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On the asymptotic behaviour of solutions of difference equations

by J. POPENDA (Poznań)

Abstract. In this paper we examine the asymptotic behaviour of the solutions of certain classes of m-th order difference equations. The results are obtained under the assumption that the solutions of the arising first order difference equation are bounded.

In this note we are concerned with the nonlinear difference equation of the form

$$\text{(E)} \qquad \Delta^m y_n = F(n, y_n, \Delta y_n, \ldots, \Delta^{m-1} y_n), \quad m \geq 1, \ n \in N,$$

where $F\colon N \times \boldsymbol{R}^m \to \boldsymbol{R}$; $N := \{n_0, n_0+1, \ldots\}$, n_0 is a given nonnegative integer.

In the case of $m = 2$, a number of results have been obtained by Golovina [1], Hooker and Patula [2], Patula [3], Popenda and Werbowski [6], Popenda and Schmeidel [5]. The general case of an m-th order difference equation has been considered in [4]. The impetus to the present investigation was given by a recent paper by Werbowski [8], in which the analogous problem was studied for differential equations.

Here $y_n = y(n)$, $\boldsymbol{R}_+ := (0, \infty)$, $\boldsymbol{R}_0 := [0, \infty)$, $\overline{k, t} := \{k, k+1, \ldots, t\}$, where k, t are any nonnegative integers, $k \leq t$. For a function $x\colon N \to \boldsymbol{R}$, we define the forward difference operator Δ^i as follows:

$$\Delta^0 x_n = x_n, \quad \Delta^k x_n = \Delta(\Delta^{k-1} x_n) = \Delta^{k-1} x_{n+1} - \Delta^{k-1} x_n \quad \text{for } k \geq 1.$$

We assume that $\sum_{j=k}^{t} x_j = 0$ for $t < k$.

We first remind a useful lemma.

LEMMA 1. *Let* a, $\Delta b\colon N \to \boldsymbol{R}_0$, $b\colon N \to \boldsymbol{R}_+$, $\lim_{n\to\infty} b_n = \infty$. *If*

$$\text{(1)} \qquad \sum_{j=n_0}^{\infty} \frac{a_j}{b_j} < \infty,$$

then

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=n_0}^{n-1} a_j = 0. \tag{2}$$

Proof. From (1) it follows that for every $\varepsilon > 0$ there exists $n_1 \in N$ such that for all $n \geqslant n_1$ we have

$$\sum_{j=n_1}^{n} \frac{a_j}{b_j} < \varepsilon.$$

Therefore we obtain for $n \geqslant n_1$

$$\frac{1}{b_n} \sum_{j=n_0}^{n-1} a_j \leqslant \frac{1}{b_n} \sum_{j=n_0}^{n_1-1} a_j + \frac{1}{b_n} \sum_{j=n_1}^{n} b_j \frac{a_j}{b_j} < \frac{1}{b_n} \sum_{j=n_0}^{n_1-1} a_j + \varepsilon$$

and (2) results.

Now we formulate the main result.

THEOREM. *Suppose that there exist functions* $g\colon N \to \boldsymbol{R}$, $h\colon N \to \boldsymbol{R}_+$ *such that*

$$\lim_{n\to\infty} \frac{1}{\Delta^{m-1} h_n} \sum_{j=n_0}^{n-1} g_j = c = \text{const}, \tag{3}$$

$$\Delta^i h\colon N \to \boldsymbol{R}_+, \quad i \in \overline{0, m-1}, \quad \Delta^m h\colon N \to \boldsymbol{R}_0, \quad \lim_{n\to\infty} \Delta^{m-1} h_n = \infty \tag{4}$$

and

$$|F(n, x_1, \ldots, x_m) - g_n| \leqslant w(n, |x_1|, \ldots, |x_m|) \quad \text{on } N \times \boldsymbol{R}^m, \tag{5}$$

where $w\colon N \times \boldsymbol{R}_0^m \to \boldsymbol{R}_0$ *is a nondecreasing function in the last* m *arguments. If all solutions of the equation*

$$\Delta z_n = \frac{1}{\Delta^{m-1} h_n} w(n, z_n h_n, \ldots, z_n \Delta^{m-1} h_n), \quad n \in N \tag{E1}$$

are bounded, then every solution $y\colon N \to \boldsymbol{R}$ *of* (E) *fulfils the asymptotic relation*

$$\lim_{n\to\infty} \frac{\Delta^i y_n}{\Delta^i h_n} = c, \quad i \in \overline{0, m-1}. \tag{AR}$$

