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On the average number of unitary factors of finite abelian groups

by

J. Wu (Nancy)

1. Introduction. Let $\mathbb{X}$ be the semigroup of all finite abelian groups with respect to the direct product $\otimes$ and let $\mathcal{E}_0$ be the identity of $\mathbb{X}$. For $\mathcal{G} \in \mathbb{X}$ and $\mathcal{H} \in \mathbb{X}$, we use $(\mathcal{G}, \mathcal{H})$ to denote the group of maximal order in $\mathbb{X}$ which is simultaneously a direct factor of $\mathcal{G}$ and $\mathcal{H}$. We say that $\mathcal{G}$ and $\mathcal{H}$ are *relatively prime* if $(\mathcal{G}, \mathcal{H}) = \mathcal{E}_0$. A direct factor $\mathcal{D}$ of $\mathcal{G}$ is called *unitary* if $\mathcal{D} \otimes \mathcal{E} = \mathcal{G}$ and $(\mathcal{D}, \mathcal{E}) = \mathcal{E}_0$. The number of unitary factors of $\mathcal{G}$ is denoted by $t(\mathcal{G})$. In 1960, Cohen [2] proved

$$\sum_{|\mathcal{G}| \le x} t(\mathcal{G}) = A_1 x \log x + A_2 x + O(\sqrt{x} \log x), \tag{1.1}$$

where the summation is over all $\mathcal{G}$ in $\mathbb{X}$ of order $|\mathcal{G}| \le x$ and the A_j are some effective constants. After a study on Dirichlet's series associated with $t(\mathcal{G})$, Krätzel [7] found a connection between (1.1) and the following three-dimensional divisor problem:

$$\sum_{n_1 n_2 n_3^2 \le x} 1 = B_1 x \log x + B_2 x + B_3 \sqrt{x} + \Delta(1,1,2;x), \tag{1.2}$$

where the B_j are some effective constants and $\Delta(1,1,2;x)$ is an error term. Using exponential sum techniques, he showed that $\Delta(1,1,2;x) \ll x^{11/29} (\log x)^2$, which implies

$$\sum_{|\mathcal{G}| \le x} t(\mathcal{G}) = A_1 x \log x + A_2 x + A_3 \sqrt{x} + \Delta(x) \tag{1.3}$$

with $\Delta(x) \ll x^{11/29} (\log x)^2$. This estimate was improved to $\Delta(x) \ll x^{3/8} (\log x)^4$ by Schmidt [11], then to $\Delta(x) \ll_\varepsilon x^{77/208+\varepsilon}$ by Liu [9] and to $\Delta(x) \ll_\varepsilon x^{29/80+\varepsilon}$ by Liu [10], where ε denotes an arbitrarily small positive number.

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In this paper, we give a better bound.

Theorem 1. *For any* $\varepsilon > 0$, *we have*

$$\Delta(1,1,2;x) \ll_\varepsilon x^{47/131+\varepsilon} \quad \text{and} \quad \Delta(x) \ll_\varepsilon x^{47/131+\varepsilon}.$$

For comparison, we have $29/80 = 0.3625$ and $47/131 \approx 0.3587$. From [10], we know that in the proof of Theorem 1 the most difficult part is to estimate the exponential sums of type

$$\sum_{h\sim H} a_h \sum_{n_1\sim N_1} \sum_{n_2\sim N_2} e(xh/(n_1^2 n_2)), \tag{1.4}$$

where $|a_h| \le 1$, $e(t) := e^{2\pi it}$ and the notation $h \sim H$ means $cH < h \le c'H$ with some positive unspecified constants c, c'. Liu [10] has treated (1.4) by combining Fouvry–Iwaniec's method [3] and Kolesnik's method [6].

We notice that via van der Corput's B-process the sum (1.4) can be transformed into bilinear exponential sums of type I,

$$T(M,N) := \sum_{m\sim M} \sum_{n\in I(m)} \varphi_m e\biggl(X\frac{m^\alpha n^\beta}{M^\alpha N^\beta}\biggr),$$

where $I(m)$ is a subinterval of $[N, 2N]$. Using the classical A, B process and the well known AB theorem of Kolesnik (see Theorem 1 of [6] and Lemma 1.5 of [8]) we shall prove an estimate for $T(M,N)$ (see Theorem 3 below). In addition we also use an idea of Jia ([5], Lemma 13) and Liu ([8], Lemma 2.4) to investigate bilinear exponential sums of type II,

$$S(M,N) := \sum_{m\sim M} \sum_{n\sim N} \varphi_m \psi_n e\biggl(X\frac{m^\alpha n^\beta}{M^\alpha N^\beta}\biggr).$$

Baker and Harman have simplified Jia–Liu's argument to obtain a slightly more general estimate for $S(M,N)$ (see Theorem 2 of [1]) than those of Jia and Liu. But all such results contain some restrictions on (X, M, N) and the number of terms is relatively large; this is not convenient in applications. Our result (see Theorem 2 below) essentially has the same power as their estimates, but it is without restriction, more general and simpler in form. Finally, it is worth indicating that we also need Theorem 7 of [12] and Lemma 2.3 of [13] for the proof of Theorem 1.

