-1)!} \{n(2k-1)P_{k-1}(n) - (k-1)P_{k-2}(n)\} \text{ from 1.6.3(1)}$$

$$= 2nx \sum_{k=1} x^{k-1} \frac{(k-1)}{(k-1)!} P_{k-1}(n) + nx \sum_{k=1} x^{k-1} \frac{P_{k-1}(n)}{(k-1)!} \\ - x^2 \sum_{k=2} \frac{x^{k-2} P_{k-2}(n)}{(k-2)!}$$

$$= 2nxL_1 + nxL_0 - x^2L_0$$

$$= 2nxL_1 + L_0(nx - x^2).$$

On multiplying this throughout by $g(n, x)$ we obtain

$$L_2^* = 2nxL_1^* + (nx - x^2) \\ = (2n^2x^2 + nx - x^2) + 2n^2x^2 \sqrt{1 - \frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1 - \frac{1}{n^2}})}{I_0(nx\sqrt{1 - \frac{1}{n^2}})} \right\}$$

3.5.3

Lemma 19

$$P_n[t-x; x] = -\frac{x}{2} \left\{ 1 - \sqrt{1 - \frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1 - \frac{1}{n^2}})}{I_0(nx\sqrt{1 - \frac{1}{n^2}})} \right\} \right\}$$

$$= -\left(\frac{x}{4n^2} + \frac{x}{16n^4} + o(n^{-4})\right) \quad (1)$$

and

$$P_n[(t-x)^2; x] = \frac{x^2}{2} + \frac{x}{4n} - \frac{x^2}{4n^2} - \frac{x^2}{2} \sqrt{1 - \frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1 - \frac{1}{n^2}})}{I_0(nx\sqrt{1 - \frac{1}{n^2}})} \right\}$$

$$\approx \frac{x^2}{2} + \frac{x}{4n} - \frac{x^2}{4n^2} - \frac{x^2}{2} \sqrt{1 - \frac{1}{n^2}}$$

$$= \frac{x}{4n} + \frac{x^2}{16n^4} + o(n^{-4}) \quad (2)$$

Proof of Lemma 19

$$P_n[t-x; x] = g(n, x) \sum_k \left(\frac{k}{2n} - x \right) \frac{x^k}{k!} P_k(n)$$

$$= \frac{L_1^*}{2n} - x.$$

Substitute the L_1^* from lemma 18 to obtain

$$P_n[t-x; x] = -\frac{x}{2} \left\{ 1 - \sqrt{1 - \frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1 - \frac{1}{n^2}})}{I_0(nx\sqrt{1 - \frac{1}{n^2}})} \right\} \right\}$$

$$\approx -\frac{x}{2} \{1 - \sqrt{1 - \frac{1}{n^2}}\}$$

$$= -\left(\frac{x}{4n^2} + \frac{x}{16n^4} + o(n^{-4})\right)$$

For the second part of the lemma;

$$P_n[(t-x)^2; x] = g(n, x) \sum_k \left(\frac{k}{2n} - x \right)^2 \frac{x^k}{k!} P_k(n)$$

$$= \frac{L_2^*}{4n^2} - \frac{xL_1^*}{n} + x^2$$

From values of L_1^* and L_2^* derived in Lemma 18, we have

$$P_n[(t-x)^2; x] = \frac{(2n^2x^2 + nx - x^2)}{4n^2} + \frac{x^2}{2} \sqrt{1 - \frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1 - \frac{1}{n^2}})}{I_0(nx\sqrt{1 - \frac{1}{n^2}})} \right\}$$

$$- x^2 - x^2 \sqrt{1 - \frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1 - \frac{1}{n^2}})}{I_0(nx\sqrt{1 - \frac{1}{n^2}})} \right\} + x^2$$

$$= \frac{x^2}{2} + \frac{x}{4n} - \frac{x^2}{4n^2} - \frac{x^2}{2} \sqrt{1 - \frac{1}{n^2}} \left\{ \frac{I_1(nx\sqrt{1 - \frac{1}{n^2}})}{I_0(nx\sqrt{1 - \frac{1}{n^2}})} \right\}$$

$$\begin{aligned} &\approx \frac{x^2}{2} + \frac{x}{4n} - \frac{x^2}{4n^2} - \frac{x^2}{2} \sqrt{1 - \frac{1}{n^2}} \\ &= \frac{x}{4n} + \frac{x^2}{16n^4} + o(n^{-4}) \end{aligned}$$

3.5.4 Lemma 20

For $x \geq 0$, and for any fixed δ , $0 < \delta < 1$,

$$\begin{aligned} \sigma_n(x) &\equiv \sum_{\left| \frac{k}{2n} - x \right| > \delta} \alpha_{n,k}(x) e^{\frac{\beta k}{2n}} \\ &= o(n^{-1}) \end{aligned}$$

Proof of Lemma 20

To avoid the repetition of a similar proof, the inequality in lemma 17 was chosen to be consistent with that in lemma 12. Obviously the values of the constants K , η , etc. will be different in each case, but these inequalities so chosen, make the proof of lemma 20 identical in form to that of lemma 16 and is thus not repeated.

