On the automatic numerical evaluation of definite integrals

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A critical examination is made of adaptive subdivision as a means of reliably and efficiently performing the automatic evaluation of definite integrals. A model is set up which embodies the basic features of adaptive schemes. Circumstances are discussed under which adaptive schemes may inspire false confidence in the result produced. The efficiency of the method is seriously impaired by any attempts to overcome this difficulty. The conclusions have been illustrated by appropriate numerical examples.

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1. Introduction

In this paper we shall consider the automatic numerical evaluation of the integral

$$I = \int_{-1}^{1} f(t)\,dt \qquad (1)$$

where $f(t)$ is an analytical function. All integrals over a finite range can be expressed in this form.

We define an ideal automatic integrator as one which when applied to an analytically given function returns a numerical value which is guaranteed correct up to a specified accuracy. Such an ideal integrator could be constructed if upper bounds to the error of a quadrature routine were available. Assessment of this integrator would be in terms of its efficiency. Since upper bounds to the error in quadrature formulae are seldom sharply defined it is probable that the integrator would be very inefficient. In practical applications one is often forced to estimate the values of the upper bounds. As a result, the integrator departs from the ideal and the reliability of its result comes into question.

There are two basic approaches to automatic quadrature. The first results when a family of quadrature rules, generally of high order (for example, Gauss quadrature formulae) are applied over the entire interval of integration while the second results when the range of integration is subdivided and a quadrature formula, generally of low order, is applied to each subinterval (an example of this would be a composite Simpson's rule). These will be referred to respectively as whole interval formulae and subdivision formulae.

Davis and Rabinowitz (1967) have classified the subdivision formulae as either *adaptive* or *non-adaptive*. When the points of subdivision of the interval are chosen according to some strategy dependent on the behaviour of the integrand, the subdivision is said to be *adaptive*. A fixed choice of subdivision points (for example, equidistant points) characterises *non-adaptive* subdivision.

The literature on numerical quadrature appears to show a general preference for adaptive subdivision. Davis and Rabinowitz (1967) in the context of automatic integration have noted that if one is confronted with an isolated integral, an adaptive Simpson's rule would seem to be best. Lyness (1965) has remarked in the context of multidimensional integration that subdivision schemes are generally used in practice. A perusal of the algorithms for numerical integration presently available (Collected algorithms from CACM, 1968) also shows a strong bias for adaptive subdivision rules and would encourage the uninitiated programmer to proceed in this direction.

There are several reasons why this preference has arisen. The whole interval method is usually equated with Gauss quadrature and the objection is raised that a result can only be improved by applying at least one other formula of different order, and consequently providing a possible check on accuracy. Since the Gauss formulae of different order have no points in common (except zero), the procedure is likely to be very inefficient. This objection is no longer very convincing mainly due to the work of Kronrod (1965) who has tabulated $2n + 1$ point rules, whose abscissae include the n point Gauss abscissae and which have degree $3n + 1$ when n is even and $3n + 2$ when n is odd. The result obtained using an n point Gauss rule can thus be considerably improved without wasting the labour already invested. Furthermore, Patterson (1968) has shown how the principle can be extended to generate families of high precision formulae having the feature that all the points of a given formula are included in the formula of next higher order. A family of n point formulae of degree $(3n + 1)/2$ has been tabulated. These formulae will be discussed in more detail later.

It is also believed that formulae of the Gauss type cannot be expected to produce good results unless applied to functions having a sufficient number of high order derivatives (Haber, 1967). While there is some truth in this, examples will be given later which show that the degree of deterioration of the performance of the formulae when singularities are present depends more on the harshness of the singularities than on their presence as such.

Adaptive subdivision of course has geometrical appeal. It seems intuitive that points should be concentrated in regions where the integrand is badly behaved. The whole interval rules can take no direct account of this.

Finally, an often levied objection to the whole interval formulae is that their weights and abscissae are generally irregular numbers which have to be stored. Henrici (1964) for example states, in connection with the Gauss formulae, that this practically (although not theoretically) limits their applicability.

The primary object of this paper is to assess adaptive subdivision as a general method for efficiently and reliably carrying out automatic integration. To avoid conclusions which apply only to a specific method a model of adaptive subdivision will be introduced later which embodies the basic features of presently available adaptive schemes.

We consider that the most serious defect of any quadrature procedure is that it inspires in the user false confidence in its result. No error estimate based on a finite amount of functional information has any validity in the absence of theoretical information about the function (Davis and Rabinowitz, 1967). Thus several independent quadrature formulae could return results* in agreement to several digits beyond their actual accuracy. We call this spurious convergence. Low order formulae are more likely to have this defect. It is generally recognised (Lyness, 1967) that the efficiency of an adaptive procedure relies heavily on its error estimates. It is not generally appreciated however that the reliability may be seriously impaired should the error in any of the earlier subintervals be seriously under-estimated. This will be the basis of our criticism of adaptive subdivision and an example will be provided later which demonstrates just how dramatic this reduction of reliability may be.

