## ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A CERTAIN FOURTH-ORDER DIFFERENTIAL EQUATION

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#### Abstract

The main purpose of this paper is to establish sufficient conditions under which any solution of (1.1) is uniformly bounded and tend to zero as $t \rightarrow \infty$.


## 1. Introduction and Statement of the Result

As we know from the relevant literature, up to now, many results have been obtained on the asymptotic behaviour of solutions of certain non-linear differential equations of the fourth- order (see, e.g., Hara [2-4], Abou-el-Ela, A.M.A and Sadek, A.I. [1], Sadek and Elaiw [7] and Tunç, C. and Tunç, E. [5], Tunç [9-10].

In this paper we investigate the asymptotic behaviour of solutions of the real non-linear ordinary differential equation of fourth order:

$$
\begin{align*}
& x^{(4)}+a(t) f_{1}(x, \dot{x}, \ddot{x}, \dddot{x})+b(t) f_{2}(x, \dot{x}, \ddot{x})+c(t) f_{3}(x, \dot{x})+d(t) f_{4}(x)  \tag{1.1}\\
& =p(t, x, \dot{x}, \ddot{x}, \dddot{x}),
\end{align*}
$$

in which the functions $a, b, c, d, f_{1}, f_{2}, f_{3}, f_{4}$, and $p$ are continuous for all values of their respective arguments. We assume that the functions $a, b, c, d$ are positive definite and differentiable in $R^{+}=[0, \infty)$, and that the derivatives $\frac{\partial}{\partial y} f_{2}(x, y, z), \frac{\partial}{\partial x} f_{3}(x, y)$, $\frac{\partial}{\partial y} f_{3}(x, y), \frac{\partial}{\partial x} f_{2}(x, y, z)$ and $f_{4}^{\prime}(x)$ exist and are continuous for all $x, y, z$ and $w$. The dots indicate differentiation with respect to $t$.

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The main purpose of this work is to prove the following

Theorem. In addition to the basic assumptions on the functions $a, b, c, d$, $f_{1}, f_{2}, f_{3}, f_{4}$, and $p$, suppose that
(i) $A \geq a(t) \geq a_{0}>0, B \geq b(t) \geq b_{0}>0, C \geq c(t) \geq c_{0}>0, D \geq d(t) \geq d_{0}>$ 0 for $t \in R^{+}$;
(ii) $0<\left[\frac{f_{1}(x, y, z, w)}{w}-\alpha_{1}\right] \leq \min \left\{\frac{c_{0} \alpha_{3}}{2 \sqrt{3} \alpha_{4} D A} \sqrt{\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} \varepsilon a_{0} \alpha_{1}}, \frac{\sqrt{6}}{3 A} \sqrt{\frac{\delta_{0} \varepsilon}{c_{0} \alpha_{3}}}\right\}$ for all $x, y, z, w ; \alpha_{1}>0, \alpha_{2}>0, \alpha_{3}>0, \alpha_{4}>0$;
(iii) $f_{3}(x, 0)=0$ and $\frac{\partial}{\partial y} f_{3}(x, y) \geq \alpha_{3}>0$ for all $x$ and $y$;
(iv) There is a finite constant $\delta_{0}>0$ such that

$$
a_{0} b_{0} c_{0} \alpha_{1} \alpha_{2} \alpha_{3}-C^{2} \alpha_{3} \frac{\partial}{\partial y} f_{3}(x, y)-A^{2} D \alpha_{1}^{2} \alpha_{4} \geq \delta_{0}
$$

for all $x, y$ and $z$;
(v) $0 \leq \frac{\partial}{\partial y} f_{3}(x, y)-\frac{f_{3}(x, y)}{y} \leq \delta_{1}<\frac{2 D \delta_{0} \alpha_{4}}{C a_{0} \alpha_{1} c_{0}^{2} \alpha_{3}^{2}}$ for all $x$ and $y \neq 0$,
(vi) $y z \frac{\partial}{\partial x} f_{2}(x, y, z) \leq 0$ for all $x, y$ and $z$
(vii) $f_{2}(x, y, 0)=0, \frac{\partial}{\partial y} f_{2}(x, y, z) \leq 0$ and $0 \leq \frac{f_{2}(x, y, z)}{z}-\alpha_{2} \leq \frac{\varepsilon_{0} c_{0}^{3} \alpha_{3}^{3}}{B D^{2} \alpha_{4}^{2}}(z \neq 0)$,
where $\varepsilon_{0}$ is a positive constant such that

$$
\begin{equation*}
\varepsilon_{0}<\epsilon=\min \left\{\frac{1}{a_{0} \alpha_{1}}, \frac{D \alpha_{4}}{c_{0} \alpha_{3}}, \frac{\delta_{0}}{4 a_{0} c_{0} \alpha_{1} \alpha_{3} \Delta_{0}}, \frac{C c_{0} \alpha_{3}}{4 D \alpha_{4} \Delta_{0}}\left(\frac{2 D \delta_{0} \alpha_{4}}{C a_{0} \alpha_{1} c_{0}^{2} \alpha_{3}^{2}}-\delta_{1}\right)\right\} \tag{1.2}
\end{equation*}
$$

with $\Delta_{0}=\frac{a_{0} b_{0} c_{0} \alpha_{1} \alpha_{2}}{C}+\frac{a_{0} b_{0} c_{0} \alpha_{2} \alpha_{3}}{A D \alpha_{4}}$;
(viii) $\frac{1}{y} \int_{0}^{y} \frac{\partial}{\partial x} f_{3}(x, \zeta) d \zeta \leq \frac{c_{0} \alpha_{3}\left(\varepsilon-\varepsilon_{0}\right)}{4 C}$ for all $x$ and $y \neq 0$, and $\left\{\frac{\partial}{\partial x} f_{3}(x, y)\right\}^{2} \leq$ $\frac{a_{0} \delta_{0} \alpha_{1}\left(\varepsilon-\varepsilon_{0}\right)}{16 C^{2}}$ for all $x$ and $y$;
(ix) $f_{4}(0)=0, f_{4}(x) \operatorname{sgn} x>0(x \neq 0), F_{4}(x) \equiv \int_{0}^{x} f_{4}(\zeta) d \zeta \rightarrow \infty$ as $|x| \rightarrow \infty$ and
$0 \leq \alpha_{4}-f_{4}^{\prime}(x) \leq \frac{\varepsilon \Delta_{0} a_{0}^{2} \alpha_{1}^{2}}{D}$ for all $x$;
(x) $\int_{0}^{\infty} \gamma_{0}(t) d t<\infty, d^{\prime}(t) \rightarrow 0$ as $t \rightarrow \infty$, where $\gamma_{0}(t):=\left|a^{\prime}(t)\right|+b_{+}^{\prime}(t)+$ $\left|c^{\prime}(t)\right|+\left|d^{\prime}(t)\right|$,
$b_{+}^{\prime}(t)=\max \left\{b^{\prime}(t), 0\right\} ;$
(xi) $|p(t, x, y, z, w)| \leq p_{1}(t)+p_{2}(t)\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]^{\delta / 2}+\Delta\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}$,
where $\delta$ and $\Delta$ are constants such that $0 \leq \delta \leq 1, \Delta \geq 0$ and $p_{1}(t), p_{2}(t)$ are nonnegative continuous functions satisfying