3.5.5 Proof of Theorem 3b

Let $\alpha=2$ in 3.1.2 so (1)

$$P_n(f; x) = f(x) + \sum_{j=1}^2 \frac{f^{(j)}(x)}{j!} P_n\{(t-x)^j; x\} + P_n\{(t-x)^2 \phi(t); x\}$$

If given $\epsilon > 0$, let δ , $0 < \delta < x$ be such that $|\phi(t)| < \epsilon$ for all t in $|t-x| < \delta$, and for all t and some $A > 0$, $|\phi(t)| < Ae^{\beta t}$.

Then

$$P_n\{(t-x)^2\phi(t);x\} = \sum_{|\frac{k}{2n}-x|\leq\delta} + \sum_{|\frac{k}{2n}-x|>\delta} \alpha_{n,k}(x) (\frac{k}{2n}-x)^2 \phi(\frac{k}{2n})$$

$$\equiv \sigma_1 + \sigma_2$$

where

$$\sigma_1 \leq \varepsilon \sum_{|\frac{k}{2n}-x|\leq\delta} \alpha_{n,k}(x) (\frac{k}{2n}-x)^2$$

$$= \varepsilon o(n^{-1}) \quad \text{from 3.5.3(2)}$$

$$\sigma_2 \leq A \sum_{|\frac{k}{2n}-x|>\delta} e^{\frac{\beta k}{2n}} \alpha_{n,k}(x)$$

$$= o(n^{-1}) \quad \text{from lemma 20}$$

Therefore

$$P_n\{(t-x)^2\phi(t);x\} = o(n^{-1})$$

Thus if $f(x)$ is twice differentiable at point x , 3.5.5(1) takes the same form as 3.4.6(3),

$$P_n(f;x) = f(x) + f^{(1)}(x)P_n(t-x;x) + \frac{f^{(2)}(x)}{2} P_n[(t-x)^2;x] + o(n^{-1}),$$

or in the asymptotic form, from lemma 19, at the point $x=\xi$

$$P_n(f;\xi) - f(\xi) = -f^{(1)}(\xi) \left\{ \frac{\xi}{4n^2} + \frac{\xi}{16n^4} + \dots \right\}$$

$$+ \frac{f^{(2)}(\xi)}{2} \left\{ \frac{\xi}{4n} + \frac{\xi^2}{16n^4} + \dots \right\} + o(n^{-1})$$

Therefore

$$\lim_{n \rightarrow \infty} n[P_n(f; \xi) - f(\xi)] = \frac{\xi f^{(2)}(\xi)}{8}$$

or

$$P_n(f; \xi) - f(\xi) = \frac{\xi f^{(2)}(\xi)}{8n} + o(n^{-1})$$

CHAPTER 4

THEOREMS ON DEGREE OF
CONVERGENCE

CHAPTER 4

Degree of Convergence

In the previous chapter we constructed various asymptotic operators, which approach the function uniformly with orders of approximation $o(n^{-1})$, $o(n^{-3})$, etc. It seems possible that if we were given the accepted order of approximation, we could generate an asymptotic operator to meet those conditions. The question which was left to be answered in this chapter is - to what order do these basic operators themselves approximate?

This chapter makes use of Theorem 2.3.1 to get expressions for the rapidity of convergence of the Szasz, Baskakov and Shah & Suryanarayana operators. Each of these operators has

$$L_n(1;x) = 1,$$

(See Szasz 1.3.1(4), Baskakov 1.4.1(3) and Shah & Suryanarayana 1.6.1(2)). Using the simplified form 2.3.4 of Shisha & Mond's theorem

$||f - L_n(f)|| \leq 2\omega(\mu_n)$ we have the following theorems

Theorem 4a

Rapidity of Convergence of the Szasz Operator

From Lemma 3

$$S_n[(t-x)^2; x] = \frac{x}{n}$$

Therefore

$$\mu_n = \left| \left| \frac{x}{n} \right| \right|^{\frac{1}{2}}$$

So that at a point $x=\xi$, $\xi \in [a,b]$, $0 < a < b < \infty$

$$|S_n(f; \xi) - f(\xi)| \leq 2\omega \left\{ \left(\frac{\xi}{n} \right)^{\frac{1}{2}} \right\}$$

Theorem 4b

Rapidity of Convergence of the Baskakov Operator

From Lemma 7

$$K_n[(t-x)^2; x] = \frac{x(1+x)}{n}$$

Therefore

$$\mu_n = \left| \left| \frac{x(1+x)}{n} \right| \right|^{\frac{1}{2}}$$

At a point $x=\xi$, $\xi \in [a,b]$, $0 < a < b < \infty$

Therefore

$$|K_n(f; \xi) - f(\xi)| \leq 2\omega \left\{ \left(\frac{\xi(1+\xi)}{n} \right)^{\frac{1}{2}} \right\}$$