Proof. Let y be any solution of (E). Summing (E) we get

$$\Delta^{m-1} y_n = \Delta^{m-1} y_{n_0} + \sum_{j=n_0}^{n-1} F(j, y_j, \ldots, \Delta^{m-1} y_j), \quad n \in N.$$

Hence, by (5),

$$|\Delta^{m-1} y_n| \leqslant |\Delta^{m-1} y_{n_0}| + \Big|\sum_{j=n_0}^{n-1} g_j\Big| + \sum_{j=n_0}^{n-1} w(j, |y_j|, \ldots, |\Delta^{m-1} y_j|), \quad n \in N.$$

We hence infer, by the monotonicity of $\Delta^{m-1}h$, that

$$(6)\qquad \frac{|\Delta^{m-1}y_n|}{\Delta^{m-1}h_n} \leqslant \frac{|\Delta^{m-1}y_{n_0}|}{\Delta^{m-1}h_n} + \frac{1}{\Delta^{m-1}h_n}\Big|\sum_{j=n_0}^{n-1} g_j\Big| + \\ + \sum_{j=n_0}^{n-1} \frac{1}{\Delta^{m-1}h_j} w(j, |y_j|, \ldots, |\Delta^{m-1}y_j|), \qquad n \in N.$$

By virtue of (3) and (4) there exist finite constants d_0, d_1 such that

$$(7)\qquad \begin{aligned} \frac{1}{\Delta^{m-1}h_n}\Big|\sum_{j=n_0}^{n-1} g_j\Big| &\leqslant \sup_{n\in N}\left|\frac{1}{\Delta^{m-1}h_n}\sum_{j=n_0}^{n-1} g_j\right| = d_0, \\ \frac{|\Delta^{m-1}y_{n_0}|}{\Delta^{m-1}h_n} &\leqslant \frac{|\Delta^{m-1}y_{n_0}|}{\Delta^{m-1}h_{n_0}} = d_1 \end{aligned}$$

holds for all $n \in N$. Write

$$(8)\qquad d = \max\left\{d_0 + d_1,\ \frac{\max\limits_{i\in\overline{0,m-1}} |\Delta^i y_{n_0}|}{\min\limits_{i\in\overline{0,m-1}} \Delta^i h_{n_0}}\right\}.$$

Using (8) it is a simple matter to get from (6) the inequality

$$\frac{|\Delta^{m-1}y_n|}{\Delta^{m-1}h_n} \leqslant d + \sum_{j=n_0}^{n-1} \frac{1}{\Delta^{m-1}h_j} w(j, |y_j|, \ldots, |\Delta^{m-1}y_j|), \qquad n \in N.$$

Consider the solution of (E1) fulfilling the initial condition $z_{n_0} = 2d$. By (8) we get

$$(9)\qquad \frac{|\Delta^i y_{n_0}|}{\Delta^i h_{n_0}} \leqslant d = z_{n_0} - d \leqslant z_{n_0}, \qquad i \in \overline{0, m-1}.$$

Hence, by the monotonicity of w, in view of (9), we obtain

$$\begin{aligned} \frac{|\Delta^{m-1}y_{n_0+1}|}{\Delta^{m-1}h_{n_0+1}} &\leqslant d + \frac{1}{\Delta^{m-1}h_{n_0}} w(n_0, |y_{n_0}|, \ldots, |\Delta^{m-1}y_{n_0}|) \\ &\leqslant d + \frac{1}{\Delta^{m-1}h_{n_0}} w(n_0, z_{n_0}h_{n_0}, \ldots, z_{n_0}\Delta^{m-1}h_{n_0}) \\ &= z_{n_0+1} - d \leqslant z_{n_0+1}. \end{aligned}$$