It is always convenient to have some means of assessing the difficulty of any integration and in this respect we used the whole interval formulae of Patterson (1968) discussed in Section 2.2, and in a few cases the Clenshaw-Curtis formulae. It is only in this respect that they are introduced. Although they generally considerably outperform the adaptive subdivision model as regards efficiency they are not in their present state of development to be regarded in the context of this paper as a competitive automatic integrator.

2. The formulae

2.1 The sub-division formulae

It is straightforward to show that when the interval $[-1, 1]$ is subdivided into n panels defined by

$$-1 = \alpha_0 < \alpha_1 < \ldots . < \alpha_{n-1} < \alpha_n = 1 \qquad (2)$$

then the integral I defined by (1) can be written as

$$I = \int_{-1}^{1} g_n(x)dx \qquad (3)$$

where

$$g_n(x) = \sum_{j=1}^{n} \left(\frac{\alpha_j - \alpha_{j-1}}{2}\right) f\left(\frac{\alpha_j - \alpha_{j-1}}{2} x + \frac{\alpha_j + \alpha_{j-1}}{2}\right) . \qquad (4)$$

$g_n(x)$ is simply a numerical transformation of the integrand. Application of an m point quadrature formula to (3) with abscissae x_i (in $[-1, 1]$) and weights w_i results in

$$I \approx \sum_{i=1}^{m} w_i g_n(x_i) \ . \qquad (5)$$

*An example (kindly supplied by the referee) of this occurs when the 7 and 15 point formulae of Patterson (1968) are applied to $\int_{-1}^{1} \exp(-6{\cdot}793\ x^2)/(1{\cdot}000001 - x^2)dx$. The results agree to 6 digits while their actual error is 1 in the second digit. Since the integrand is even about half of the abscissae are wasted so that in this case these are actually low order formulae.

This transformation is defined to be either adaptive or non-adaptive depending on whether the α_j respectively do or do not depend on the behaviour of the integrand. It is reasonable to require that non-adaptive subdivision should be optimal in some sense. For example, when the subdivision points are equidistant, (4) takes the form

$$g_n(x) = \frac{1}{n} \sum_{j=1}^{n} f([x + 2j - 1 - n]/n) \qquad (6)$$

which is well known (Krylov, 1962) to reduce trignometric polynomials of degree $n - 1$ to a constant which can be numerically integrated exactly by the simplest quadrature rule. This transformation is best applied when the integrand has a rapidly converging Fourier expansion. In addition, when used in conjunction with the midpoint rule, the transformation is optimal if the first derivative of the integrand is not continuous. There appears to be no other choice of the α_j in the literature which attempts to optimise (4) in any other sense. It should be noted that (4) does not lower the degree of an algebraic polynomial but only reduces the coefficients of the higher powers.

Another numerical transformation is the Romberg scheme which relies on a knowledge of the error functional to make the transformation effective (Lyness, 1967). When the error functional has a power series expansion, the Romberg scheme can be directly assessed in terms of algebraic precision and is thus inferior to the well-known high precision formulae.

Adaptive subdivision schemes should attempt to optimise the transformation with respect to the particular integrand rather than a general class of integrands. The effectiveness of these schemes depends heavily on the numerical information accurately reflecting the behaviour of the integrand. Adaptive schemes, as does the model to be discussed next, rely on a numerical error estimate being available. Should this be inaccurate, the transformation will at best be inefficient.

A model of adaptive subdivision will now be described based on an algorithm proposed by O'Hara and Smith (1968, 1969). Suppose that I defined by (1) has to be evaluated with maximum absolute error ε and that a quadrature rule Q and absolute error estimate E_Q are available. At step s in the application of the scheme let $[-1, 1]$ be subdivided into s panels for each of which the result of applying Q and E_Q is known. If the sum of E_Q over the panels is $\leq \varepsilon$ then the integration is terminated and the sum of Q over the panels taken as the adaptive subdivision result. Otherwise that panel on which E_Q is greatest is halved and Q and E_Q applied to each half. This takes us to step $s + 1$ with $[-1, 1]$ subdivided into $s + 1$ panels and the procedure for step s can be repeated. Step 1 of the scheme consists of applying Q and E_Q over the entire interval $[-1, 1]$. No panel of $[-1, 1]$ is excluded from possible further subdivision at any time by this scheme, in contrast with the O'Hara and Smith algorithm which, as the integration proceeds, excludes an increasing portion of the left-hand side of $[-1, 1]$ from further subdivision according to a criterion depending on ε. The termination of subdivision in certain regions could lead to serious errors if the interval used at the time of termination was insufficiently fine to resolve rapidly varying parts of the function. It is clear that the strategy is heavily dependent on the accuracy of E_Q.

