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ON THE ASYMPTOTICITY ASPECT OF HYERS-ULAM STABILITY OF MAPPINGS

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Abstract. The object of the present paper is to prove an asymptotic analogue of Th. M. Rassias' theorem obtained in 1978 for the Hyers-Ulam stability of mappings.

1. Introduction

In [15] Rassias generalized the result of Hyers [9] by allowing growth of the form $\varepsilon \cdot (\|x\|^p + \|y\|^p)$ for the norm of the Cauchy difference $f(x+y) - f(x) - f(y)$, where $0 \leq p < 1$, and still obtained the formula

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for the additive mapping approximating f. Other developments of this idea are described in [10] (see also [1], [5], [7], [8], [12], [13], [16]). In the present article we obtain an asymptotic analogue of this result of Th. M. Rassias.

Several authors have used asymptotic conditions in stating approximations to Cauchy's functional equation

$$f(x+y) = f(x) + f(y).$$

P.D.T.A. Elliott [6] showed that if the real function f belongs to the class $L^p(0,z)$ for every $z \geq 0$, where $p \geq 1$, and satisfies the asymptotic condition

$$\lim_{z\to\infty} \frac{\int_0^z \int_0^z |f(x+y) - f(x) - f(y)|^p \, dx \, dy}{z} = 0,$$

then there is a constant c such that $f(x) = cx$ almost everywhere on $\mathbb{R}^+$. One of the theorems of J. R. Alexander, C. E. Blair and L. A. Rubel [1] states that if $f \in L^1(0,b)$ for all $b > 0$, and if for almost all $x > 0$

$$\lim_{u\to\infty} \frac{\int_0^y [f(x+y) - f(x) - f(y)] \, dy}{u} = 0,$$

then for some real number $c, f(x) = cx$ for almost all $x \geq 0$.

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F. Skof [17] proved that given real normed spaces X and E and a mapping $f : X \to E$ satisfying the condition

$$\|f(x+y) - f(x) - f(y)\| \to 0 \quad \text{as } \|x\| + \|y\| \to \infty,$$

then $f(x+y) = f(x) + f(y)$ for all x and y in X. In a later article [18] the same author showed that a real-valued function f defined on a real normed space X is additive proving that $f(0) = 0$ and $|f(x+y)| - |f(x) + f(y)| \to 0$ when $\|x\| + \|y\| \to 0$. In [12] is shown an interesting relation between the Hyers-Ulam stability and the asymptotic derivability. This relation is applied to the study of some important nonlinear problems (cf. [13]).

In the present paper we consider the asymptoticity aspect of Hyers-Ulam stability close to the asymptotic derivability. The asymptotic derivability is very important in nonlinear analysis (cf. [2], [3], [4], [11], [14]).

2. Main result

Theorem 1. *Given a real normed vector space E_1 and a real Banach space E_2, let numbers $M > 0, \varepsilon > 0$ and p with $0 < p < 1$ be chosen. Let the mapping $f : E_1 \to E_2$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all x, y in E_1 such that

$$\|x\|^p + \|y\|^p > M^p. \tag{2}$$

Then there exists an additive mapping $\varphi : E_1 \to E_2$ such that

$$\|\varphi(x) - f(x)\| < \beta(p)\varepsilon\|x\|^p \tag{3}$$

for all $x \in E_1$ with $\|x\| > \frac{M}{2^{1/p}}$, where $\beta(p) = \frac{2}{2-2^p}$ and $\varphi(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

Proof. When $\|x\| > \frac{M}{2^{1/p}}$, that is, when $2\|x\|^p > M^p$, we may put $y = x$ in (1) to obtain

$$\|2^{-1} f(2x) - f(x)\| \le \varepsilon\|x\|^p. \tag{4}$$

Of course we can replace x by $2x$ in (4) since $\|2x\|$ is also greater than $\frac{M}{2^{1/p}}$. Thus, we can use the argument given in [15] to arrive at the inequality

$$\|2^{-n} f(2^n x) - f(x)\| \le \beta(p)\varepsilon\|x\|^p \quad \text{when } \|x\| > \frac{M}{2^{1/p}} \text{ for } n \in N \tag{5}$$

and thus to show that the limit

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{6}$$

exists when $\|x\| > \frac{M}{2^{1/p}}$. Therefore

$$\|g(x) - f(x)\| \le \beta(p)\varepsilon\|x\|^p. \tag{7}$$

Clearly, when $\|x\| > \frac{M}{2^{1/p}}, g(2x) = \lim_{n\to\infty} \frac{f(2^{n+1}x)}{2^n} = 2\lim_{n\to\infty} \frac{f(2^{n+1}x)}{2^{n+1}}$, so that

$$g(2x) = 2g(x) \quad \text{for } \|x\| > \frac{M}{2^{1/p}}. \tag{8}$$

Now suppose that $\|x\|, \|y\|$ and $\|x+y\|$ are all greater than $\frac{M}{2^{1/p}}$. Then by (1) we find that for all $n \in N$,

$$\|2^{-n} f(2^n(x+y)) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y)\| \le \varepsilon 2^{-n(1-p)}(\|x\|^p + \|y\|^p). \tag{9}$$

