On The Asymptotic Capacity of Gaussian Relay Networks

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Abstract — **In this paper, we determine the asymptotic capacity of a Gaussian multiple-relay channel as the number of relays tends to infinity. The upper bound is an application of the cut-set theorem, and the lower bound follows from an argument involving uncoded transmission. Hence, this paper gives one more example where the cut-set bound is achievable, and one more example where uncoded transmission achieves optimal performance. In the latter sense, the result is an extension to [1]. The arguments of this paper are also relevant to wireless networks, yielding an asymptotic capacity result [2].**

I. Relay Network Model

Various relay network models have been studied in the literature, some of the most recent examples being [3, 4, 5]. However, capacity results are rare. The simple Gaussian relay network studied in this paper is obtained from the non-degraded Gaussian single-relay channel by adding $M-1$ relay branches, as shown in Figure 1. Note that even for the case $M=1$, capacity is not known for this channel. The W_k's denote additive

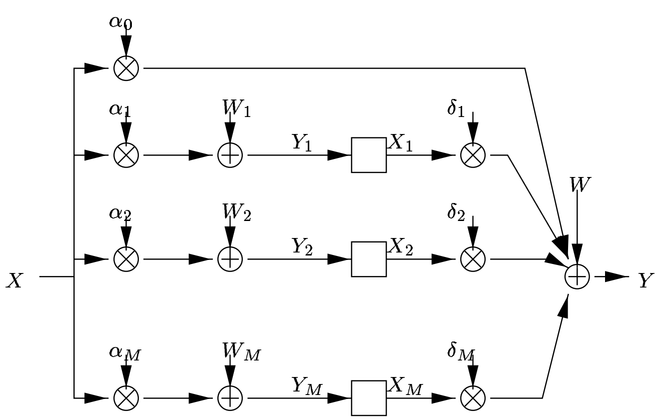

Fig. 1: The considered Gaussian M-relay channel.

white Gaussian noise and are assumed to be independent. In this paper, we assume that all W_k's are of the same variance N. The general case is studied in [6]. Not shown in the figure are the power constraints. For the scope of this paper, those are $EX^2 \leq P$ and $\sum_{k=1}^{M} X_k^2 \leq c(M)$. An interesting choice is for example $c(M) = MQ$ for some constant Q.

II. Asymptotic Capacity

To obtain an upper bound on the capacity of the network of Figure 1, we apply the cut-set bound [7, Thm. 14.10.1]; in particular, we separate the source node and connect the rest of the nodes ideally. The resulting system is a multi-antenna channel with one transmit and $M+1$ receive antennas. Its capacity is an upper bound to the capacity of our network; we denote it by C_{upper}.

The lower bound is found by fixing the relay operation to be direct forwarding: The output at time t is simply a scaled version of the input at time $t-1$. Under this assumption, the relay network becomes a point-to-point channel whose capacity can be determined. This is a lower bound to the capacity of our network; we denote it by C_{lower}.

Define the auxiliary functions

$$a(M) = \sum_{k=0}^{M} \alpha_k^2 \quad \text{and} \quad b(M) = \sum_{k=1}^{M} \alpha_k^2 \frac{\alpha_k^2 P + N}{\delta_k^2}. \quad (1)$$

With this, our main result can be phrased as follows:
Theorem. *If*

$$\lim_{M\to\infty} \frac{1}{a(M)} = 0 \quad \text{and} \quad \lim_{M\to\infty} \frac{b(M)}{a(M)c(M)} = \delta < \infty, \quad (2)$$

then

$$\lim_{M\to\infty} (C_{upper} - C_{lower}) \leq \frac{1}{2}\log_2\left(1 + \delta \frac{\alpha_0^2 P + N}{N}\right). \quad (3)$$

The proof is reported in [6].

III. Illustrations and Extensions

Example: Suppose $\alpha_k = \delta_k = 1$, for all k, and $c(M) = MQ$. The conditions of the theorem are satisfied, and $\delta = 0$. Hence, the bounds are tight, and it can be shown that the asymptotic capacity is

$$C = \frac{1}{2}\log_2\left(1 + \frac{(M+1)P}{N}\right). \quad (4)$$

The result has extensions to wireless networks and to sensor networks, reported in part in [2], as well as to fading relay channels.

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