$$
\begin{equation*}
\int_{0}^{\infty} p_{i}(t) d t<\infty \quad(i=1,2) . \tag{1.3}
\end{equation*}
$$

If $\Delta$ is sufficiently small, then every solution $x(t)$ of (1.1) is uniformly bounded and satisfies

$$
\begin{equation*}
x(t) \rightarrow 0, \dot{x}(t) \rightarrow 0, \ddot{x}(t) \rightarrow 0, \dddot{x}(t) \rightarrow 0, \quad \text { as } t \rightarrow \infty \tag{1.4}
\end{equation*}
$$

Remark. Our result includes those of Abou-el-Ela and Sadek [1], Sadek and AL-Elaiw [7].
2. The function $V_{0}(t, x, y, z, w)$

In what follows it will be convenient to use the equivalent differential system

$$
\begin{align*}
& \dot{x}=y, \dot{y}=z, \dot{z}=w \\
& \dot{w}=-a(t) f_{1}(x, y, z, w)-b(t) f_{2}(x, y, z)-c(t) f_{3}(x, y)-d(t) f_{4}(x)+p(t, x, y, z, w), \tag{2.1}
\end{align*}
$$

which is obtained from (1.1) by setting $\dot{x}=y, \ddot{x}=z$ and $\dddot{x}=w$.
For the proof of the theorem our main tool is the function $V_{0}=V_{0}(t, x, y, z, w)$ defined as follows:

$$
\begin{align*}
2 V_{0}= & 2 \Delta_{2} d(t) \int_{0}^{x} f_{4}(\zeta) d \zeta+2 c(t) \int_{0}^{y} f_{3}(x, \zeta) d \zeta \\
& +\left[\Delta_{2} \alpha_{2} b(t)-\Delta_{1} \alpha_{4} d(t)\right] y^{2}+a(t) \alpha_{1} z^{2}+2 \Delta_{1} b(t) \int_{0}^{z} f_{2}(x, y, \zeta) d \zeta  \tag{2.2}\\
& -\Delta_{2} z^{2}+\Delta_{1} w^{2}+2 d(t) y f_{4}(x)+2 \Delta_{1} d(t) z f_{4}(x) \\
& +2 \Delta_{2} a(t) \alpha_{1} y z+2 \Delta_{1} c(t) z f_{3}(x, y)+2 \Delta_{2} y w+2 z w+k
\end{align*}
$$

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where

$$
\begin{equation*}
\Delta_{1}=\frac{1}{a_{0} \alpha_{1}}+\varepsilon, \quad \Delta_{2}=\frac{\alpha_{4} D}{c_{0} \alpha_{3}}+\varepsilon \tag{2.3}
\end{equation*}
$$

and $k$ is a positive constant to be determined later in the proof.
Now we will obtain some basic inequalities which will be used in the proof of the result.

By noting (2.3), (i) and (iii) we obtain

$$
\begin{gather*}
\Delta_{1}-\frac{1}{a(t) \alpha_{1}} \geq \varepsilon, \text { for all } x, y, z \text { and all } t \in R^{+},  \tag{2.4}\\
\Delta_{2}-\frac{D \alpha_{4} y}{c(t) f_{3}(x, y)} \geq \varepsilon, \text { for all } x, y \neq 0 \text { and all } t \in R^{+} . \tag{2.5}
\end{gather*}
$$

In view of (2.3), (i) and (iv) it follows that

$$
\begin{aligned}
& \alpha_{2} b(t)-\Delta_{1} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\Delta_{2} a(t) \alpha_{1} \\
& \geq \frac{1}{a_{0} c_{0} \alpha_{1} \alpha_{3}}\left[a_{0} b_{0} c_{0} \alpha_{1} \alpha_{2} \alpha_{3}-C^{2} \alpha_{3} \frac{\partial}{\partial y} f_{3}(x, y)-A^{2} D \alpha_{1}^{2} \alpha_{4}\right] \\
& -\left[c(t) \frac{\partial}{\partial y} f_{3}(x, y)+a(t) \alpha_{1}\right] \varepsilon \\
& \geq \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}-\left[c(t) \frac{\partial}{\partial y} f_{3}(x, y)+a(t) \alpha_{1}\right] \varepsilon .
\end{aligned}
$$

Also (iv) implies that

$$
\begin{equation*}
\frac{\partial}{\partial y} f_{3}(x, y)<\frac{a_{0} b_{0} c_{0} \alpha_{1} \alpha_{2}}{C^{2}}, \quad \alpha_{1}<\frac{a_{0} b_{0} c_{0} \alpha_{2} \alpha_{3}}{A^{2} D \alpha_{4}} \tag{2.6}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\alpha_{2} b(t)-\Delta_{1} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\Delta_{2} a(t) \alpha_{1} \geq \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}-\varepsilon \Delta_{0} \tag{2.7}
\end{equation*}
$$

for all $x, y, z$ and all $t \in R^{+}$.
Let $\Phi_{3}$ be the function defined by

$$
\Phi_{3}(x, y)=\left\{\begin{array}{l}
\frac{f_{3}(x, y)}{y}, \quad y \neq 0  \tag{2.8}\\
\frac{\partial}{\partial y} f_{3}(x, 0), \quad y=0
\end{array}\right.
$$

Then from (iii) and (v) we have

$$
\begin{gather*}
\Phi_{3}(x, y) \geq \alpha_{3} \text { for all } x \text { and } y  \tag{2.9}\\
0 \leq \frac{\partial}{\partial y} f_{3}(x, y)-\Phi_{3}(x, y) \leq \delta_{1} \text { for all } x \text { and } y \tag{2.10}
\end{gather*}
$$