Theorem 4c

Rapidity of Convergence of Shah & Suryanarayana Operators

When the coefficients are Chebychev polynomials of degree n , from Lemma 15

$$\begin{aligned} P_n[(t-x)^2; x] &\leq \frac{x}{2n} - \frac{x}{8n^3} \\ &\leq \frac{x}{2n} \end{aligned}$$

At a point ξ in $[a,b]$, $0 < a < b < \infty$

$$|P_n(f; \xi) - f(\xi)| \leq 2\omega\left(\left(\frac{\xi}{2n}\right)^{\frac{1}{2}}\right).$$

Theorem 4d

When the coefficients are Legendre polynomials of degree n ,

$$P_n[(t-x)^2; x] \leq \frac{x}{4n} + \frac{x^2}{16n^4}$$

At a point ξ in $[a,b]$, $0 < a < b < \infty$

$$|P_n(f; \xi) - f(\xi)| \leq 2\omega\left(\left(\frac{\xi}{4n} + \frac{\xi^2}{16n^4}\right)^{\frac{1}{2}}\right).$$

CHAPTER 5

APPROXIMATION OF THE
DERIVATIVES OF FUNCTIONS

Derivatives of the Operators

So far we have examined how the various operators approximate the functions $f(x)$. In this chapter we look at the derivatives of the approximating operators to see how they approximate the derivatives of the functions.

Before we attempt the first theorem, on derivatives of the Baskakov operator, we will state some obvious lemmas.

§5.1 Theorem 5

If $K_n(f;x)$ is the positive linear Baskakov operator and $f(x)$ is v times differentiable then

$$\lim_{n \rightarrow \infty} K_n^{(v)}(f;x) = f^{(v)}(x) \quad v=1,2,\dots$$

in any bounded closed interval $[a,b]$, $0 < a < b$.

5.1.1

Lemma 21

$$k \binom{n+k}{k} = (n+k) \binom{n+k-1}{k-1} \quad (1)$$

Lemma 22

$$k \binom{n+k-1}{k} = (n+k-1) \binom{k-1+n-1}{k-1} \quad (2)$$

Lemma 23

$$k \binom{n+k+v-1}{k} = (n+k+v-1) \binom{n+k+v-2}{k-1} \quad (3)$$

5.1.2 Proof of Theorem 5

Assume that $f \in C_0^v[a, b]$, i.e. f vanishes outside of $[a, b]$ and possesses any derivatives of orders up to v . The proof is inductive:

$$\begin{aligned} \text{As } K_n(f; x) &\equiv \sum_k h_{n,k}(x) f\left(\frac{k}{n}\right) \\ &\equiv \sum_k \binom{k+n-1}{k} x^k (1+x)^{-(n+k)} f\left(\frac{k}{n}\right) \end{aligned}$$

then

$$\begin{aligned} \frac{d}{dx} K_n(f; x) &\equiv K_n^{(1)}(f; x) \\ &= \sum_k \binom{k+n-1}{k} f\left(\frac{k}{n}\right) \{kx^{k-1}(1+x)^{-(n+k)} - (n+k)x^k(1+x)^{-(n+k+1)}\} \\ &= \sum_k \{f\left(\frac{k}{n}\right)(n+k-1)\binom{k-1+n-1}{k-1}x^{k-1}(1+x)^{-(n+k)} \\ &\quad - (n+k)\binom{k+n-1}{k}f\left(\frac{k}{n}\right)x^k(1+x)^{-(n+k+1)}\} \end{aligned}$$

from lemmas 21 and 22

$$\begin{aligned} &= \sum_k f\left(\frac{k+1}{n}\right)(n+k)\binom{k+n-1}{k}x^k(1+x)^{-(n+k+1)} \\ &\quad - \sum_k f\left(\frac{k}{n}\right)(n+k)\binom{k+n-1}{k}x^k(1+x)^{-(n+k+1)} \\ &= \sum_k (n+k)\binom{n+k-1}{k}x^k(1+x)^{-(n+k+1)} \{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\} \\ &= \sum_k \binom{n+k}{k}x^k(1+x)^{-(n+k+1)} f^{(1)}\left(\frac{k+\theta_{n,k}}{n}\right) \quad 0 \leq \theta_{n,k} < 1 \\ &\equiv \sum_k h_{n+1,k}(x) f^{(1)}\left(\frac{k+\theta_{n,k}}{n}\right) \end{aligned}$$

$$= \sum_k h_{n+1,k}(x) f^{(1)}\left(\frac{k}{n+1}\right) + \sum_k h_{n+1,k}(x) \left\{ f^{(1)}\left(\frac{k+\theta_{n,k}}{n}\right) - f^{(1)}\left(\frac{k}{n+1}\right) \right\}$$

