142. On the Automorphism Group of the Full Transformation Semigroups

By Nobuko IWAHORI and Hirosi NAGAO

(Comm. by Kunihiko KODAIRA, M. J. A., Nov. 13, 1972)

Let $\Sigma(\Omega)$ be the semigroup of all the mapping of a set Ω into Ω and $\mathfrak{S}(\Omega)$ the symmetric group on Ω. For $a \in \Omega$ we denote by ε_a the element of $\Sigma(\Omega)$ such that $\varepsilon_a(x)=a$ for every $x \in \Omega$. Let $E=\{\varepsilon_a \mid a \in \Omega\}$.

The purpose of this note is to prove the following:

Theorem. *Let S be a subsemigroup of $\Sigma(\Omega)$ containing E and let θ be an isomorphism of S into $\Sigma(\Omega)$. If $\theta(S)$ contains E, then θ can be extended to an inner automorphism of $\Sigma(\Omega)$, i.e., there is an element σ of $\mathfrak{S}(\Omega)$ such that $\theta(\gamma)=\sigma\gamma\sigma^{-1}$ for all $\gamma \in S$. In case Ω is finite, the weaker assumption that $\theta(S)\cap E\neq\phi$ implies the same conclusion.*

Denote by $\Sigma^*(\Omega)$ the subsemigroup $\Sigma(\Omega)-\mathfrak{S}(\Omega)$ of $\Sigma(\Omega)$. From the theorem we obtain, immediately the following:

Corollary[1]**.** *Every automorphism of $\Sigma(\Omega)$ or of $\Sigma^*(\Omega)$ is inner. Thus the automorphism groups of $\Sigma(\Omega)$ and $\Sigma^*(\Omega)$ are both isomorphic to $\mathfrak{S}(\Omega)$.*

Before proving the theorem, we remark that an element ε_a of E satisfies the following equations for all $\gamma \in \Sigma(\Omega)$:

(1) $$\varepsilon_a\gamma=\varepsilon_a$$

(2) $$\gamma\varepsilon_a=\varepsilon_{\gamma(a)}$$

It follows from (2) that if an element δ of $\Sigma(\Omega)$ satisfies $\delta\varepsilon_a=\delta$ for some $a \in \Omega$, then $\delta=\varepsilon_{\delta(a)}$ is an element of E.

Proof of theorem. Let θ be an isomorphism of S into $\Sigma(\Omega)$ such that $\theta(S)\cap E\neq\phi$. Then there is an element $\beta \in S$ such that $\theta(\beta)=\varepsilon_b$ for some $b \in \Omega$. Since $\varepsilon_a\beta=\varepsilon_a$ for every $a \in \Omega$, $\theta(\varepsilon_a)\varepsilon_b=\theta(\varepsilon_a)$. By the above remark $\theta(\varepsilon_a)\in E$ and hence $\theta(E)\subset E$. If Ω is finite then this implies $\theta(E)=E$. In the case Ω is infinite, if we assume that $\theta(S)\supset E$, then in the same way as above, we get $\theta^{-1}(E)\subset E$ and hence $\theta(E)=E$. Let $\theta(\varepsilon_a)=\varepsilon_{\sigma(a)}$. Then, since $\theta(E)=E$, it follows that σ is in $S(\Omega)$. Now from (2) we have

$$\theta(\gamma)\varepsilon_{\sigma(a)}=\varepsilon_{\sigma\gamma(a)}$$
$$=\varepsilon_{\theta(\gamma)\sigma(a)}$$

for any $\gamma \in S$. Thus $\theta(\gamma)\sigma=\sigma\gamma$ and hence $\theta(\gamma)=\sigma\gamma\sigma^{-1}$.

Remark. The following two examples show that the assumptions in the theorem are not superfluous.

---

1) This result was first obtained by the first author by making use of the result in [1].

Example 1. Let $\Omega=\{1,2,3,4\}$, and consider the following elements of $\Sigma(\Omega)$;

$$f_1: \{1,2\}\to 1, \qquad \{3,4\}\to 3$$
$$f_2: \{1,2\}\to 1, \qquad \{3,4\}\to 4$$
$$f_3: \{1,2\}\to 2, \qquad \{3,4\}\to 3$$
$$f_4: \{1,2\}\to 2, \qquad \{3,4\}\to 4$$

Then $f_a f_b = f_a$. Thus the mapping $\theta: E\to\Sigma(\Omega)$ defined by $\theta(\varepsilon_a)=f_a$ is a monomorphism. Now $\theta(E)\cap E=\phi$ and obviously θ can not be extended to an inner automorphism of $\Sigma(\Omega)$.

Example 2. Let $\Omega=\{1,2,3,\cdots\}$, and let $\theta: E\to\Sigma(\Omega)$ be the monomorphism defined by $\theta(\varepsilon_a)=\varepsilon_{a+1}$. Then $\theta(E)\subset E$ but $\theta(E)\neq E$. Thus θ can not be extended to an inner automorphism of $\Sigma(\Omega)$.

Reference

[1] N. Iwahori and N. Iwahori: On a set of generating relations of the full transformation semigroup (to appear).