With adaptive subdivision there is likely to be some wastage of computational labour as an integration proceeds since when an interval is halved the new points in the half intervals at which the integrand has now to be evaluated may not include all of the points at which the integrand was previously evaluated in the whole interval. If l is the number of integrand evaluations lost each time an interval is halved, then after n panels have been generated using a closed m point rule $n(m - 1 + l) - l$

+ 1 integrand evaluations will have been carried out. It is thus clear that the choice of the quadrature formula to be used on each panel has an important influence on the efficiency of the integration. If a high order quadrature were used on each interval then most of the computational labour would be lost when the interval was halved. The efficiency E of a formula can be defined as the number of integrand evaluations needed to apply the m point quadrature rule to n panels divided by the number of integrand evaluations actually required to generate the n panels by the adaptive subdivision scheme. Thus,

$$E = [n(m-1)+1]/[n(m-1+l)-l+1] \ . \qquad (7)$$

The Clenshaw-Curtis quadrature formulae are particularly suited to the adaptive algorithm since they are usually not only very accurate but also have easily applied error estimates available. As $n \to \infty$ the values of E for the 4, 5, 7, 9, and 13 point Clenshaw-Curtis formulae (which respectively have $l = 0$, 2, 2, 6, and 8) are respectively 1·0, 0·67, 0·75, 0·57, and 0·60. The error estimate of the 4-point rule was found to be too crude to make effective use of the adaptive algorithm so that the 7-point formula was adopted. **Table 1** gives the weights and abscissae of the rule and its error estimate. To generate n panels the rule requires $8n - 1$ integrand evaluations.

Table 1 Clenshaw-Curtis 7-point formula and error estimate

$$Q = \sum_{i=1}^{7} w_i f(x_i) \ , \ E_Q = \sum_{i=1}^{7} w_i' f(x_i)$$

$\pm x_i$	w_i	w_i'
1.0	1/35	16/945
$\sqrt{3}/2$	16/63	-32/945
1/2	16/35	32/945
0	164/315	-32/945

2.2 The whole interval formulae

The family of whole interval formulae we shall principally use in this paper are those given by Patterson (1968). Formulae using 3, 7, 15, 31, 63, 127, and 255 points have been tabulated,* the m point formula having degree $(3m + 1)/2$. The formulae were derived using a procedure for augmenting an m point quadrature rule with $m + 1$ points chosen so as to gain the greatest increase in integrating degree. We shall refer to these as the optimal degree common point formulae. The above set of formulae were based on an initial 3-point Gauss formula and have the following important features:

1. The formulae form a common point family, that is, the abscissae of a given member of the set are all included in and interlace the abscissae of the member of next higher order; thus no integrand evaluations are lost as one proceeds to higher order, a common objection to the Gauss formulae.
2. The weights of all the formulae are positive so that cancellation effects and non-uniform convergence of the quadrature rules are less likely.
3. The formulae generally integrate high powers of the integration variable more accurately than the Gauss formula beyond their theoretical integrating degree. Although the formal integrating degree of the m point member of the set is $(3m + 1)/2$, with actual machine precision (say, 16 digits) the integrating degree of the high order members of the set up to power $2m - 1$ is indistinguishable from the Gauss m point formula.

*The last member of this family has not yet been published.

The Clenshaw-Curtis formulae whose abscissae and weights are easily calculated also have the important features (1) and (2) and could be adopted as suitable common point whole interval formulae. Their general performance, however, is inferior to the above formulae as later examples will show.

3. Results and discussion

There are two requirements which must be demanded of any automatic integrator:

(*a*) Some attempt should be made to minimise the computational labour. The procedure should be fairly efficient in comparison with presently available quadrature schemes (e.g. Clenshaw and Curtis, 1960; O'Hara and Smith, 1968; Patterson, 1968) on smooth integrands without requiring an excessive amount of labour on what are regarded as awkward integrands.

(*b*) Although, as we have observed earlier, spurious convergence cannot be ruled out, it should be possible to improve confidence in any result without performing an inordinate amount of labour.