From (2.9), (i) and (2.3) we get

$$
\begin{equation*}
\Delta_{2}-\frac{D \alpha_{4}}{c(t) \Phi_{3}(x, y)} \geq \varepsilon, \quad \text { for all } x, y \text { and all } t \in R^{+} \tag{2.11}
\end{equation*}
$$

To prove the present theorem we need the following two lemmas:

Lemma 1. Subject to the conditions (i)-(ix) of the theorem, there are positive constants $D_{1}$ and $D_{2}$ such that

$$
\begin{equation*}
D_{1}\left[F_{4}(x)+y^{2}+z^{2}+w^{2}+k\right] \leq V_{0} \leq D_{2}\left[F_{4}(x)+y^{2}+z^{2}+w^{2}+k\right] \tag{2.12}
\end{equation*}
$$

for all $x, y, z$ and $w$.
Proof. Since $f_{2}(x, y, 0)=0$ and $\frac{f_{2}(x, y, z)}{z} \geq \alpha_{2}(z \neq 0)$, it is clear that

$$
2 \Delta_{1} b(t) \int_{0}^{z} f_{2}(x, y, \zeta) d \zeta \geq \Delta_{1} b(t) \alpha_{2} z^{2}
$$

Therefore it follows from (2.2) that

$$
\begin{aligned}
2 V_{0} \geq & 2 \Delta_{2} d(t) \int_{0}^{x} f_{4}(\zeta) d \zeta+2 c(t) \int_{0}^{y} f_{3}(x, \zeta) d \zeta+\left[\Delta_{2} \alpha_{2} b(t)-\Delta_{1} \alpha_{4} d(t)\right] y^{2} \\
& +a(t) \alpha_{1} z^{2}+\Delta_{1} b(t) \alpha_{2} z^{2}-\Delta_{2} z^{2}+\Delta_{1} w^{2}+2 d(t) y f_{4}(x)+2 \Delta_{1} d(t) z f_{4}(x) \\
& +2 \Delta_{2} a(t) \alpha_{1} y z++2 \Delta_{1} c(t) z f_{3}(x, y)+2 \Delta_{2} y w+2 z w+k
\end{aligned}
$$

Rewrite above inequality as follows:

$$
\begin{gathered}
2 V_{0} \geq \frac{c(t)}{\Phi_{3}(x, y)}\left[\frac{d(t)}{c(t)} f_{4}(x)+y \Phi_{3}(x, y)+\Delta_{1} z \Phi_{3}(x, y)\right]^{2} \\
+\frac{a(t)}{\alpha_{1}}\left[\frac{w}{a(t)}+\alpha_{1} z+\Delta_{2} \alpha_{1} y\right]^{2}+\left[2 \Delta_{2} d(t) \int_{0}^{x} f_{4}(\zeta) d \zeta-\frac{d^{2}(t) f_{4}^{2}(x)}{c(t) \Phi_{3}(x, y)}\right]
\end{gathered}
$$

$$
\begin{aligned}
& +\left[\Delta_{2} b(t) \alpha_{2}-\Delta_{1} d(t) \alpha_{4}-\Delta_{2}^{2} a(t) \alpha_{1}\right] y^{2}+2 c(t) \int_{0}^{y} f_{3}(x, \zeta) d \zeta-c(t) \Phi_{3}(x, y) y^{2} \\
& +\left[\Delta_{1} \alpha_{2} b(t)-\Delta_{2}-\Delta_{1}^{2} c(t) \Phi_{3}(x, y)\right] z^{2}+\left[\Delta_{1}-\frac{1}{a(t) \alpha_{1}}\right] w^{2}+k
\end{aligned}
$$

From (2.4) we get

$$
\left[\Delta_{1}-\frac{1}{a(t) \alpha_{1}}\right] w^{2} \geq \varepsilon w^{2}
$$

Then

$$
\begin{equation*}
2 V_{0} \geq V_{1}+V_{2}+V_{3}+\varepsilon w^{2}+k, \tag{2.13}
\end{equation*}
$$

where

$$
\begin{gathered}
V_{1}:=2 \Delta_{2} d(t) \int_{0}^{x} f_{4}(\zeta) d \zeta-\frac{d^{2}(t) f_{4}^{2}(x)}{c(t) \Phi_{3}(x, y)}, \\
V_{2}:=\left[\Delta_{2} \alpha_{2} b(t)-\Delta_{1} \alpha_{4} d(t)-\Delta_{2}^{2} a(t) \alpha_{1}\right] y^{2}+2 c(t) \int_{0}^{y} f_{3}(x, \zeta) d \zeta-c(t) \Phi_{3}(x, y) y^{2}, \\
V_{3}:=\left[\Delta_{1} \alpha_{2} b(t)-\Delta_{2}-\Delta_{1}^{2} c(t) \Phi_{3}(x, y)\right] z^{2} .
\end{gathered}
$$

From (2.3), (2.9) and (i) we find

$$
\begin{aligned}
V_{1} & \geq 2 \varepsilon d(t) \int_{0}^{x} f_{4}(\zeta) d \zeta+\frac{D d(t)}{c_{0} \alpha_{3}}\left[2 \alpha_{4} \int_{0}^{x} f_{4}(\zeta) d \zeta-f_{4}^{2}(x)\right] \\
& \geq 2 \varepsilon d(t) \int_{0}^{x} f_{4}(\zeta) d \zeta+\frac{2 D d(t)}{c_{0} \alpha_{3}} \int_{0}^{x}\left[\alpha_{4}-f_{4}^{\prime}(\zeta)\right] f_{4}(\zeta) d \zeta
\end{aligned}
$$

Since the second integral on the right hand side is non-negative by (ix), it clear that

$$
\begin{equation*}
2 \alpha_{4} \int_{0}^{x} f_{4}(\zeta) d \zeta-f_{4}^{2}(x) \geq 0 \tag{2.14}
\end{equation*}
$$