However
$$\left| \frac{k+\theta_{n,k}}{n} - \frac{k}{n+1} \right| < \left| \frac{k+1}{n} - \frac{k}{n+1} \right|$$

$$< \frac{1}{n} \left| 1 + \frac{k}{n+1} \right|$$

$$< \frac{b+1}{n} \quad \text{since } a \leq \frac{k}{n} \leq b$$

Therefore

$$\left| f^{(1)}\left(\frac{k+\theta_{n,k}}{n}\right) - f^{(1)}\left(\frac{k}{n+1}\right) \right| \leq \omega\left(f^{(1)}; \frac{b+1}{n}\right)$$

where $\omega(f^{(1)}, \delta)$ is the modulus of continuity, $\omega(f^{(1)}; \frac{b+1}{n})$ converges uniformly to zero as $n \rightarrow \infty$

Therefore

$$\left| \sum_k h_{n+1,k}(x) \left\{ f^{(1)}\left(\frac{k+\theta_{n,k}}{n}\right) - f^{(1)}\left(\frac{k}{n+1}\right) \right\} \right| \leq \omega\left(f^{(1)}; \frac{b+1}{n}\right)$$

$\rightarrow 0$ as $n \rightarrow \infty$

Therefore

$$\lim_{n \rightarrow \infty} K_n^{(1)}(f; x) = \lim_{n \rightarrow \infty} \sum_k h_{n+1,k}(x) f^{(1)}\left(\frac{k}{n+1}\right)$$

$$\rightarrow f^{(1)}(x).$$

Assume that in the closed interval $[a, b]$

$$\begin{aligned} K_n^{(v)}(f; x) &\equiv \sum_k h_{n+v, k}(x) f^{(v)}\left(\frac{k}{n+v}\right) \\ &\equiv \sum_k \binom{k+n+v-1}{k} x^k (1+x)^{-(n+v+k)} f^{(v)}\left(\frac{k}{n+v}\right) \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dx} K_n^{(v)}(f; x) &\equiv K_n^{(v+1)}(f; x) \\ &= \sum_k \binom{k+n+v-1}{k} f^{(v)}\left(\frac{k}{n+v}\right) \{k x^{k+1} (1+x)^{-(n+v+k)} \\ &\quad - (n+v+k) x^k (1+x)^{-(n+v+k+1)}\} \\ &= \sum_k \{f^{(v)}\left(\frac{k}{n+v}\right) k \binom{k+n+v-1}{k} x^{k-1} (1+x)^{-(n+v+k)} \\ &\quad - (n+v+k) x^k (1+x)^{-(n+v+k+1)} f^{(v)}\left(\frac{k}{n+v}\right) \binom{k+n+v-1}{k}\} \\ &= \sum_k \{f^{(v)}\left(\frac{k}{n+v}\right) (n+k+v-1) \binom{n+k+v-2}{k-1} x^{k-1} (1+x)^{-(n+v+k)} \\ &\quad - (n+v+k) x^k (1+x)^{-(n+v+k+1)} f^{(v)}\left(\frac{k}{n+v}\right) \binom{k+n+v-1}{k}\} \\ &= \sum_{k=0} \{f^{(v)}\left(\frac{k+1}{n+v}\right) (n+k+v) \binom{n+k+v-1}{k} x^k (1+x)^{-(n+v+k+1)} \\ &\quad - (n+k+v) \binom{n+k+v-1}{k} x^k (1+x)^{-(n+v+k+1)} f^{(v)}\left(\frac{k}{n+v}\right)\} \\ &= \sum_k (n+k+v) \binom{n+k+v-1}{k} x^k (1+x)^{-(n+v+k+1)} \{f^{(v)}\left(\frac{k+1}{n+v}\right) - f^{(v)}\left(\frac{k}{n+v}\right)\} \\ &= \sum_k \frac{(n+k+v)}{(n+v)} \binom{n+k+v-1}{k} x^k (1+x)^{-(n+v+k+1)} f^{(v+1)}\left(\frac{k+\theta_{n+v, k}}{n+v}\right) \end{aligned}$$

where $0 \leq \theta_{n+v, k} < 1$

$$\begin{aligned}
 &= \sum_k \binom{n+k+v-2}{k} x^k (1+x)^{-(n+v+k+1)} f^{(v+1)}\left(\frac{k+\theta_{n+v,k}}{n+v}\right) \\
 &\equiv \sum_k h_{n+v+1}(x) f^{(v+1)}\left(\frac{k+\theta_{n+v,k}}{n+v}\right) \\
 &= \sum_k h_{n+v+1,k}(x) f^{(v+1)}\left(\frac{k}{n+v+1}\right) \\
 &\quad + \sum_k h_{n+v+1,k}(x) \left\{ f^{(v+1)}\left(\frac{k+\theta_{n+v,k}}{n+v}\right) - f^{(v+1)}\left(\frac{k}{n+v+1}\right) \right\}
 \end{aligned}$$