It should be observed that the solution z is a nondecreasing function, because w is nonnegative. From the equality

$$\Delta^i x_n = \Delta^{i+1} x_{n-1} + \Delta^i x_{n-1}, \qquad i \geqslant 0,\ n \geqslant 1,$$

holding for any function on N, we get

$$\begin{aligned}\frac{|\Delta^i y_{n_0+1}|}{\Delta^i h_{n_0+1}} &\leqslant \frac{|\Delta^{i+1} y_{n_0}|+|\Delta^i y_{n_0}|}{\Delta^i h_{n_0+1}} \\ &\leqslant \frac{(z_{n_0}-d)\,\Delta^{i+1} h_{n_0}+(z_{n_0}-d)\,\Delta^i h_{n_0}}{\Delta^i h_{n_0+1}} \\ &= (z_{n_0}-d)\frac{\Delta^i h_{n_0+1}}{\Delta^i h_{n_0+1}} \leqslant z_{n_0+1}-d \leqslant z_{n_0+1}.\end{aligned}$$

An inductive argument shows that

$$\frac{|\Delta^i y_n|}{\Delta^i h_n} \leqslant z_n-d \leqslant z_n, \quad i\in\overline{0,\, m-1},\ n\in N. \tag{10}$$

As observed, z is nondecreasing and bounded, hence convergent. Let

$$\lim_{n\to\infty} z_n = M.$$

Clearly, $M=0$ if and only if $z_n=0$ for all $n\in N$, since nonnegative solutions are only taken into account. Therefore $z_{n_0}=d=0$. This means that $g_n=0$ for all $n\in N$ and hence $c=0$. It can be checked that $\Delta^i y_n=0$ ($n\in N$, $i\in\overline{0,\, m-1}$), on account of the former relations. Thus the solution y satisfies (AR) and the theorem holds. Now we consider the case of $M>0$. Summing (E1) we obtain

$$z_n = z_{n_0}+\sum_{j=n_0}^{n-1}\frac{1}{\Delta^{m-1} h_j} w(j,\, z_j h_j,\, \ldots,\, z_j \Delta^{m-1} h_j), \qquad n\in N.$$

The left-hand side tends to M as $n\to\infty$, hence so does the right-hand side:

$$M = z_{n_0}+\sum_{j=n_0}^{\infty}\frac{1}{\Delta^{m-1} h_j} w(j,\, z_j h_j,\, \ldots,\, z_j \Delta^{m-1} h_j).$$

This implies that the series below converges,

$$\sum_{j=n_0}^{\infty}\frac{1}{\Delta^{m-1} h_j} w(j,\, z_j h_j,\, \ldots,\, z_j \Delta^{m-1} h_j) = M-2d < \infty.$$

We take any $\varepsilon>0$ and choose $n_1\in N$ such that

$$z_n \geqslant M-\varepsilon > 0 \quad \text{for all } n\geqslant n_1.$$

By the monotonicity of w we get

$$\sum_{j=n_1}^{n} \frac{1}{\Delta^{m-1} h_j} w(j, (M-\varepsilon) h_j, \ldots, (M-\varepsilon) \Delta^{m-1} h_j)$$
$$\leqslant \sum_{j=n_1}^{n} \frac{1}{\Delta^{m-1} h_j} w(j, z_j h_j, \ldots, z_j \Delta^{m-1} h_j), \qquad n \geqslant n_1.$$

Passing with n to infinity, we conclude that the series

$$\sum_{j=n_0}^{\infty} \frac{1}{\Delta^{m-1} h_j} w(j, (M-\varepsilon) h_j, \ldots, (M-\varepsilon) \Delta^{m-1} h_j)$$

is convergent for an arbitrary positive $\varepsilon < M$. Thus, by (10), also the series

$$\sum_{j=n_0}^{\infty} \frac{1}{\Delta^{m-1} h_j} w(j, |y_j|, \ldots, |\Delta^{m-1} y_j|)$$

converges. Applying Lemma 1, we obtain

$$\lim_{n\to\infty} \frac{1}{\Delta^{m-1} h_n} \sum_{j=n_0}^{n-1} w(j, |y_j|, \ldots, |\Delta^{m-1} y_j|) = 0. \tag{11}$$