So $V_{1} \geq 2 \varepsilon d_{0} \int_{0}^{x} f_{4}(\zeta) d \zeta$. Also from (2.3), (iii), (i) and (2.7) we obtain

$$
\begin{aligned}
& \Delta_{2} \alpha_{2} b(t)-\Delta_{1} \alpha_{4} d(t)-\Delta_{2}^{2} a(t) \alpha_{1} \\
= & \Delta_{2}\left[\alpha_{2} b(t)-\Delta_{1} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\Delta_{2} a(t) \alpha_{1}\right] \\
& +\Delta_{1}\left[\Delta_{2} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\alpha_{4} d(t)\right] \\
> & \Delta_{2}\left[\alpha_{2} b(t)-\Delta_{1} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\Delta_{2} a(t) \alpha_{1}\right] \\
> & \frac{D \alpha_{4}}{c_{0} \alpha_{3}}\left(\frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}-\varepsilon \Delta_{0}\right) .
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{0}^{y} \zeta \frac{\partial}{\partial \zeta} f_{3}(x, \zeta) d \zeta & \equiv y f_{3}(x, y)-\int_{0}^{y} f_{3}(x, \zeta) d \zeta \\
& =y^{2} \Phi_{3}(x, y)-\int_{0}^{y} f_{3}(x, \zeta) d \zeta
\end{aligned}
$$

then

$$
\begin{aligned}
2 c(t) \int_{0}^{y} f_{3}(x, \zeta) d \zeta-c(t) \Phi_{3}(x, y) y^{2} & =c(t)\left[\int_{0}^{y} f_{3}(x, \zeta) d \zeta-\int_{0}^{y} \zeta \frac{\partial}{\partial \zeta} f_{3}(x, \zeta) d \zeta\right] \\
& =c(t) \int_{0}^{y}\left[\Phi_{3}(x, y)-\frac{\partial}{\partial \zeta} f_{3}(x, \zeta)\right] \zeta d \zeta \\
& \geq-\frac{C \delta_{1}}{2} y^{2}, \quad \text { by }(2.10) .
\end{aligned}
$$

Therefore we have

$$
V_{2} \geq\left[\frac{D \alpha_{4}}{c_{0} \alpha_{3}}\left(\frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}-\varepsilon \Delta_{0}\right)-\frac{C \delta_{1}}{2}\right] y^{2} \geq \frac{C}{4}\left(\frac{2 \alpha_{4} D \delta_{0}}{C a_{0} \alpha_{1} c_{0}^{2} \alpha_{3}^{2}}-\delta_{1}\right) y^{2}, \text { by (1.2). }
$$

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Similarly, from (2.3), (i), (2.10) and (2.7) we obtain

$$
\begin{aligned}
& \Delta_{1} \alpha_{2} b(t)-\Delta_{2}-\Delta_{1}^{2} c(t) \Phi_{3}(x, y) \\
= & \Delta_{1}\left[\alpha_{2} b(t)-\Delta_{1} c(t) \Phi_{3}(x, y)-\Delta_{2} a(t) \alpha_{1}\right]+\Delta_{2}\left[\Delta_{1} a(t) \alpha_{1}-1\right] \\
> & \Delta_{1}\left[\alpha_{2} b(t)-\Delta_{1} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\Delta_{2} a(t) \alpha_{1}\right] \\
> & \frac{1}{a_{0} \alpha_{1}}\left(\frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}-\varepsilon \Delta_{0}\right) .
\end{aligned}
$$

Therefore we obtain

$$
V_{3} \geq \frac{1}{a_{0} \alpha_{1}}\left(\frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}-\varepsilon \Delta_{0}\right) z^{2}, \text { by }(1.2)
$$

Combining the estimates for $V_{1}, V_{2}$ and $V_{3}$ with (2.13) we find

$$
2 V_{0} \geq 2 \varepsilon d_{0} F_{4}(x)+\frac{C}{4}\left(\frac{2 \alpha_{4} D \delta_{0}}{C a_{0} \alpha_{1} c_{0}^{2} \alpha_{3}^{2}}-\delta_{1}\right) y^{2}+\left(\frac{3 \delta_{0}}{4 a_{0}^{2} c_{0} \alpha_{1}^{2} \alpha_{3}}\right) z^{2}+\varepsilon w^{2}+k
$$

Then there exists a positive constant $D_{1}$ such that

$$
V_{0} \geq D_{1}\left[F_{4}(x)+y^{2}+z^{2}+w^{2}+k\right] .
$$

Easily, by noting the hypothesis of the theorem, it can be followed that there exists a positive constant $D_{2}$ such that

$$
V_{0} \leq D_{2}\left[F_{4}(x)+y^{2}+z^{2}+w^{2}+k\right] .
$$

Therefore (2.12) is verified.

Lemma 2. Under the conditions of the theorem there exist positive constants $D_{4}, D_{5}$ and $D_{6}$ such that

$$
\begin{align*}
\dot{V}_{0} \leq & -D_{5}\left(y^{2}+z^{2}+w^{2}\right)+\sqrt{3} D_{6}\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}\left[p_{1}(t)+p_{2}(t)\right] \\
& +\sqrt{3} D_{6} p_{2}(t)\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]+D_{4} \gamma_{0} V_{0} . \tag{2.15}
\end{align*}
$$

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Proof. An easy calculation from (2.2) and (2.1) yields that

$$
\begin{gathered}
\frac{d}{d t} V_{0}=\frac{\partial V_{0}}{\partial w} \dot{w}+\frac{\partial V_{0}}{\partial z} w+\frac{\partial V_{0}}{\partial y} z+\frac{\partial V_{0}}{\partial x} y+\frac{\partial V_{0}}{\partial t} \\
=-\Delta_{1} a(t) w f_{1}(x, y, z, w)-\Delta_{2} b(t) y f_{2}(x, y, z)-\Delta_{2} c(t) y f_{3}(x, y)-b(t) z f_{2}(x, y, z) \\
+w^{2}+c(t) y \int_{0}^{y} \frac{\partial}{\partial x} f_{3}(x, \zeta) d \zeta+\Delta_{1} b(t) z \int_{0}^{z} \frac{\partial}{\partial y} f_{2}(x, y, \zeta) d \zeta+\Delta_{1} b(t) y \int_{0}^{z} \frac{\partial}{\partial x} f_{2}(x, y, \zeta) d \zeta \\
+\Delta_{2} a(t) \alpha_{1} z^{2}+\left[\Delta_{2} \alpha_{2} b(t)-\Delta_{1} \alpha_{4} d(t)\right] y z+\Delta_{1} c(t) z^{2} \frac{\partial}{\partial y} f_{3}(x, y) \\
+\Delta_{1} c(t) y z \frac{\partial}{\partial x} f_{3}(x, y)+d(t) y^{2} f_{4}^{\prime}(x)+\Delta_{1} d(t) y z f_{4}^{\prime}(x) \\
-\Delta_{2} a(t) y f_{1}(x, y, z, w)+\Delta_{2} a(t) \alpha_{1} y w \\
-a(t) z f_{1}(x, y, z, w)+a(t) \alpha_{1} z w+\left(\Delta_{2} y+z+\Delta_{1} w\right) p(t, x, y, z, w)+\frac{\partial V_{0}}{\partial t} .
\end{gathered}
$$