Any adaptive subdivision scheme which excludes a subinterval from further consideration when it has satisfied a convergence criterion cannot possibly meet requirement (*b*). The model of adaptive subdivision described earlier does not have this drawback in that the entire domain of integration is under surveillance. The result produced by an adaptive scheme at any point depends on what is usually a long series of decisions each conditional on the validity of the previous decision. The inaccuracies introduced by an incorrect decision are compounded until such time as the strategy checks the validity of this decision. The lag between making an incorrect decision and discovering it may be so great that sufficient confidence is inspired to terminate the procedure. It is not difficult to find integrals which practically illustrate this defect. Such an integral is:

$$\int_0^1 [\text{sech}^2 10(x-{\cdot}2)+\text{sech}^4 100(x-{\cdot}4)+\text{sech}^6 1000(x-{\cdot}6)]\,dx$$
$$= {\cdot}2108027354 \ .$$

Fig. 1 shows how the whole interval and adaptive subdivision formulae perform on this integral. Both the true error of the adaptive formulae and the requested error are given (that is ε as described in Section 2.1). The convergence of the adaptive formulae is completely spurious. The reason for the failure, of course, is that discussed above. Most of the abscissae are concentrated into the peaks at $x = {\cdot}2$ and $x = {\cdot}4$. An incorrect decision is made in the region of $x = {\cdot}6$ and consequently the abscissae are too sparsely distributed to detect the peak there. A slight shift in the third peak to $x = {\cdot}593$ dramatically changes the situation. The result is shown in **Fig. 2**. The spurious convergence is partially removed although false confidence in the result would be inspired until about seven digit accuracy is requested when the earlier incorrect decision is detected. It is, of course, always possible to contrive examples which produce poor performance from any integration formula. However, the point we wish to emphasise is that under conditions which should exploit the useful properties of adaptive subdivision, results are produced whose accuracies are several orders of magnitude inferior to those indicated by the convergence criteria. It might be argued that a more accurate error estimate would avert this situation. The reasons for the failure we have discussed are, however, in essence unrelated to the actual error estimate but are a result of a basic failure of the strategy of adaptive subdivision.

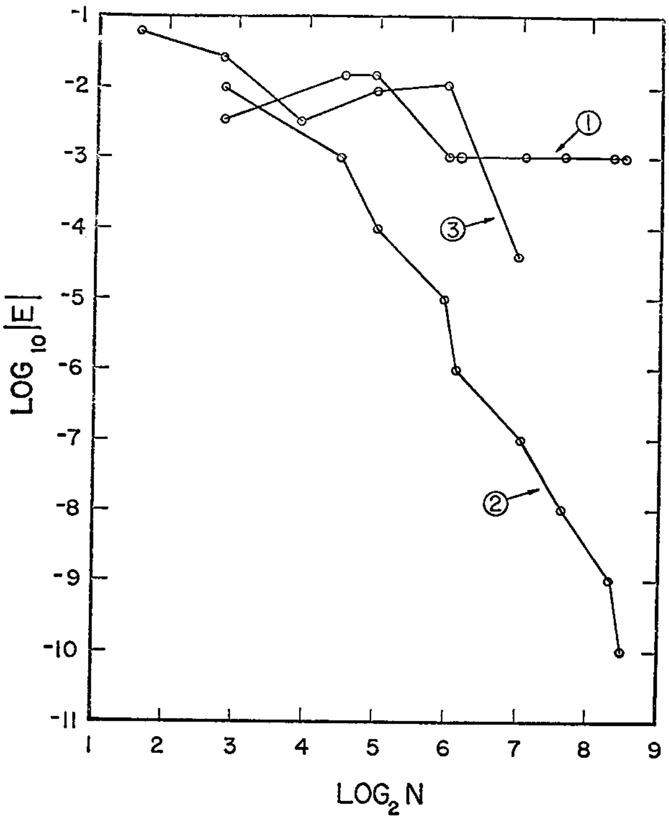

Fig. 1. Results of integrating

$$\int_0^1 [\text{sech}^2\, 10(x - \cdot 2) + \text{sech}^4\, 100(x - \cdot 4) + \text{sech}^6\, 1000(x - \delta)]\, dx$$

with $\delta = \cdot 6$. $\log_{10} |E|$ versus $\log_2 N$ is plotted where E is the error and N the number of integrand evaluations required to obtain the result. Curve 1 shows the true error of the adaptive subdivision algorithm while Curve 2 shows the error actually requested from the algorithm (that is ϵ of Section 2.1 of the text). It is notable that the actual error is several orders of magnitude greater than the indicated error for $\delta = \cdot 6$. Curve 3 gives the true errors produced by the common point whole interval formulae (Patterson, 1968). On all the curves only the circles have significance. They have been joined to improve clarity