Under the conditions stated it follows by (6) that

$$g(x+y) = g(x) + g(y). \tag{9a}$$

Using an extension method of F. Skof [18] we will define a mapping $\varphi : E_1 \to E_2$ to be an extension of the mapping g to the whole space E_1. Given any $x \in E_1$ with $0 < \|x\| < \frac{M}{2^{1/p}}$, let $k = k(x)$ denote the largest integer such that

$$\frac{M}{2^{1/p}} < 2^k \|x\| \le M. \tag{10}$$

Define the mapping φ as follows:

$$\begin{cases} \varphi(0) = 0, \\ \varphi(x) = 2^{-k} g(2^k x) & \text{for } 0 < \|x\| < \frac{M}{2^{1/p}}, \text{ where } k = k(x), \\ \varphi(x) = g(x) & \text{for } \|x\| > \frac{M}{2^{1/p}}. \end{cases}$$

Lemma. *For all x in E_1*

$$\varphi(x) = \lim_{s\to\infty} 2^{-s} f(2^s x) \tag{11}$$

and

$$\varphi(-x) = -\varphi(x). \tag{12}$$

Proof. Take any x in E_1 with $0 < \|x\| < \frac{M}{2^{1/p}}$, and let $k = k(x)$, so that k is the largest integer satisfying (10). Thus $k-1$ is the largest integer satisfying

$$\frac{M}{2^{1/p}} < \|2^{k-1}(2x)\| \le M,$$

and we have

$$\varphi(2x) = 2^{-(k-1)} g(2^{k-1}(2x)) = 2^{-k} \cdot 2g(2^k x) = 2\varphi(x) \quad \text{for } 0 < \|x\| < \frac{M}{2^{1/p}}.$$

From the definition of φ and property (8) of g it follows that $\varphi(2x) = 2\varphi(x)$ for all x in E_1. Given x in E_1 with $x \ne 0$, choose a positive integer m so large that $\|2^m x\| > \frac{M}{2^{1/p}}$.

By the definition of φ we have

$$\varphi(x) = 2^{-m}\varphi(2^m x) = 2^{-m} g(2^m x),$$

and by (6) this implies that

$$\varphi(x) = \lim_{n\to\infty} 2^{-(m+n)} f(2^{m+n} x),$$

which demonstrates (11) for $x \ne 0$.

Since $\varphi(0) = 0$, the same is true for $x = 0$. Equation (12) is obvious for $x = 0$. Take any x in E_1 with $x \ne 0$ and choose $n \in N$ large enough so that $\|2^n x\| > \frac{M}{2^{1/p}}$.

Then by (1) with $y = -x$ we obtain

$$\|2^{-n} f(2^n x) + 2^{-n} f(-2^n x)\| \le 2\varepsilon 2^{-n(1-p)} \|x\|^p + 2^{-n}\|f(0)\|.$$

When $n \to \infty$ it follows from (11) that (12) holds. The lemma is proved.

In proving the additivity of φ we note that the equation

$$\varphi(x+y) = \varphi(x) + \varphi(y) \tag{13}$$

holds when either x or y is zero.

Assume then that $x \ne 0$ and $y \ne 0$. If $x + y = 0$, i.e. $y = -x$, then (12) shows that (13) holds. The only remaining case is when x, y and $x+y$ are all different from zero. In this case we may choose an n in N such that $\|2^n x\|, \|2^n y\|$ and $\|2^n(x+y)\|$ are all greater than $\frac{M}{2^{1/p}}$. Then by (1) we have

$$\|f(2^n(x+y)) - f(2^n x) - f(2^n y)\| \leq \varepsilon 2^{np}(\|x\|^p + \|y\|^p).$$

If we divide both sides of this inequality by 2^n and then let $n \to \infty$, we find by (11) that (13) is true, thus φ is additive.

By definition $\varphi(x) = g(x)$ when $\|x\| > \frac{M}{2^{1/p}}$, thus (3) follows from (7) and the proof of Theorem 1 is complete. Q.E.D.

For convenience in applications we give the following modified version of Theorem 1.

Theorem 2. *Given a real normed vector space E_1 and a real Banach space E_2, let numbers $m > 0, \varepsilon > 0$ and p with $0 \leq p < 1$ be chosen. Suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all x and y in E_1 such that $\|x\| > m, \|y\| > m$ and $\|x+y\| > m$. Then there exists an additive mapping $\varphi : E_1 \to E_2$ which satisfies

$$\|\varphi(x) - f(x)\| \leq 2\varepsilon(2 - 2^p)^{-1}\|x\|^p$$

for all x in E_1 such that $\|x\| > m$. Moreover, φ is given by the formula

$$\varphi(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

for all x in E_1.