Since

$$
z \int_{0}^{z} \frac{\partial}{\partial y} f_{2}(x, y, \zeta) d \zeta \leq 0, \text { by (vii) and } y \int_{0}^{z} \frac{\partial}{\partial x} f_{2}(x, y, \zeta) d \zeta, \quad \text { by (vi). }
$$

Then we find that

$$
\begin{align*}
\frac{d}{d t} V_{0} & =-\left(V_{4}+V_{5}+V_{6}+V_{7}+V_{8}\right)-\Delta_{2} a(t) y f_{1}(x, y, z, w)+\Delta_{2} a(t) \alpha_{1} y w \\
& -a(t) z f_{1}(x, y, z, w)+a(t) \alpha_{1} z w+\left(\Delta_{2} y+z+\Delta_{1} w\right) p(t, x, y, z, w)+\frac{\partial V_{0}}{\partial t}, \tag{2.16}
\end{align*}
$$

where

$$
\begin{gathered}
V_{4}:=\Delta_{2} c(t) y f_{3}(x, y)-\alpha_{4} d(t) y^{2}-c(t) y \int_{0}^{y} \frac{\partial}{\partial x} f_{3}(x, \zeta) d \zeta-\Delta_{1} c(t) y z \frac{\partial}{\partial x} f_{3}(x, y), \\
V_{5}:=\left[\alpha_{2} b(t)-\Delta_{1} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\Delta_{2} a(t) \alpha_{1}\right] z^{2} \\
V_{6}:=\left[\Delta_{1} a(t) \frac{f_{1}(x, y, z, w)}{w}-1\right] w^{2} \\
V_{7}:=z b(t) f_{2}(x, y, z)-\alpha_{2} b(t) z^{2}+\Delta_{2} b(t) y f_{2}(x, y, z)-\Delta_{2} \alpha_{2} b(t) y z \\
V_{8}:=\alpha_{4} d(t) y^{2}-d(t) f_{4}^{\prime}(x) y^{2}+\Delta_{1} \alpha_{4} d(t) y z-\Delta_{1} d(t) f_{4}^{\prime}(x) y z .
\end{gathered}
$$

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But

$$
\begin{gather*}
V_{4}=c(t) \Phi_{3}(x, y)\left[\Delta_{2}-\frac{D \alpha_{4}}{c(t) \Phi_{3}(x, y)}\right] y^{2}-c(t) y \int_{0}^{y} \frac{\partial}{\partial x} f_{3}(x, \zeta) d \zeta-\Delta_{1} c(t) y z \frac{\partial}{\partial x} f_{3}(x, y) \\
\geq \varepsilon c_{0} \alpha_{3} y^{2}-C y \int_{0}^{y} \frac{\partial}{\partial x} f_{3}(x, \zeta) d \zeta-\Delta_{1} C y z \frac{\partial}{\partial x} f_{3}(x, y), \tag{2.17}
\end{gather*}
$$

by (i), (2.9) and (2.11).

$$
\begin{align*}
V_{5} & =\left[\alpha_{2} b(t)-\Delta_{1} c(t) \frac{\partial}{\partial y} f_{3}(x, y)-\Delta_{2} a(t) \alpha_{1}\right] z^{2}  \tag{2.18}\\
& \geq\left(\frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}-\varepsilon \Delta_{0}\right) z^{2}, \quad \text { by }(2.7), \\
V_{6} & =\left[\Delta_{1} a(t) \frac{f_{1}(x, y, z, w)}{w}-1\right] w^{2} \geq \varepsilon \alpha_{0} \alpha_{1} w^{2}, \tag{2.19}
\end{align*}
$$

by (i), (ii) and (2.3).

$$
\begin{aligned}
V_{7} & =b(t)\left[\frac{f_{2}(x, y, z)}{z}-\alpha_{2}\right]\left(z^{2}+\Delta_{2} y z\right), \text { for } z \neq 0 \\
& \geq-\frac{\Delta_{2}^{2}}{4} b(t)\left[\frac{f_{2}(x, y, z)}{z}-\alpha_{2}\right] y^{2}, \text { by (vii). }
\end{aligned}
$$

By using (vii) and (2.3) we get for $z \neq 0$

$$
\begin{gathered}
\frac{\Delta_{2}^{2}}{4} b(t)\left[\frac{f_{2}(x, y, z)}{z}-\alpha_{2}\right] \leq \frac{1}{4} b(t)\left(\frac{D \alpha_{4}}{c_{0} \alpha_{3}}+\varepsilon\right)^{2} \frac{\varepsilon_{0} c_{0}^{3} \alpha_{3}^{3}}{B D^{2} \alpha_{4}^{2}} \\
=\frac{1}{4} b(t)\left(1+\frac{c_{0} \alpha_{3}}{D \alpha_{4}} \varepsilon\right)^{2} \frac{\varepsilon_{0} c_{0} \alpha_{3}}{B} \leq \varepsilon_{0} c_{0} \alpha_{3},
\end{gathered}
$$

since $\varepsilon<\frac{D \alpha_{4}}{c_{0} \alpha_{3}}$ by (1.2). Then

$$
V_{7} \geq-\varepsilon_{0} c_{0} \alpha_{3} y^{2} \text { for all } x, y \text { and } z \neq 0
$$

but $V_{7}=0$ when $z=0$, so

$$
\begin{equation*}
V_{7} \geq-\varepsilon_{0} c_{0} \alpha_{3} y^{2} \text { for all } x, y \text { and } z \tag{2.20}
\end{equation*}
$$

By (ix)

$$
V_{8}=d(t)\left[\alpha_{4}-f_{4}^{\prime}(x)\right]\left(y^{2}+\Delta_{1} y z\right) \geq-\frac{\Delta_{1}^{2}}{4} d(t)\left[\alpha_{4}-f_{4}^{\prime}(x)\right] z^{2} .
$$