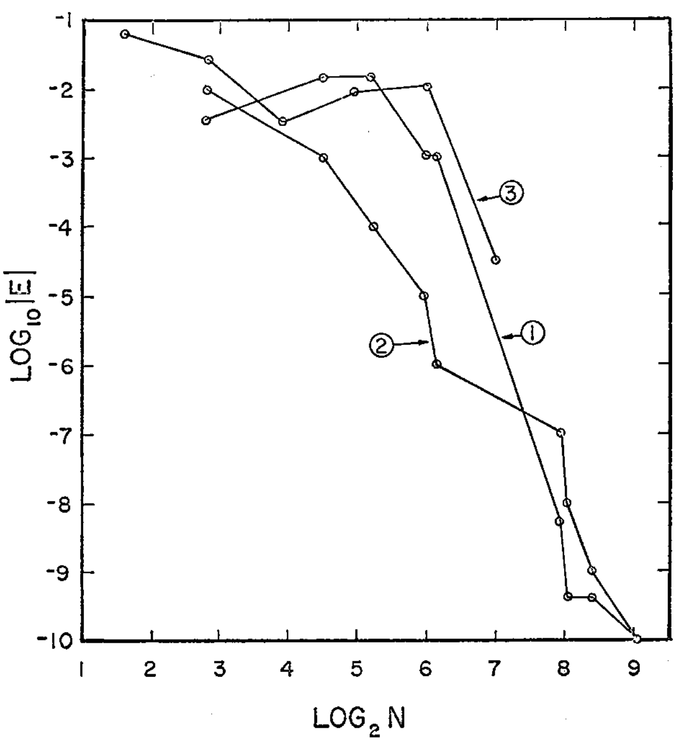

Fig. 2. As for Fig. 1 but with $\delta = \cdot 593$.

Obviously the better the error estimator one uses, the less likely it is that such a situation can arise but, in principle, a failure can always occur. Any attempt to improve the error estimator would almost certainly lead to a reduction in efficiency. The model of adaptive subdivision described in this paper is likely to be unique in that the error estimate requires no additional function evaluations.

It is of interest in relation to remark (*a*) to investigate the efficiency of the model with respect to common point whole interval formulae. Sets of test integrals with various analytical properties have been given and discussed by O'Hara and Smith (1968) and Davis and Rabinowitz (1967). These test integrands range from smooth functions to functions which might be regarded as awkward and are selected to avoid intentionally favouring one method over another.

The results of the various schemes for these integrals are shown in **Table 2**. To make comparison easier the same number of points has been used for the Clenshaw-Curtis formulae as for the formulae of Patterson (1968). The Clenshaw-Curtis formulae will no longer have interlacing abscissae but their integration properties should not be affected. It is clear that the actual accuracies of the subdivision model are at best comparable to and typically require from two to four times as much work as the whole interval formulae.

Additionally, the actual errors in the model are frequently several orders of magnitude better than the tolerance requested. It might be concluded, on the basis of the criteria for automatic integration proposed by Lyness (1969) that the model was 'over cautious'. The results are typical of many tests undertaken including all the examples of O'Hara and Smith (1968).

It is clear that with the present model the probability of spurious convergence is unacceptably high. Although the introduction of more conservative criteria in the strategy would lessen this probability, the inevitable decrease in efficiency, which is already unimpressive, would make it difficult to satisfy requirement (*a*).

It is interesting to note that if the sign of the error estimate were reliable, considerable improvements in the rates of convergence of the subdivision model could be expected as a result of cancellation effects. It was found in some cases that when the estimated error was replaced by its true value, then the efficiency was improved by about a factor of two if account was taken of the sign of the error estimate. However, the signs of the presently available error estimates are not regarded as being trustworthy and so the absolute scheme was adopted. These error cancellation effects which might have been exploited can be seen in Table 2 where there is a large oscillation in the magnitude of error in some cases

$$\left(\text{e.g. } \int_0^\pi dx/(5+4\cos x)\right.$$

with 39 points).

The whole interval formulae are not specifically designed to deal with functions with a low order of differentiability or analytic functions with singular points close to the interval of integration. Special optimal formulae have been developed (cf. Krylov, 1962; Sard, 1949; Meyer and Sard, 1950; Stern, 1967) to deal with integrands which have specific analytic properties. However, there are many such integrands which can be better handled by the whole interval formulae than by the so-called optimal formulae. **Tables 3, 4 and 5** show examples of this situation on integrands for which the midpoint rule and the second order formula of Stern (1967) (referred to by him as Formula 2) would be optimal. It can be seen that only in the first two examples do the optimal formulae even compete with the optimal degree common point formulae. In the first example for $K = \cdot 01$, it is notable that the midpoint rule exhibits spurious convergence. Thus, although the special formulae are optimal for a complete class of integrands, their behaviour may