Now $\left| \frac{k+\theta_{n+v,k}}{n+v} - \frac{k}{n+v+1} \right| \leq \left| \frac{k+1}{n+v} - \frac{k}{n+v+1} \right|$

$$< \frac{1}{n+v} \left| 1 + \frac{k}{n+v+1} \right|$$

$$< \frac{b+1}{n+v}$$

$$< \frac{b+1}{n} \quad \text{since } a \leq \frac{k}{n} \leq b$$

Therefore we have

$$\left| f^{(v+1)}\left(\frac{k+\theta_{n+v,k}}{n+v}\right) - f^{(v+1)}\left(\frac{k}{n+v+1}\right) \right| < \omega(f^{(v+1)}; \frac{b+1}{n})$$

$$\rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and thus

$$\begin{aligned}
 \lim_{n \rightarrow \infty} K_n^{(v+1)}(f; x) &= \lim_{n \rightarrow \infty} \sum_k h_{n+v+1,k}(x) f^{(v+1)}\left(\frac{k}{n+v+1}\right) \\
 &\rightarrow f^{(v+1)}(x).
 \end{aligned} \tag{1}$$

Theorem 5 is true then for all values of v , $f \in C_0^v[a, b]$.

5.1.3 Corollary to Theorem 5

Define $F = f, x \in [a-2\varepsilon, b+2\varepsilon], f \in C^V[a,b]$
 $= 0$ otherwise (2)

and let \tilde{F} be a regularization of F such that the support of F is contained in $[a-\varepsilon, b+\varepsilon]$ i.e.

$$\begin{aligned}\tilde{F} &= f, x \in [a,b] \\ &= 0 \text{ outside } (a-2\varepsilon, b+2\varepsilon)\end{aligned}$$

Then $\lim_{n \rightarrow \infty} K_n^{(v)}(\tilde{F}; x) = f^{(v)}(x), v=0,1,2,\dots f \in C^V[a,b]$

Proof:

We know $(\widetilde{D^{(v)} F})(t) = D^{(v)}(\tilde{F}(t))$ from 2.4.2

and $\lim_{n \rightarrow \infty} D^{(v)}[K_n(g;t)] = D^{(v)}g(t) \equiv$ from theorem 5

Thus $\lim_{n \rightarrow \infty} D^{(v)} K_n(\tilde{F}; t) = D^{(v)} \tilde{F}(t)$
 $= \widetilde{D^{(v)} F(t)}$

$$= f^{(v)}(t) \quad f \in C^V[a,b]$$

We now consider a similar situation with the Szasz operator as

§ 5.2 Theorem 6

If $S_n(f;x)$ is the positive linear Szasz operator

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

then if $f(x)$ is v times differentiable

$$\lim_{n \rightarrow \infty} S_n^{(v)}(f; x) = f^{(v)}(x)$$

for all x in the bounded closed interval $C_0^v [a, b]$.

5.2.1 Proof of Theorem 6

$$\begin{aligned} S_n(f; x) &\equiv \sum_k s_{n,k}(x) f\left(\frac{k}{n}\right) \\ &\equiv e^{-nx} \sum_k \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dx} S_n(f; x) &= S_n^{(1)}(f; x) \\ &= \sum \frac{f\left(\frac{k}{n}\right)}{k!} \{nk(nx)^{k-1} e^{-nx} - ne^{-nx} (nx)^k\} \\ &= \sum ne^{-nx} \left\{ f\left(\frac{k}{n}\right) \frac{(nx)^{k-1}}{(k-1)!} - f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} \right\} \\ &= \sum ne^{-nx} \frac{(nx)^k}{k!} \left\{ f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right\} \\ &= \sum e^{-nx} \frac{(nx)^k}{k!} f^{(1)}\left(\frac{k+\theta}{n}\right) \quad 0 \leq \theta \leq 1 \\ &= \sum s_{n,k}(x) f^{(1)}\left(\frac{k+\theta}{n}\right) \\ &= \sum s_{n,k}(x) f^{(1)}\left(\frac{k}{n}\right) + \sum s_{n,k}(x) \left\{ f^{(1)}\left(\frac{k+\theta}{n}\right) - f^{(1)}\left(\frac{k}{n}\right) \right\} \end{aligned}$$

and

$$\left| \sum s_{n,k}(x) \left\{ f^{(1)}\left(\frac{k+\theta}{n}\right) - f^{(1)}\left(\frac{k}{n}\right) \right\} \right| \leq \left| \sum s_{n,k}(x) \right| \cdot \left| f^{(1)}\left(\frac{k+\theta}{n}\right) - f^{(1)}\left(\frac{k}{n}\right) \right|$$

$$\leq \varepsilon$$

as $\left| f^{(1)}\left(\frac{k+\theta}{n}\right) - f^{(1)}\left(\frac{k}{n}\right) \right| \leq \varepsilon$ whenever $\frac{\theta}{n} < \delta$

Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n^{(1)}(f; x) &= \lim_{n \rightarrow \infty} \sum_k s_{n,k}(x) f^{(1)}\left(\frac{k}{n}\right) \\ &= f^{(1)}(x) \end{aligned}$$

