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ON THE ASYMPTOTIC DISTRIBUTION OF WATSON'S STATISTIC

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An integral representation of the positive stable distributions is used to give a tabulation of the asymptotic test statistic for Watson's optimal invariant test for uniformity of distribution on a circle.

1. A test for isotropy of directions. Let $X_1, X_2, \cdots, X_n$ be i.i.d. observations with density $h(x)$, $0 \le x \le 1$. The well known tests of the hypothesis $h(x) = 1$, $0 \le x \le 1$, associated with the names of Kolmogorov, Smirnov, etc., have no known optimality properties, and since their test statistics are not invariant under translations (mod 1) of the data, they are not suitable for a test of uniformity of distribution on a circle.

Among the tests which are invariant in this sense is the following test of Watson [7], which is optimal against "distant" alternatives. Let $F_n(t)$ be the empirical distribution function of the $\{X_i\}$ and let $\bar{X}_n$ be their mean. Set

$$G_n = \sqrt{n}\, \sup_{0 \le t \le 1}(F_n(t) - t + \bar{X}_n - \tfrac{1}{2}).$$

The hypothesis is to be rejected if G_n is too large. The "distant" alternatives, as contrasted with the "local" alternatives, consist of isolated deviations from uniformity – see [7] for a discussion and proof of optimality.

The distribution of G_n seems difficult to obtain, but it is not hard to deduce that if the hypothesis is true the limiting distribution is that of the random variable

$$G = \sup_{0 \le t \le 1} X(t) - \int_0^1 X(t)\, dt \tag{1.1}$$

where $X(t)$ is the "tied down" Wiener process, $X(0) = X(1) = 0$. This follows since if $E_n(t) = \sqrt{n}\,(F_n(t) - t)$, then $G_n = \sup_{0 \le t \le 1} E_n(t) - \int_0^1 E_n(t)\, dt$, $E_n(t)$ converges in distribution to $X(t)$, and the invariance principle is known to apply to such functionals as (1.1) – see, e.g., Breiman [2] Section 13.9, problem 9. Thus if we could find the distribution of G we would have an asymptotic test of the hypothesis.

The distribution of G given by (1.1) has recently been obtained [4] in the following form. Let $0 < \alpha_1 < \alpha_2 \cdots$ be the zeros of the function $J_{1/3}(x) + J_{-1/3}(x)$ where J_ν is the standard Bessel function, and let $v(x)$ be the density of the positive stable distribution of exponent $\tfrac{2}{3}$. Then

$$F(x) = P(G \le x) = \frac{4\sqrt{\pi}}{3} \sum_{n=1}^{\infty} \frac{1}{\alpha_n}\, v\left(\frac{\sqrt{8}\, x}{3\, \alpha_n}\right). \tag{1.2}$$

(This is the distribution of the supremum of a stationary non-Markov Gaussian process. In [4] it was stated that as of 1976 there were only 3 cases in which such distributions were known. Subsequently another was deduced by Cressie [3].)

The chief difficulty in using this formula to obtain a numerical evaluation is in finding a representation of $v(x)$ which is amenable to numerical calculations for the relevant range of values of x. There exist asymptotic formulas for large and small x, but none of these seems suitable for numerical calculations. The theorem which is given in Section 3 yields, however, a representation which enables $v(x)$ in (1.2) to be numerically evaluated.

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The details of the numerical work are given in Section 2, together with a table of values of $F(x)$.

2. Numerical computation of $F(x)$. The formula for the positive stable density of exponent ⅔, denoted by $v(x)$, is obtained from the corollary of Section 3 by setting $\gamma =$ ⅔ in (3.7) and (3.2), yielding

$$v(x) = \frac{2}{\pi x^3} \int_0^\pi \exp\left(\frac{-g(\theta)}{x^2}\right) g(\theta)\, d\theta \tag{2.1}$$

where

$$g(\theta) = \sin^2\left(\frac{2\theta}{3}\right)\sin\left(\frac{\theta}{3}\right) \bigg/ \sin^3\theta.$$

The integrand in (2.1) is quite smooth and an excellent approximation to $v(x)$ can be obtained by numerical integration. A 20 point Gaussian integration was used [1]. The values of α_n, which are simply related to the zeros of the Airy function $Ai(-x)$, are well known and extensively tabulated. The values of α_n from table VII, page 751 of Watson [6] were used in the series. (A misprint in this table caused some trouble in calculations. The second zero should be $\alpha = 5.5101956$.) For each x in Table 1 and each α_n the numerical integration was used to approximate the terms in the series (1.2).

The series (1.2) converges very rapidly, and the contribution of the sum of the terms beyond the fourth does not exceed 10^{-8} for the range of x considered. The integration method yielded 10 place accuracy for a few functions whose integrals could be explicitly found. A modest APL program yielded the table in less than a minute run time. The significance points were found by successive interpolations.