From the equality

$$\Delta^{m-1} y_n - \sum_{j=n_0}^{n-1} g_j = \Delta^{m-1} y_{n_0} + \sum_{j=n_0}^{n-1} [F(j, y_j, \ldots, \Delta^{m-1} y_j) - g_j]$$

it follows by (4) and (5) that

$$\left| \frac{\Delta^{m-1} y_n}{\Delta^{m-1} h_n} - \frac{\sum_{j=n_0}^{n-1} g_j}{\Delta^{m-1} h_n} \right| \leqslant \frac{|\Delta^{m-1} y_{n_0}|}{\Delta^{m-1} h_n} + \frac{1}{\Delta^{m-1} h_n} \sum_{j=n_0}^{n-1} w(j, |y_j|, \ldots, |\Delta^{m-1} y_j|)$$

and consequently, by (11),

$$\lim_{n\to\infty} \left| \frac{\Delta^{m-1} y_n}{\Delta^{m-1} h_n} - \frac{\sum_{j=n_0}^{n-1} g_j}{\Delta^{m-1} h_n} \right| = 0.$$

Hence, in view of (3), we get

$$\lim_{n\to\infty} \frac{\Delta^{m-1} y_n}{\Delta^{m-1} h_n} = c.$$

By virtue of (4) we have $\lim_{n\to\infty} \Delta^i h_n = \infty$ and $\Delta^i h_n > 0$, $i \in \overline{0, m-1}$, $n \geqslant n_2$. We obtain by the Stolz theorem

$$\lim_{n\to\infty}\frac{\Delta^{m-2}y_n}{\Delta^{m-2}h_n}=\lim_{n\to\infty}\frac{\Delta(\Delta^{m-2}y_n)}{\Delta(\Delta^{m-2}h_n)}=c.$$

Repeating this reasoning we get finally

$$\lim_{n\to\infty}\frac{\Delta^{i}y_n}{\Delta^{i}h_n}=c,\qquad i\in\overline{0,\,m-1}.\qquad \text{Q.E.D.}$$

COROLLARY 1. *Let* (3) *and* (4) *hold and let*

$$|F(n,\,x_1,\,\ldots,\,x_m)-g_n|\leqslant a_n w\left(\sum_{i=1}^{m}\frac{|x_i|}{\Delta^{i-1}h_n}\right)\quad \text{on } N\times R_+^m,$$

where $a\colon N\to R_+$, $w\colon R_+\to R_+$, w *is nondecreasing and continuous on* R_+,

$$\sum_{j=n_0}^{\infty}\frac{a_j}{\Delta^{m-1}h_j}<\infty,\qquad \int_{\varepsilon}^{\infty}\frac{ds}{w(s)}=\infty\qquad \text{for any } \varepsilon>0.$$

Then every solution of (E) *satisfies the asymptotic equality* (AR).

To prove this, note that the respective equation (E1) is of the form

$$\Delta z_n=\frac{a_n}{\Delta^{m-1}h_n}w(mz_n). \tag{12}$$

A discrete version of Gronwall's lemma [7] leads to the observation that every solution of (12) with positive initial data is bounded. Therefore Corollary 1 follows from the Theorem.

For the next corollary we need a lemma.

LEMMA 2. *Let* $a_i\in R_0$, $b_i\in R_+$, $e_i\in[0,\,1]$, $i\in\overline{1,\,m}$. *Then*

$$\sum_{i=1}^{m}a_i(b_i)^{e_i}\leqslant\Big[\sum_{i=1}^{m}a_i\Big]\Big[m+\sum_{i=1}^{m}b_i\Big].$$

Proof. The proof follows immediately from the inequality

$$\sum_{i=1}^{m}(b_i)^{e_i}<m+\sum_{i=1}^{m}b_i.$$

COROLLARY 2. *Suppose, in addition to* (3), (4), *that*

$$|F(n,\,x_1,\,\ldots,\,x_m)-g_n|\leqslant\sum_{i=1}^{m}a_n^i(|x_i|)^{e_i}\quad \text{on } N\times R_+^m,$$

where $a^i\colon N\to R_+$, $e_i\in[0,\,1]$ ($e_i=\text{const}$), $i\in\overline{1,\,m}$, *and*

$$\sum_{j=n_0}^{\infty}\Big[\sum_{i=1}^{m}a_j^i(\Delta^{i-1}h_j)^{e_i}\Big]\frac{1}{\Delta^{m-1}h_j}<\infty. \tag{13}$$

Then every solution of (E) *fulfils the asymptotic relation* (AR).