Table 2 **Errors of quadrature formulae**

$\int_0^1 dx/(1+x^2)$

No. of integrand evaluations	3	7	15
Patterson	1·3 (−4)*	4·6 (−9)	exact**
Clenshaw-Curtis	2·1 (−3)	3·4 (−7)	7·5 (−14)

No. of integrand evaluations	7	15	31
Actual subdivision error	3·3 (−7)	6·0 (−10)	1·0 (−14)
Error requested	1·0 (−2) −1·0 (−4)	1·0 (−5) −1·0 (−6)	1·0 (−8) −1·0 (−9)

*Denotes $1·3 \times 10^{-4}$. **Indicates accuracy in excess of 16 digits.

$\int_0^1 dx/(1+100x^2)$

No. of integrand evaluations	3	7	15	31
Patterson	4·2 (−3)	6·0 (−4)	6·0 (−7)	5·0 (−13)
Clenshaw-Curtis	4·7 (−2)	1·9 (−3)	6·6 (−7)	2·3 (−12)

No. of integrand evaluations	7	15	23	31	39	55	95
Actual subdivision error	2·0 (−3)	1·0 (−4)	1·0 (−5)	1·1 (−7)	5·5 (−9)	2·3 (−9)	5·0 (−12)
Error requested	1·0 (−2)	1·0 (−3)	1·0 (−4)	1·0 (−5)	1·0 (−6)	1·0 (−7) −1·0 (−8)	1·0 (−9)

$\int_0^1 dx/(1-·98x^4)$

No. of integrand evaluations	3	7	15	31	63	127
Patterson	4 (−1)	1 (−1)	4 (−4)	2 (−6)	1·5 (−11)	exact
Clenshaw-Curtis	7·3	2·9 (−1)	1·7 (−3)	4·1 (−4)	5·5 (−7)	6·0 (−13)

No. of integrand evaluations	39	55	63	71	111	143	215
Actual subdivision error	1·2 (−3)	1·1 (−5)	3·0 (−7)	7·0 (−8)	6·0 (−9)	4·1 (−10)	2·0 (−11)
Error requested	1 (−2)	1 (−3)	1 (−4)	1 (−5)	1 (−6)	1 (−7)	1 (−8)

$\int_0^1 dx/(1+x)$

No. of integrand evaluations	3	7	15
Patterson	2·5 (−5)	2·4 (−10)	exact
Clenshaw-Curtis	1·3 (−3)	6·3 (−8)	2·7 (−15)

No. of integrand evaluations	7	15	23	39
Actual subdivision error	6·3 (−8)	5·0 (−10)	2·4 (−11)	4·5 (−13)
Error requested	1 (−2) −1 (−5)	1 (−6) −1 (−7)	1 (−8)	1 (−9)

$\int_0^1 dx/(1-\cdot 5x^4)$

No. of integrand evaluations	3	7	15	31
Patterson	4·5 (−3)	6·0 (−6)	1·1 (−11)	exact
Clenshaw-Curtis	4·4 (−2)	1·1 (−4)	8·7 (−9)	4·0 (−15)

No. of integrand evaluations	7	15	23	31	47	63	95
Actual subdivision error	1·1 (−4)	5·4 (−6)	1·1 (−7)	3·6 (−9)	4·1 (−10)	2·2 (−11)	2·1 (−12)
Error requested	1 (−2) −1 (−3)	1 (−4)	1 (−5)	1 (−6)	1 (−7)	1 (−8)	1 (−9)

$\int_0^1 e^x dx$

No. of integrand evaluations	3	7
Patterson	8·2 (−7)	exact
Clenshaw-Curtis	5·8 (−4)	6·5 (−11)

No. of integrand evaluations	7	15	23
Actual sub-division error	6·4 (−11)	2 (−13)	1 (−13)
Error requested	1 (−2) −1 (−6)	1 (−7) −1 (−8)	1 (−9)

$\int_0^1 dx/(1+25x^2)$

No. of integrand evaluations	3	7	15	31
Patterson	1·1 (−2)	8 (−5)	2·5 (−9)	exact
Clenshaw-Curtis	9·7 (−3)	1·9 (−4)	8·3 (−8)	2·3 (−13)

No. of integrand evaluations	7	15	31	39	63	87
Actual subdivision error	1·8 (−4)	2·7 (−5)	2·7 (−11)	4·7 (−9)	1·7 (−12)	4·3 (−12)
Error requested	1 (−2)	1 (−3) −1 (−4)	1 (−5) −1 (−6)	1 (−7)	1 (−8)	1 (−9)