From (ix) and (2.3) we find

$$
\begin{gathered}
\frac{\Delta_{1}^{2}}{4} d(t)\left[\alpha_{4}-f_{4}^{\prime}(x)\right] \leq \frac{1}{4} d(t)\left(\frac{1}{a_{0} \alpha_{1}}+\varepsilon\right)^{2} \frac{\varepsilon \Delta_{0} a_{0}^{2} \alpha_{1}^{2}}{D} \\
=\frac{1}{4} d(t)\left(1+a_{0} \alpha_{1} \varepsilon\right)^{2} \frac{\varepsilon_{0} \Delta_{0}}{D} \leq \varepsilon \Delta_{0}
\end{gathered}
$$

since $\varepsilon<\frac{1}{a_{0} \alpha_{1}}$ by (1.2). Thus it follows that

$$
\begin{equation*}
V_{8} \geq-\varepsilon \Delta_{0} z^{2} \tag{2.21}
\end{equation*}
$$

From (2.17) and (2.20) we have, for $y \neq 0$,

$$
\begin{aligned}
V_{4}+V_{7} & \geq\left[\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3}-\frac{C}{y} \int_{0}^{y} \frac{\partial}{\partial x} f_{3}(x, \zeta) d \zeta\right] y^{2}-\Delta_{1} C y z \frac{\partial}{\partial x} f_{3}(x, y) \\
& \geq \frac{3}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\Delta_{1} C y z \frac{\partial}{\partial x} f_{3}(x, y), \quad \text { by }(\text { viii }) \\
& =\frac{1}{2}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}+\frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3}\left[y^{2}-\frac{4 \Delta_{1} C}{\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3}} y z \frac{\partial}{\partial x} f_{3}(x, y)\right] \\
& \geq \frac{1}{2}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\frac{\Delta_{1}^{2} C^{2}}{\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3}}\left[\frac{\partial}{\partial x} f_{3}(x, y)\right]^{2} z^{2} \\
& \geq \frac{1}{2}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\frac{\delta_{0}}{4 a_{0} \alpha_{1} c_{0} \alpha_{3}} z^{2}
\end{aligned}
$$

by using (vii), (2.3) and (1.2). But $V_{4}+V_{7}=0$, when $y=0$,by (2.17) and (2.20); therefore we have

$$
\begin{equation*}
V_{4}+V_{7} \geq \frac{1}{2}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\frac{\delta_{0}}{4 a_{0} \alpha_{1} c_{0} \alpha_{3}} z^{2}, \text { for all } y \text { and } z . \tag{2.22}
\end{equation*}
$$

From the estimates given by (2.18), (2.19), (2.21) and (2.22) we get

$$
\begin{gather*}
\dot{V}_{0} \leq-\frac{1}{2}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\left(\frac{3 \delta_{0}}{4 a_{0} c_{0} \alpha_{1} \alpha_{3}}-2 \varepsilon \Delta_{0}\right) z^{2} \\
-\varepsilon a_{0} \alpha_{1} w^{2}-a(t) z f_{1}(x, y, z, w)+a(t) \alpha_{1} z w \\
-\Delta_{2} a(t) y f_{1}(x, y, z, w)+\Delta_{2} a(t) \alpha_{1} y w+\left(\Delta_{2} y+z+\Delta_{1} w\right) p(t, x, y, z, w)+\frac{\partial V_{0}}{\partial t} \\
\leq-\frac{1}{2}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\frac{1}{4} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2}-\varepsilon a_{0} \alpha_{1} w^{2}-a(t) z f_{1}(x, y, z, w)+a(t) \alpha_{1} z w \\
-\Delta_{2} a(t) y f_{1}(x, y, z, w)+\Delta_{2} a(t) \alpha_{1} y w+\left(\Delta_{2} y+z+\Delta_{1} w\right) p(t, x, y, z, w)+\frac{\partial V_{0}}{\partial t}, \tag{2.23}
\end{gather*}
$$

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since $\varepsilon<\frac{\delta_{0}}{4 a_{0} c_{0} \alpha_{1} \alpha_{3} \Delta_{0}}$ by (1.2).Consider the expressions

$$
\begin{gathered}
W_{1}=-\frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\frac{1}{3} \varepsilon a_{0} \alpha_{1} w^{2} \\
-\Delta_{2} a(t)\left[\frac{f_{1}(x, y, z, w)}{w}-\alpha_{1}\right] y w
\end{gathered}
$$

and

$$
W_{2}=-\frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2}-\frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2}-\frac{1}{3} \varepsilon a_{0} \alpha_{1} w^{2}-a(t)\left[\frac{f_{1}(x, y, z, w)}{w}-\alpha_{1}\right] z w
$$

which is contained in (2.23). Because of the inequalities

$$
\begin{aligned}
& -W_{1}=\frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}+\frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}+\frac{1}{3} \varepsilon a_{0} \alpha_{1} w^{2} \\
& +\Delta_{2} a(t)\left[\frac{f_{1}(x, y, z, w)}{w}-\alpha_{1}\right] y w \\
& \geq \frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}+\left[\frac{1}{2} \sqrt{\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3}}|y| \pm \sqrt{\frac{1}{3} \varepsilon a_{0} \alpha_{1}}|w|\right]^{2} \\
& \geq 0, \quad \text { by (ii), }
\end{aligned}
$$

and

$$
\begin{aligned}
& -W_{2}=\frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2}+\frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2}+\frac{1}{3} \varepsilon a_{0} \alpha_{1} w^{2}+a(t)\left[\frac{f_{1}(x, y, z, w)}{w}-\alpha_{1}\right] z w \\
& \geq \frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2}+\left[\sqrt{\frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}}}|z| \pm \sqrt{\frac{1}{3} \varepsilon a_{0} \alpha_{1}}|w|\right]^{2} \\
& \geq 0, \quad \text { by }(\mathrm{ii}),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
W_{1} & \leq-\frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}, \\
W_{2} & \leq-\frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2} .
\end{aligned}
$$

Hence, a combination of the estimates $W_{1}$ and $W_{2}$ with (2.23) yields that

$$
\begin{aligned}
\dot{V}_{0} & \leq-\frac{1}{4}\left(\varepsilon-\varepsilon_{0}\right) c_{0} \alpha_{3} y^{2}-\frac{1}{2} \frac{\delta_{0}}{a_{0} c_{0} \alpha_{1} \alpha_{3}} z^{2}-\frac{1}{3} \varepsilon a_{0} \alpha_{1} w^{2} \\
& +\left(\Delta_{2} y+z+\Delta_{1} w\right) p(t, x, y, z, w)+\frac{\partial V_{0}}{\partial t}
\end{aligned}
$$