Proof. Assume that $\|x\| > m$. Then as in the proof of Theorem 1 we obtain (4)–(8) inclusive, but now all these formulas are satisfied for $\|x\| > m$. In particular,

$$g(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \quad \text{when } \|x\| > m.$$

Also, if $\|x\| > m, \|y\| > m$ and $\|x+y\| > m$, then by hypothesis we see that (9) and (9a) also hold. To apply Skof's extension procedure in the present case, let x in E_1 be given with $0 < \|x\| \leq m$ and define $k = k(x)$ to be the unique positive integer such that

$$m < 2^k\|x\| \leq 2m. \tag{14}$$

Now define the mapping $\varphi : E_1 \to E_2$ as follows:

$$\begin{cases} \varphi(0) = 0, \\ \varphi(x) = 2^{-k} g(2^k x) & \text{for } 0 < \|x\| \leq m, \\ \varphi(x) = g(x) & \text{for } \|x\| > m. \end{cases}$$

The proof of the Lemma used in the proof of Theorem 1, follows as before with the obvious changes.

Indeed, we start with x in E_1 satisfying $0 < \|x\| \leq m$ and let $k = k(x)$ as defined by (14), etc. Thus the Lemma holds under the conditions of Theorem 2. The proof of the additivity of φ also follows as before. Therefore the proof of Theorem 2 is complete. Q.E.D.

3. p-ASYMPTOTICAL ADDITIVITY

We apply the main theorem, precisely Theorem 2, to the study of p-asymptotical derivatives.

Let E_1 and E_2 be Banach spaces. Let T be a mapping from E_1 into E_2 satisfying eventually a special property such as, for example, additivity, linearity, etc. Let $0 < p < 1$ be arbitrary.

Definition 1. A mapping $f : E_1 \to E_2$ is *p-asymptotically close to T* if and only if $\lim_{\|x\|\to\infty} \frac{\|f(x)-T(x)\|}{\|x\|^p} = 0$.

Remark 1. If in Definition 1, $T \in L(E_1, E_2)$, then we say that T is a p-asymptotical derivative of f and if such a T exists, then f is *p-asymptotically derivable*.

Remark 2. Since for x such that $\|x\| \geq 1$ we have $\|x\|^p \leq \|x\|$, one obtains that every p-asymptotical derivative of f is an asymptotical derivative. Indeed, if $T \in L(E_1, E_2)$ is a p-asymptotical derivative of f, then

$$0 \leq \lim_{\substack{\|x\|\to\infty \\ \|x\|>1}} \frac{\|f(x) - T(x)\|}{\|x\|} \leq \lim_{\substack{\|x\|\to\infty \\ \|x\|>1}} \frac{\|f(x) - T(x)\|}{\|x\|^p} = 0.$$

Definition 2. A mapping $f : E_1 \to E_2$ is *p-asymptotically additive* if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$ such that $\|x\|^p, \|y\|^p, \|x + y\|^p > \delta$.

Definition 3. A mapping $T : E_1 \to E_2$ is *additive outside a ball* if there exists $r > 0$, such that $T(x + y) = T(x) + T(y)$ for all $x, y \in E_1$ with $\|x\|, \|y\| \geq r$ and $\|x + y\| \geq r$.

Example. Let $T : E_1 \to E_2$ be defined by

$$T(x) = \begin{cases} L(x) & \text{if } \|x\| \geq r, \\ \varphi(x) & \text{if } \|x\| < r \end{cases}$$

where $L : E_1 \to E_2$ is a linear mapping and

$$\varphi : B(0, r) \to E_2$$

is a nonlinear mapping where $B(0, r) = \{x \in E_1 | \, \|x\| < r\}$. It follows that if $x, y \in E_1$ with $\|x\| \geq r, \|y\| \geq r$, and $\|x + y\| \geq r$, then $T(x + y) = T(x) + T(y)$.

We have the following result.

Theorem 3. *If $f : E_1 \to E_2$ is p-asymptotically close to an additive mapping outside a ball $T : E_1 \to E_2$, then f is p-asymptotically additive.*

Theorem 4. *If $f : E_1 \to E_2$ is p-asymptotically close to an additive outside a ball mapping $T : E_1 \to E_2$, then f is p_*-asymptotically close to an additive mapping, where $0 < p < p_* < 1$.*

Corollary. *If $f : E_1 \to E_2$ is p-asymptotically close to an additive outside a ball mapping $T : E_1 \to E_2$, then f has an additive asymptotical derivative.*

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