Proof. Observe that (E1) now becomes

$$\Delta z_n = \frac{1}{\Delta^{m-1} h_n} \sum_{i=1}^{m} a_n^i (\Delta^{i-1} h_n)^{e_i} (z_n)^{e_i}. \tag{14}$$

Consequently, if $z_{n_0} > 0$ then $z_n > 0$ for all $n \in N$. Hence by Lemma 2 we get the inequality

$$\Delta z_n \leqslant \left\{ m \frac{1}{\Delta^{m-1} h_n} \sum_{i=1}^{m} a_n^i (\Delta^{i-1} h_n)^{e_i} \right\} (1+z_n), \qquad n \in N.$$

By (13), applying again Gronwall's lemma, we get that all solutions of the above inequality, hence also the solutions of equation (14) with positive initial data, are bounded.

Remark. If u and v are any two solutions of (E1) satisfying $0 \leqslant u_{n_0} < v_{n_0}$, then by the monotonicity of w we get

$$\begin{aligned} v_{n_0+1} - u_{n_0+1} &= v_{n_0} - u_{n_0} + \frac{1}{\Delta^{m-1} h_{n_0}} [w(n_0, v_{n_0} h_{n_0}, \ldots, v_{n_0} \Delta^{m-1} h_{n_0}) - \\ &\qquad - w(n_0, u_{n_0} h_{n_0}, \ldots, u_{n_0} \Delta^{m-1} h_{n_0})] > 0. \end{aligned}$$

Repeating the argument we obtain $v_n > u_n$ for all $n \in N$. By the boundedness of v, the behaviour of u must be the same. Examining the proof of the Theorem, we deduce the existence of a solution of (E) with the asymptotic behaviour (AR) when any solution of (E1) (with the initial condition $z_{n_0} = b > d$, d being defined by (8)) is bounded, even if (E1) has other unbounded solutions. Suppose that the solution z of (E1) (with $z_{n_0} = b$) is bounded. Let $d_0 < b$ hold for d_0 defined by (7). Then every solution of (E) with the initial data $\Delta^i y_{n_0}$, $i \in \overline{0, m-1}$, such that

$$|\Delta^i y_{n_0}| < b \min_{i \in \overline{0, m-1}} \Delta^i h_{n_0}, \qquad i \in \overline{0, m-1},$$

$$|\Delta^{m-1} y_{n_0}| < (b - d_0) \Delta^{m-1} h_{n_0},$$

satisfies the asymptotic equality (AR).

As a special example we consider the equation

$$\Delta^m y_n = a_n |y_n|^\alpha + g_n, \qquad n \in N, \tag{15}$$

where $a \colon N \to R_0$, $\alpha > 1$, $g \colon N \to R$, and (3), (4) hold. The respective equation (E1) is of the form

$$\Delta z_n = \frac{1}{\Delta^{m-1} h_n} a_n (z_n h_n)^\alpha, \qquad n \in N. \tag{16}$$

Applying for instance Gronwall's inequality we get the boundedness of all solutions of (16) satisfying the initial condition

$$z_{n_0} = b < \left[(\alpha-1)\sum_{j=n_0}^{\infty} \frac{(h_j)^{\alpha}}{\Delta^{m-1} h_j} a_j\right]^{-1/(\alpha-1)} < \infty .$$

Therefore, if

$$d_0 < \left[(\alpha-1)\sum_{j=n_0}^{\infty} \frac{(h_j)^{\alpha}}{\Delta^{m-1} h_j} a_j\right]^{-1/(\alpha-1)}$$

and if (3) holds, then there exists a solution of (15) fulfilling the asymptotic relation (AR).

We remark that the result presented here generalizes some of the results of [2], [4], [6]; however, our approach to the problem is different.

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TECHNICAL UNIVERSITY, INSTITUTE OF MATHEMATICS, POZNAŃ

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