From (2.2) we obtain

$$
\begin{aligned}
\frac{\partial V_{0}}{\partial t}= & a^{\prime}(t)\left[\frac{1}{2} \alpha_{1} z^{2}+\frac{1}{2} \Delta_{2} \alpha_{1} y z\right] \\
& +b^{\prime}(t)\left[\Delta_{1} \int_{0}^{z} f_{2}(x, y, \zeta) d \zeta+\frac{1}{4} \Delta_{2} \alpha_{2} y^{2}\right]+c^{\prime}(t)\left[\int_{0}^{y} f_{3}(x, \zeta) d \zeta+\Delta_{1} z f_{3}(x, y)\right] \\
& +d^{\prime}(t)\left[\Delta_{2} \int_{0}^{x} f_{4}(\zeta) d \zeta-\frac{1}{2} \Delta_{1} \alpha_{4} y^{2}+y f_{4}(x)+\Delta_{1} z f_{4}(x)\right]
\end{aligned}
$$

From the assumptions in the theorem, (2.6) and (2.14) we have a positive constant $D_{3}$ satisfying

$$
\frac{\partial V_{0}}{\partial t} \leq D_{3}\left[\left|a^{\prime}(t)\right|+b_{+}^{\prime}(t)+\left|c^{\prime}(t)\right|+\left|d^{\prime}(t)\right|\right]\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right] \leq D_{4} \gamma_{0} V_{0}
$$

by using the inequality (2.12), where $D_{4}=\frac{D_{3}}{D_{1}}$. Therefore one can find a positive constant $D_{5}$ such that

$$
\dot{V}_{0} \leq-2 D_{5}\left(y^{2}+z^{2}+w^{2}\right)+\left(\Delta_{2} y+z+\Delta_{1} w\right) p(t, x, y, z, w)+D_{4} \gamma_{0} V_{0} .
$$

Let $D_{6}=\max \left(\Delta_{2}, 1, \Delta_{1}\right)$, then

$$
\begin{aligned}
\dot{V}_{0} & \leq-2 D_{5}\left(y^{2}+z^{2}+w^{2}\right)+\sqrt{3} D_{6}\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}|p(t, x, y, z, w)|+D_{4} \gamma_{0} V_{0} \\
& \leq-2 D_{5}\left(y^{2}+z^{2}+w^{2}\right)+\sqrt{3} D_{6}\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}\left\{p_{1}(t)\right. \\
& \left.+p_{2}(t)\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]^{\delta / 2}+\Delta\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}\right\}+D_{4} \gamma_{0} V_{0} .
\end{aligned}
$$

Let $\Delta$ be fixed, in what follows, to satisfy $\Delta=\frac{D_{5}}{\sqrt{3} D_{6}}$ with this limitation on $\Delta$ we have

$$
\begin{align*}
\dot{V}_{0} \leq & -D_{5}\left(y^{2}+z^{2}+w^{2}\right)+\sqrt{3} D_{6}\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}\left\{p_{1}(t)\right.  \tag{2.24}\\
& \left.+p_{2}(t)\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]^{\delta / 2}\right\}+D_{4} \gamma_{0} V_{0} .
\end{align*}
$$

Note that

$$
\begin{equation*}
\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]^{\delta / 2} \leq 1+\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]^{1 / 2} \tag{2.25}
\end{equation*}
$$

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From (2.24) and (2.25) we find

$$
\begin{aligned}
\dot{V}_{0} \leq & -D_{5}\left(y^{2}+z^{2}+w^{2}\right)+\sqrt{3} D_{6}\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}\left[p_{1}(t)+p_{2}(t)\right] \\
& +\sqrt{3} D_{6} p_{2}(t)\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]+D_{4} \gamma_{0} V_{0}
\end{aligned}
$$

## 3. Completion of the Proof

We define

$$
\begin{equation*}
V(t, x, y, z, w)=\exp \left(-\int_{0}^{t} \gamma(\tau) d \tau\right) V_{0}(t, x, y, z, w) \tag{3.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma(t)=D_{4} \gamma_{0}+\frac{2 \sqrt{3} D_{6}}{D_{1}}\left[p_{1}(t)+p_{2}(t)\right] . \tag{3.2}
\end{equation*}
$$

Then it is easy to see that there exist two functions $U_{1}(r), U_{2}(r)$ satisfying

$$
\begin{equation*}
U_{1}(\|\bar{x}\|) \leq V(t, x, y, z, w) \leq U_{2}(\|\bar{x}\|), \tag{3.3}
\end{equation*}
$$

for all $\bar{x} \in R^{4}$ and $t \in R^{+}$where $U_{1}(r)$ is a continuous increasing positive definite function, $U_{1}(r) \rightarrow \infty$ as $r \rightarrow \infty$ and $U_{2}(r)$ is a continuous increasing function.

From (3.1), (2.15), (3.2) and (2.12) we have

$$
\begin{gathered}
\dot{V}=\exp \left(-\int_{0}^{t} \gamma(\tau) d \tau\right)\left[\dot{V}_{0}-\gamma(t) V_{0}\right] \\
\leq \exp \left(-\int_{0}^{t} \gamma(\tau) d \tau\right)\left\{-D_{5}\left(y^{2}+z^{2}+w^{2}\right)+\sqrt{3} D_{6}\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}\left[p_{1}(t)+p_{2}(t)\right]\right. \\
\left.-\sqrt{3} D_{6}\left[p_{1}(t)+p_{2}(t)\right]\left[F_{4}(x)+y^{2}+z^{2}+w^{2}+2 k\right]\right\} \\
\leq \exp \left(-\int_{0}^{t} \gamma(\tau) d \tau\right)\left\{-D_{5}\left(y^{2}+z^{2}+w^{2}\right)\right. \\
\left.\left.-\sqrt{3} D_{6}\left[p_{1}(t)+p_{2}(t)\right]\left[\left(\sqrt{y^{2}+z^{2}+w^{2}}-\frac{1}{2}\right)^{2}-\frac{1}{4}+2 k\right]\right]\right\}
\end{gathered}
$$

Setting $k \geq \frac{1}{8}$, we can find a positive constant $D_{7}$ such that

$$
\begin{equation*}
\dot{V} \leq-D_{7}\left(y^{2}+z^{2}+w^{2}\right)=-U(\|\bar{x}\|) . \tag{3.4}
\end{equation*}
$$

From inequalities (3.3) and (3.4) it follows that all the solutions $(x(t), y(t), z(t), w(t))$ of (2.1) are uniformly bounded [12; Theorem 10.2].