Assume that in this closed interval

$$\begin{aligned} S_n^{(v)}(f; x) &\equiv \sum_k s_{n,k}(x) f^{(v)}\left(\frac{k}{n}\right) \\ &= e^{-nx} \sum_k \frac{(nx)^k}{k!} f^{(v)}\left(\frac{k}{n}\right) \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dx} S_n^{(v)}(f; x) &\equiv S_n^{(v+1)}(f; x) \\ &= \sum_k \frac{f^{(v)}\left(\frac{k}{n}\right)}{k!} \{ nk(nx)^{k-1} e^{-nx} - ne^{-nx}(nx)^k \} \\ &= e^{-nx} \sum_k \frac{(nx)^k}{k!} n \{ f^{(v)}\left(\frac{k+1}{n}\right) - f^{(v)}\left(\frac{k}{n}\right) \} \\ &= e^{-nx} \sum_k \frac{(nx)^k}{k!} f^{(v+1)}\left(\frac{k+\theta}{n}\right) \quad 0 \leq \theta_v \leq 1 \\ &\equiv \sum_k s_{n,k}(x) f^{(v+1)}\left(\frac{k+\theta}{n}\right) \\ &= \sum_k s_{n,k}(x) f^{(v+1)}\left(\frac{k}{n}\right) + \sum_k s_{n,k}(x) \left\{ f^{(v+1)}\left(\frac{k+\theta}{n}\right) - f^{(v+1)}\left(\frac{k}{n}\right) \right\} \end{aligned}$$

$$|\sum s_{n,k}(x) \{f^{(v+1)}(\frac{k+\theta_v}{n}) - f^{(v+1)}(\frac{k}{n})\}| \leq |\sum s_{n,k}(x)| \cdot |f^{(v+1)}(\frac{k+\theta_v}{n}) - f^{(v+1)}(\frac{k}{n})|$$

$$\leq \varepsilon$$

where $|f^{(v+1)}(\frac{k+\theta_v}{n}) - f^{(v+1)}(\frac{k}{n})| \leq \varepsilon$ whenever $\frac{\theta_v}{n} < \delta$

Therefore

$$\lim_{n \rightarrow \infty} S_n^{(v+1)}(f; x) = f^{(v+1)}(x) \quad (1)$$

The theorem will therefore hold for all v , $f \in C_0^v[a, b]$.

5.2.2 Corollary to Theorem 6

Define $F = f, \quad x \in [a-2\varepsilon, b+2\varepsilon], \quad f \in C^v[a, b]$
 $= 0 \quad \text{otherwise}$

then $\lim_{n \rightarrow \infty} D^{(v)}(S_n(\tilde{F}; x)) = f^{(v)}(x), \quad f \in C^v[a, b]$

where $D^{(v)} \equiv \frac{d^v}{dx^v}$

and \tilde{F} is the regularization of f such that

$$\tilde{F} = f, \quad x \in [a-2\varepsilon, b+2\varepsilon]$$

$$= 0 \quad \text{outside } (a-\varepsilon, b+\varepsilon)$$

Proof $\tilde{F} \in C_0^\infty(-\infty, \infty)$

We know $\widetilde{(D^{(v)}F)}(t) = D^{(v)}(\widetilde{F(t)})$ from 2.4.2

and $\lim_{n \rightarrow \infty} D^{(v)} [S_n(g;t)] = D^{(v)} g(t) \equiv g^{(v)}(t), \quad g \in C_O^v[a,b]$

from Theorem 6.

Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} D^{(v)} (S_n(\tilde{F};x)) &= D^{(v)} \tilde{F}(x) \\ &= \widetilde{D^{(v)} F(x)} \\ &= f^{(v)}(x) \quad \text{for all } f \in C^v[a,b]. \end{aligned}$$

§5.3 Theorem 7

For the positive linear Meyer-König and Zeller operator, $M_n(f;x)$, with $f(x)$ which has a derivative of order v , $v=1,2,\dots$ in the interval $[0,1)$,

$$\lim_{n \rightarrow \infty} M_n^{(v)}(f;x) = f^{(v)}(x) \quad v=1,2,\dots$$

uniformly in the interval.