$\int_0^{\pi/2} dx/(1+\cos x)$

No. of integrand evaluations	3	7	15
Patterson	1·6 (−4)	2·8 (−9)	exact
Clenshaw-Curtis	6·1 (−3)	5·3 (−7)	4·1 (−14)

No. of integrand evaluations	7	15	23
Actual sub-division error	5·3 (−7)	4·7 (−9)	1·8 (−10)
Error requested	1 (−2) −1 (−4)	1 (−5) −1 (−6)	1 (−7)

$\int_0^{\pi} dx/(5+4\cos x)$

No. of integrand evaluations	3	7	15	31
Patterson	3·1 (−2)	2·1 (−4)	1·0 (−8)	exact
Clenshaw-Curtis	4·7 (−2)	2·3 (−5)	1·3 (−7)	4·0 (−14)

No. of integrand evaluations	7	15	31	39	55
Actual subdivision error	2·3 (−5)	4·2 (−5)	4·3 (−10)	1·0 (−8)	5 (−10)
Error requested	1 (−2)	1 (−3) −1 (−4)	1 (−5)	1 (−6)	1 (−7) −1 (−8)

$\int_0^{1} 4dx/(1+256[x-3/8]^2)$

No. of integrand evaluations	3	7	15	31	63	127
Patterson	2·9 (−1)	2·2 (−1)	3·3 (−2)	2 (−4)	8·4 (−7)	exact
Clenshaw-Curtis	1·6 (−1)	2·8 (−1)	3·2 (−2)	5·4 (−4)	1·8 (−7)	exact

No. of integrand evaluations	7	39	71	87	111
Actual subdivision error	2·7 (−1)	9 (−6)	2 (−7)	2 (−9)	8 (−10)
Error requested	1 (−2) −1 (−3)	1 (−4)	1 (−5)	1 (−6)	1 (−7)

$\int_0^1 dx/(1-{\cdot}998x^4)$

No. of integrand evaluations	3	7	15	31	63	127
Patterson	1·0	5·5 (−1)	1·6 (−1)	2·5 (−3)	2 (−5)	5 (−10)
Clenshaw-Curtis	8·2 (1)	6·2	7·4 (−1)	3·7 (−2)	2·8 (−3)	5·3 (−5)

No. of integrand evaluations	63	79	87	95	167	231
Actual subdivision error	1·8 (−3)	3 (−5)	7 (−6)	1·2 (−7)	5 (−9)	4 (−10)
Error requested	1 (−2)	1 (−3)	1 (−4)	1 (−5)	1 (−6)	1 (−7)

$\int_0^1 \sqrt{x}\, dx$

No. of integrand evaluations	3	7	15	31	63	127
Patterson	2·5 (−3)	1·4 (−4)	7·0 (−6)	3·3 (−7)	1·6 (−8)	7·9 (−10)
Clenshaw-Curtis	2·9 (−2)	5·5 (−4)	4·0 (−5)	4·0 (−6)	4·6 (−7)	5·4 (−8)

No. of integrand evaluations	7	15	31	47	63	95	127
Actual subdivision error	5·5 (−4)	1·9 (−4)	2·4 (−5)	3·0 (−6)	3·8 (−7)	1·7 (−8)	2·1 (−9)
Error requested	1 (−2)	1 (−3)	1 (−4)	1 (−5)	1 (−6)	1 (−7)	1 (−8)

$\int_0^1 x^{3/2} dx$

No. of integrand evaluations	3	7	15	31	63	127
Patterson	1·9 (−4)	1 (−6)	1·8 (−9)	2·1 (−11)	2·7 (−13)	2·0 (−15)
Clenshaw-Curtis	2·4 (−3)	2·9 (−6)	5·7 (−8)	1·2 (−9)	3·2 (−11)	9·3 (−13)

No. of integrand evaluations	7	15	31	39	55	79	127
Actual subdivision error	2·9 (−6)	5·1 (−7)	1·6 (−8)	2·9 (−9)	5·1 (−10)	1·9 (−11)	6·7 (−13)
Error requested	1 (−2) −1 (−4)	1 (−5)	1 (−6)	1 (−7)	1 (−8)	1 (−9)	1 (−10)

$\int_0^1 dx/(1+\cdot 5\sin 10\pi\, x)$

No. of integrand evaluations	3	7	15	31	63	127
Patterson	1·3 (−1)	1·1 (−1)	8·9 (−2)	1·2 (−3)	1·6 (−5)	3 (−9)
Clenshaw-Curtis	1·5 (−1)	5·5 (−2)	9·1 (−2)	5·9 (−3)	2·1 (−5)	1 (−9)