## Auxiliary Lemma

We consider a system of differential equations

$$
\begin{equation*}
\dot{\bar{x}}=F(t, \bar{x})+G(t, \bar{x}), \tag{3.5}
\end{equation*}
$$

where $F(t, \bar{x})$ and $G(t, \bar{x})$ are continuous vector functions on $R^{+} \times Q(Q$ is an open set in $\left.R^{n}\right)$. We assume

$$
\|G(t, \bar{x})\| \leq G_{1}(t, \bar{x})+G_{2}(\bar{x})
$$

where $G_{1}(t, \bar{x})$ is non-negative continuous scalar function on $R^{+} \times Q$ and $\int_{0}^{t} G_{1}(\tau, \bar{x}) d \tau$ is bounded for all $t$ whenever $\bar{x}$ belongs to any compact subset of $Q$ and $G_{2}(\bar{x})$ is a non-negative continuous scalar function on $Q$.

The following lemma is a simple extension of the well-known result obtained by Yoshizawa [12; Theorem 14.2].

Lemma 3. Suppose that there exists a non-negative continuously differentiable scalar function $V(t, \bar{x})$ on $R^{+} \times Q$ such that $\dot{V}_{(3.5)}(t, \bar{x}) \leq-U(\|\bar{x}\|)$, where $U(\|\bar{x}\|)$ is positive definite with respect to a closed set $\Omega$ of $Q$. Moreover, suppose that $F(t, \bar{x})$ of system (3.5) is bounded for all $t$ when $\bar{x}$ belongs to an arbitrary compact set in $Q$ and that $F(t, \bar{x})$ satisfies the following two conditions with respect to $\Omega$
(1) $F(t, \bar{x})$ tends to a function $H(\bar{x})$ for $\bar{x} \in \Omega$ as $t \rightarrow \infty$, and on any compact set in $\Omega$ this convergence is uniform;
(2) Corresponding to each $\varepsilon>0$ and each $\bar{y} \in \Omega$, there exist a $\delta, \delta=\delta(\varepsilon, \bar{y})$ and a $T=T(\varepsilon, \bar{y})$ such that if $t \geq T$ and $\|\bar{x}-\bar{y}\| \delta$, we have $\|F(t, \bar{x})-F(\varepsilon, \bar{y})\|<\varepsilon$. And suppose that
(3) $G_{2}(\bar{x})$ is positive definite with respect to a closed set $\Omega$ of $Q$.

Then every bounded solution of (3.5) approaches the largest semi-imvariant set of the system $\dot{\bar{x}}=H(\bar{x})$ contained in $\Omega$ as $t \rightarrow \infty$.

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Proof. (See [7]) From (2.1) we set $F$ and $G$ in (3.5) as follows

$$
\begin{gathered}
F(t, \bar{x})=\left[\begin{array}{c}
y \\
z \\
w \\
-a(t) f_{1}(x, y, z, w) w-b(t) f_{2}(x, y, z)-c(t) f_{3}(x, y)-d(t) f_{4}(x)
\end{array}\right], \\
G(t, \bar{x})=\left[\begin{array}{c}
0 \\
0 \\
0 \\
p(t, x, y, z, w)
\end{array}\right]
\end{gathered}
$$

Thus from (xi) we find

$$
\|G(t, \bar{x})\| \leq p_{1}(t)+p_{2}(t)\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]^{\delta / 2}+\Delta\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2} .
$$

Let

$$
G_{1}(t, \bar{x})=p_{1}(t)+p_{2}(t)\left[F_{4}(x)+y^{2}+z^{2}+w^{2}\right]^{\delta / 2} \text { and } G_{2}(\bar{x})=\Delta\left(y^{2}+z^{2}+w^{2}\right)^{1 / 2}
$$

Then $F(t, \bar{x})$ and $G(t, \bar{x})$ clearly satisfy the conditions of Lemma 3.
Now $U(\|\bar{x}\|)$ in (3.4) is positive definite with respect to the closed set $\Omega=$ $\left\{(x, y, z, w) \mid x \in R^{+}, y=0, z=0, w=0\right\}$, it follows that, in $\Omega$,

$$
F(t, \bar{x})=\left[\begin{array}{c}
0 \\
0 \\
0 \\
-d(t) f_{4}(x)
\end{array}\right]
$$

From (i) and (x), we have $d(t) \rightarrow d_{\infty}$ as $t \rightarrow \infty$ where $0 \leq d_{0}<d_{\infty} \leq D$. If we set

$$
H(\bar{x})=\left[\begin{array}{c}
0  \tag{3.6}\\
0 \\
0 \\
-d_{\infty} f_{4}(x)
\end{array}\right]
$$

then the conditions on $H(\bar{x})$ of Lemma 3 are satisfied. Moreover $G_{2}(\bar{x})$ is positive definite with respect to a closed set $\Omega$.

Since all of the solutions of (2.1) are bounded, it follows from Lemma 3 that every solution of (2.1) approaches the largest semi-imvariant set of the system $\dot{\bar{x}}=H(\bar{x})$ contained in $\Omega$ as $t \rightarrow \infty$. From (3.6), $\dot{\bar{x}}=H(\bar{x})$ is the system

$$
\dot{x}=0, \dot{y}=0, \dot{z}=0, \dot{w}=-d_{\infty} f_{4}(x)
$$

which has the solutions $x=k_{1}, y=k_{2}, z=k_{3}, w=k_{4}-d_{\infty} f_{4}\left(k_{1}\right)\left(t-t_{0}\right)$. To remain in $\Omega ; k_{2}=k_{3}=0$ and $k_{4}-d_{\infty} f_{4}\left(k_{1}\right)\left(t-t_{0}\right)=0$ for all $t \geq t_{0}$ which implies $k_{1}=k_{4}=0$.

Therefore the only solution of $\dot{\bar{x}}=H(\bar{x})$ remaining in $\Omega$ is $\bar{x}=\overline{0}$, that is, the largest semi-invariant set of $\dot{\bar{x}}=H(\bar{x})$ contained in $\Omega$ is the point $(0,0,0,0)$. Then it follows that

$$
x(t) \rightarrow 0, y(t) \rightarrow 0, z(t) \rightarrow 0, w(t) \rightarrow 0 \text { as } t \rightarrow \infty
$$

which are equivalent to (1.4).
This completes the proof of the theorem.

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