5.3.1

Lemma 24

$$k \binom{k+n}{k} = (n+1) \binom{k+n}{k-1} \quad (1)$$

Lemma 25

$$\binom{k+n+1}{k} = \binom{k+n}{k} + \binom{k+n}{k-1} \quad (2)$$

Lemma 26

$$\frac{\binom{k+n+1}{k}}{k+n+1} = \frac{\binom{k+n}{k}}{n+1} \quad (3)$$

5.3.2

Lemma 27

$$k \binom{k+n-v}{k} = (n+1-v) \binom{k-1+n-v+1}{k-1} \quad (1)$$

Lemma 28

$$(n+k+1-v) \binom{k+n-v}{k} = (n+1-v) \binom{k+n-v+1}{k} \quad (2)$$

Lemma 29

$$\frac{(n+v+1)(n-v)}{(k+n-v+1)(k+n-v)} \binom{k+n-v+1}{k} = \binom{k+n-v-1}{k} \quad (3)$$

5.3.3 Proof of Theorem 7

From the M-K and Z operator

$$\begin{aligned} M_n(f; x) &\equiv \sum m_{n,k}(x) f\left(\frac{k}{k+n}\right) \\ &\equiv (1-x)^{n+1} \sum f\left(\frac{k}{k+n}\right) \binom{k+n}{k} x^k \end{aligned}$$

differentiated yields

$$\begin{aligned} \frac{d}{dx} M_n(f; x) &\equiv M_n^{(1)}(f; x) \\ &= -(n+1)(1-x)^n \sum f\left(\frac{k}{k+n}\right) \binom{k+n}{k} x^k \\ &\quad + (1-x)^{n+1} \sum f\left(\frac{k}{k+n}\right) \binom{k+n}{k} k x^{k-1} \end{aligned}$$

Using lemma 24 gives

$$\begin{aligned}
 M_n^{(1)}(f; x) &= -(n+1)(1-x)^n \left[f\left(\frac{k}{k+n}\right) \binom{k+n}{k} x^k \right. \\
 &\quad \left. + (n+1)(1-x)^{n+1} \left[f\left(\frac{k}{k+n}\right) \binom{k+n}{k-1} x^{k-1} \right] \right. \\
 &= (n+1)(1-x)^n \left[- \left[f\left(\frac{k}{k+n}\right) \binom{k+n}{k} x^k + \left[f\left(\frac{k}{k+n}\right) \binom{k+n}{k-1} x^{k-1} \right. \right. \right. \\
 &\quad \left. \left. \left. - \left[f\left(\frac{k}{k+n}\right) \binom{k+n}{k-1} x^k \right] \right] \right] \right. \\
 &= (n+1)(1-x)^n \left[\left[f\left(\frac{k+1}{k+1+n}\right) \binom{k+1+n}{k} x^k \right. \right. \\
 &\quad \left. \left. - \left[x^k f\left(\frac{k}{k+n}\right) \left\{ \binom{k+n}{k} + \binom{k+n}{k-1} \right\} \right] \right] \right. \\
 &= (n+1)(1-x)^n \left[\left[f\left(\frac{k+1}{k+n+1}\right) \binom{k+1+n}{k} x^k - \left[x^k f\left(\frac{k}{k+n}\right) \binom{k+1+n}{k} \right] \right] \right]
 \end{aligned}$$

from lemma 25

$$\begin{aligned}
 &= (n+1)(1-x)^n \left[\binom{k+n+1}{k} x^k \left\{ f\left(\frac{k+1}{k+n+1}\right) - f\left(\frac{k}{k+n}\right) \right\} \right. \\
 &= (n+1)(1-x)^n \left[\frac{\binom{k+n+1}{k}}{(k+n+1)} \cdot \frac{n}{(k+n)} x^k f^{(1)}(\phi_{n-1,k}) \right]
 \end{aligned}$$

$$\text{where } \frac{k+1}{k+n+1} > \phi_{n-1,k} > \frac{k}{k+n}$$

$$= (1-x)^n \left[x^k \binom{k+n-1}{k} f^{(1)}(\phi_{n-1,k}) \right]$$

$$\equiv \sum_{m_{n-1,k}}(x) f^{(1)}(\phi_{n-1,k})$$

$$\begin{aligned}
 &= \sum_{n-1,k}^m (x) f^{(1)}\left(\frac{k}{k+n-1}\right) + \sum_{n-1,k}^m (x) \{f^{(1)}(\phi_{n-1,k}) - f^{(1)}\left(\frac{k}{k+n-1}\right)\} \\
 &= \sum_1 + \sum_2
 \end{aligned}$$

However

$$\begin{aligned}
 \left| \frac{k}{k+n-1} - \phi_{n-1,k} \right| &\leq \left| \frac{k}{k+n-1} - \frac{k}{k+n} \right| + \left| \frac{k}{k+n} - \phi_{n-1,k} \right| \\
 &\leq \left| \frac{k}{(k+n)(k+n-1)} \right| + \left| \frac{k}{k+n} - \frac{k+1}{k+n+1} \right| \\
 &\leq \frac{1}{k+n} \left| \frac{k}{k+n-1} + \frac{n}{k+n+1} \right| \\
 &\leq \frac{2}{n}
 \end{aligned}$$

With the modulus of continuity $\omega(f^{(1)}; \delta)$ for $f^{(1)}$ then

$$\left| f^{(1)}(\phi_{n-1,k}) - f^{(1)}\left(\frac{k}{k+n-1}\right) \right| \leq \omega\left(f^{(1)}; \frac{2}{n}\right)$$