No. of integrand evaluations	7	31	63	127
Actual subdivision error	5·5 (−2)	4·3 (−3)	2·2 (−5)	<1 (−9)
Error requested	1 (−2)	1 (−3)	1 (−4) −1 (−5)	1 (−6) −1 (−10)

$\int_0^1 dx/(1+e^x)$

No. of integrand evaluations	3	7
Patterson	1·8 (−7)	1·7 (−14)
Clenshaw-Curtis	3·5 (−5)	8·3 (−11)

No. of integrand evaluations	7	15
Actual subdivision error	8·3 (−11)	2·6 (−13)
Error requested	1 (−2) −1 (−7)	1 (−8) −1 (−10)

$\int_0^1 x\,dx/(e^x-1)$

No. of integrand evaluations	3	7
Patterson	9·8 (−9)	1 (−16)
Clenshaw-Curtis	1·0 (−5)	9·1 (−13)

No. of integrand evaluations	7	15
Actual subdivision error	9·1 (−13)	3·3 (−15)
Error requested	1 (−2) −1 (−8)	1 (−9) −1 (−10)

Table 3 Fractional errors (error/true value) in

$$\int_{-1}^{1} \exp\left[K\left|x+\frac{\sqrt{3}}{20}\right|\right]dx.$$

No. of integrand evaluations		3	7	15	31	63	127	255
$K = 10^{-2}$	Mid-point	9·2 (−3)	7·2 (−3)	6·5 (−3)	6·2 (−3)	6·0 (−3)	5·9 (−3)	5·9 (−3)
	Patterson	3·5 (−4)	1·5 (−5)	1·7 (−5)	7·1 (−7)	1·3 (−6)	3·8 (−7)	4·0 (−8)
$K = 10^{-1}$	Mid-point	3·1 (−3)	1·9 (−4)	2·6 (−5)	6·7 (−6)	2·9 (−6)	1·0 (−7)	6·3 (−7)
	Patterson	3·3 (−3)	1·4 (−4)	1·7 (−4)	6·8 (−6)	1·2 (−5)	3·6 (−6)	3·5 (−7)
$K = 1$	Mid-point	3·6 (−2)	4·3 (−3)	8·5 (−4)	2·0 (−4)	5·7 (−5)	1·0 (−5)	6·3 (−6)
	Patterson	2·0 (−2)	8·2 (−4)	1·0 (−3)	4·2 (−5)	7·5 (−5)	2·1 (−5)	2·1 (−6)
$K = 10$	Mid-point	7·6 (−1)	2·7 (−1)	7·1 (−2)	1·7 (−2)	4·2 (−3)	1·0 (−3)	2·6 (−4)
	Patterson	4·2 (−1)	1·1 (−3)	5·0 (−6)	2·3 (−7)	4·2 (−7)	1·3 (−7)	1·2 (−8)

Table 4 **Errors in** $\int_{-1}^{1} f(x)dx = \cdot 5881513703$

$$f(x) = (x+1)^7 \quad -1 \leq x \leq \cdot 13$$
$$= [1\cdot 13(1-x)/\cdot 87]^7 \quad \cdot 13 < x \leq 1$$

No. of integrand evaluations	3	7	15	31	63	127	255
Mid-point rule	8·1 (−2)	1·0 (−1)	4·1 (−2)	1·1 (−2)	1·4 (−3)	9·5 (−5)	8·0 (−5)
Patterson	3·0 (−1)	8·3 (−2)	6·4 (−2)	7·9 (−3)	2·4 (−3)	8·9 (−4)	7·1 (−5)

Table 5 **Errors in** $16 \int_{0}^{1} x^3 \ln x \, dx = \mathbf{1 \cdot 0}$

No. of integrand evaluations	3	7	15	31	63	127	255
Stern	2·7 (−2)	2·4 (−3)	2·5 (−4)	3·0 (−5)	3·6 (−6)	4·4 (−7)	5·4 (−8)
Patterson	1·3 (−3)	4·3 (−7)	6·3 (−10)	3·1 (−13)	exact	exact	exact

be inferior to the high precision formulae on many of these.

It would therefore seem justifiable to use the whole interval formulae to test efficiency.

4. Conclusions

This paper has concentrated mainly on an assessment of adaptive subdivision as an automatic integrator. In the light of our findings we suggest that a more profitable approach to automatic integration may be found using families of whole interval formulae. We have not attempted an analysis of the whole interval scheme referred to in the paper but would note only that in its present state of development the comparison of successive members of the family would be used to check convergence. Although, as an earlier example has indicated, this can lead to spurious convergence the unconditional and essentially independent way in which successive results are obtained, unlike adaptive subdivision, makes it possible to meet requirement (*b*) of Section 3 in a fairly satisfactory manner.

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