TABLE 1
Distribution $F(x) = P(G \le x)$.

x	$F(x)$	x	$F(x)$	x	$F(x)$
0.41	.0510	0.66	.6294	0.91	9494
0.42	.0634	0.67	.6516	0.92	.9542
0.43	.0774	0.68	.6730	0.93	.9585
0.44	.0930	0.69	.6936	0.94	.9625
0.45	.1103	0.70	.7133	0.95	.9661
0.46	.1291	0.71	.7322	0.96	.9694
0.47	.1493	0.72	.7502	0.97	.9725
0.48	.1709	0.73	.7673	0.98	.9752
0.49	.1936	0.74	.7836	0.99	.9778
0.50	.2175	0.75	.7991	1.00	.9801
0.51	.2422	0.76	.8137	1.01	.9821
0.52	.2667	0.77	.8275	1.02	.9840
0.53	.2938	0.78	.8405	1.03	.9857
0.54	.3203	0.79	.8528	1.04	.9873
0.55	.3472	0.80	.8643	1.05	.9886
0.56	.3742	0.81	.8751	1.06	.9899
0.57	.4013	0.82	.8852	1.07	.9910
0.58	.4283	0.83	.8946	1.08	.9920
0.59	.4551	0.84	.9034	1.09	.9929
0.60	.4816	0.85	.9115	1.10	.9937
0.61	.5077	0.86	.9191	1.11	.9945
0.62	.5333	0.87	.9262	1.12	.9951
0.63	.5583	0.88	.9327	1.13	.9957
0.64	.5827	0.89	.9387	1.14	.9962
0.65	.6064	0.90	.9443	1.15	.9967

$F(.8361) = .9000$; $F(.9112) = .9500$; $F(1.0609) = .9900$

3. The positive stable distributions. Let $V(x)$ be the c. d. f. of the positive stable law of index $0 < \gamma < 1$. This is normalized so that the Lévy representation of its Laplace transform is

$$\int_0^\infty e^{-sx}\, dV(x) = \exp(-s^\gamma). \tag{3.1}$$

The following integral representation of $V(x)$ is due to Zolotarev [9], who obtained similar formulas for the other stable laws.

THEOREM. *Let*

$$g(\theta) = \left(\frac{\sin \gamma\theta}{\sin \theta}\right)^{\gamma/(1-\gamma)} \frac{\sin((1-\gamma)\theta)}{\sin \theta}, \quad 0 < \theta < \pi \tag{3.2}$$

Then for $x > 0$,

$$V(x) = \frac{1}{\pi} \int_0^\pi \exp(-\, x^{-\gamma/(1-\gamma)} g(\theta))\, d\theta. \tag{3.3}$$

We give a proof of this theorem, considerably simpler than the proof of Zolotarev, for the sake of completeness and to illustrate the utility of the method of constant phase.

The standard inversion formula for the Laplace transform (3.1) is

$$V(x) = \frac{1}{2\pi i} \int_C \exp(f(z))\, \frac{dz}{z} \tag{3.4}$$

where

$$f(z) = xz - z^\gamma \tag{3.5}$$

for z in the complex plans cut along the negative real axis, $f(z)$ being real for z real and positive. The contour C is the straight line joining $k - i\infty$ and $k + i\infty$, k being any positive number. Cf Widder [8].

We modify this contour by requiring $f(z)$ to be real on it. This requirement will minimize the oscillation of the integrand, making it more suitable for numerical approximation. For further applications and discussion of this "principle of constant phase" cf Watson [6], sec. 7.4.

Setting $z = re^{i\theta}$ in (3.5), the condition that $f(z)$ be real is that $xr \sin \theta - r^\gamma \sin \gamma\theta = 0$, which yields

$$r = x^{-1/(1-\gamma)} \left(\frac{\sin \gamma\theta}{\sin \theta}\right)^{1/(1-\gamma)}, \quad -\pi < \theta < \pi. \tag{3.6}$$

If we denote by C_0 the contour whose equation in polar coordinates is given by (3.6) it is easy to deduce that we can deform the contour C in (3.4) to C_0 — this will, in fact, improve the convergence of (3.4).

Using (3.5) and (3.6) a minor calculation shows that for z on C_0

$$f(z) = -\, x^{-\gamma/(1-\gamma)} g(\theta)$$

where $g(\theta)$ is given by (3.2). Using this expression for $f(z)$ in (3.4) we get

$$V = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \exp(1 - x^{-\gamma/(1-\gamma)} g(\theta)) \left(\frac{dr}{r} + i\, d\theta\right)$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-\, x^{-\gamma/(1-\gamma)} g(\theta))\, d\theta$$

and (3.3) follows on noting that $g(\theta)$ is an even function.

By taking the derivative of (3.3) we have the

COROLLARY. *The density of the positive stable law of exponent* γ, $0 < \gamma < 1$, $x > 0$, is

$$v(x) = \frac{\gamma}{\pi(1-\gamma)x^{1/(1-\gamma)}} \int_0^\pi \exp(-x^{-\gamma/(1-\gamma)} g(\theta)) g(\theta)\, d\theta. \tag{3.7}$$

This result was stated (with a slight misprint) in Ibragimov and Chernin [5].

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