So \sum_2 is never greater than $\omega\left(f^{(1)}; \frac{2}{n}\right)$, which converges uniformly to 0 as $n \rightarrow \infty$, in $[0,1)$

$$\begin{aligned}
 \text{Thus } \lim_{n \rightarrow \infty} M_n^{(1)}(f; x) &= \lim_{n \rightarrow \infty} \sum_{n-1,k}^m (x) f^{(1)}(\phi_{n-1,k}) \\
 &= f^{(1)}(x)
 \end{aligned}$$

Now assume

$$\begin{aligned}
 M_n^{(v)}(f; x) &\equiv \sum_{n-v,k}^m (x) f^{(v)}\left(\frac{k}{k+n-v}\right) \\
 &= \sum (1-x)^{n-v+1} \binom{k+n-v}{k} x^k f^{(v)}\left(\frac{k}{k+n-v}\right)
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{d}{dx} M_n^{(v)}(f; x) &\equiv M_n^{(v+1)}(f; x) \\
 &= \sum \binom{k+n-v}{k} f^{(v)} \left(\frac{k}{k+n-v} \right) \{ kx^{k-1}(1-x)^{n+1-v} - (n+1-v)(1-x)^{n-v} x^k \} \\
 &= (1-x)^{n-v} \sum \binom{k+n-v}{k} f^{(v)} \left(\frac{k}{k+n-v} \right) \{ kx^{k-1}(1-x) - (n+1-v)x^k \} \\
 &= (1-x)^{n-v} \sum \binom{k+n-v}{k} f^{(v)} \left(\frac{k}{k+n-v} \right) \{ kx^{k-1} - (n+k+1-v)x^k \} \\
 &= (1-x)^{n-v} \sum f^{(v)} \left(\frac{k}{k+n-v} \right) \{ k \binom{k+n-v}{k} x^{k-1} - (n+k+1-v) \binom{k+n-v}{k} x^k \} \\
 &= (1-x)^{n-v} \sum f^{(v)} \left(\frac{k}{k+n-v} \right) (n+1-v) \{ x^{k-1} \binom{k-1+n-v+1}{k-1} - x^k \binom{k+n-v+1}{k} \}
 \end{aligned}$$

from Lemmas 27 and 28

$$= (1-x)^{n-v} \sum (n+1-v) x^k \binom{k+n-v+1}{k} \{ f^{(v)} \left(\frac{k+1}{k+n+1-v} \right) - f^{(v)} \left(\frac{k}{k+n-v} \right) \}$$

$$\text{where } \frac{k}{k+n-v} < \phi_{n-(v+1), k} < \frac{k+1}{k+n-v+1}$$

$$\equiv \sum m_{n-v-1, k}(x) f^{(v+1)}(\phi_{n-v-1, k})$$

$$= \sum m_{n-v-1, k}(x) f^{(v+1)} \left(\frac{k}{k+n-v-1} \right)$$

$$+ \sum m_{n-v-1, k}(x) \{ f^{(v+1)}(\phi_{n-v-1, k}) - f^{(v+1)} \left(\frac{k}{k+n-v-1} \right) \}$$

$$= \sum_1^* + \sum_2^*$$

Now

$$\begin{aligned}
 \left| \frac{k}{k+n-v-1} - \phi_{n-v-1,k} \right| &\leq \left| \frac{k}{k+n-v-1} - \frac{k}{k+n-v} \right| + \left| \frac{k}{k+n-v} - \phi_{n-v-1,k} \right| \\
 &\leq \left| \frac{k}{k+n-v-1} - \frac{k}{k+n-v} \right| + \left| \frac{k}{k+n-v} - \frac{k+1}{k+n-v+1} \right| \\
 &\leq \frac{1}{k+n-v} \left\{ \left| \frac{k}{k+n-v-1} \right| + \left| \frac{v-n}{k+n-v+1} \right| \right\} \\
 &\leq \frac{2}{n}
 \end{aligned}$$

With the modulus of continuity $\omega(f^{(v+1)}; \delta)$ for $f^{(v+1)}$ the following inequality holds

$$\left| f^{(v+1)}(\phi_{n-v-1,k}) - f^{(v+1)}\left(\frac{k}{k+n-v-1}\right) \right| \leq \omega(f^{(v+1)}; \frac{2}{n}).$$

\sum_2^* is therefore never larger than $\omega(f^{(v+1)}; \frac{2}{n})$ and this converges uniformly to 0 as $n \rightarrow \infty$, in $[0,1)$.

Therefore

$$\begin{aligned}
 \lim_{n \rightarrow \infty} M_n^{(v+1)}(f; x) &= \lim_{n \rightarrow \infty} \sum_k m_{n-v-1,k}(x) f^{(v+1)}\left(\frac{k}{k+n-v-1}\right) \\
 &= f^{(v+1)}(x) \quad v=1,2,\dots
 \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} M_n^{(v)}(f; x) \rightarrow f^{(v)}(x) \text{ is true for all } v \text{ and } f \in C^v[0,1).$$

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