2. Estimates for exponential sums. We first prove two estimates for $S = S(M,N)$, defined as in Section 1. In the sequel, the letter ε_0 denotes a suitably small positive number (depending on α, β and α_j at most).

Theorem 2. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha-1)(\beta-1) \ne 0$, $X > 0$, $M \ge 1$, $N \ge 1$, $|\varphi_m| \le 1$, $|\psi_n| \le 1$ *and* $\mathcal{L} := \log(2+XMN)$. *If* (κ, λ) *is an exponent pair*, *then*

$$
\begin{aligned}
(2.1)\qquad S(M,N) \ll \{&(X^{2+4\kappa}M^{8+10\kappa}N^{9+11\kappa+\lambda})^{1/(12+16\kappa)}\\
&+ X^{1/6}M^{2/3}N^{3/4+\lambda/(12+12\kappa)}\\
&+ (XM^3N^4)^{1/5} + (XM^7N^{10})^{1/11}\\
&+ M^{2/3}N^{11/12+\lambda/(12+12\kappa)}\\
&+ MN^{1/2} + (X^{-1}M^{14}N^{23})^{1/22} + X^{-1/2}MN\}\mathcal{L}^2,\\
(2.2)\qquad S(M,N) \ll \{&(XM^3N^4)^{1/5} + (X^4M^{10}N^{11})^{1/16} + (XM^7N^{10})^{1/11}\\
&+ MN^{1/2} + (X^{-1}M^{14}N^{23})^{1/22} + X^{-1/2}MN\}\mathcal{L}^2.
\end{aligned}
$$

Proof. We begin in the same way as Jia [5], Liu [8], Baker and Harman [1]. Without loss of generality, we suppose that $\beta > 0$ and $\mathcal{L}$ is sufficiently large. Let $Q \in [\mathcal{L}, N/\mathcal{L}]$ be a parameter to be chosen later. By Cauchy's inequality and a "Weyl shift" ([4], Lemma 2.5), we have

$$
\begin{aligned}
|S|^2 \ll{}& \frac{(MN)^2}{Q}\\
&+ \frac{M^{3/2}N}{Q}\sum_{1\le|q_1|<Q}\Big(1-\frac{|q_1|}{Q}\Big)\sum_{n+q_1,n\sim N}\psi_{n+q_1}\overline{\psi}_n\sum_{m\sim M} m^{-1/2}e(Am^\alpha t),
\end{aligned}
$$

where $t = t(n,q_1) := (n+q_1)^\beta - n^\beta$ and $A := X/(M^\alpha N^\beta)$. Splitting the range of q_1 into dyadic intervals and removing $1 - q_1/Q$ by partial summation, we get

$$
|S|^2 \ll (MN)^2Q^{-1} + \mathcal{L}M^{3/2}NQ^{-1}\max_{1\le Q_1\le Q}|S(Q_1)|, \tag{2.3}
$$

where

$$
S(Q_1) := \sum_{q_1\sim Q_1}\sum_{n+q_1,n\sim N}\psi_{n+q_1}\overline{\psi}_n\sum_{m\sim M} m^{-1/2}\,e(Am^\alpha t).
$$

If $X(MN)^{-1}Q_1 \ge \varepsilon_0$, by Lemma 2.2 of [12] we can transform the innermost sum to a sum over l and then using Lemma 2.3 of [12] with $n = m$ we can estimate the corresponding error term. As a result, we obtain

$$
\begin{aligned}
S(Q_1) \ll{}& \sum_{q_1\sim Q_1}\sum_{n+q_1,n\sim N}\psi_{n+q_1}\overline{\psi}_n\sum_{l\in I} l^{-1/2}\,e(\widetilde{\alpha}(At)^\gamma l^{1-\gamma})\\
&+ \{(XM^{-1}N^{-1}Q_1^3)^{1/2} + M^{-1/2}NQ_1\\
&+ (X^{-1}MNQ_1)^{1/2} + (X^{-2}MN^4)^{1/2}\}\mathcal{L},
\end{aligned}
$$

where $\gamma := 1/(1-\alpha)$, $\widetilde{\alpha} = |1-\alpha|\cdot|\alpha|^{\alpha/(1-\alpha)}$, $I := [c_1AM^{\alpha-1}|t|,\, c_2AM^{\alpha-1}|t|]$ and $c_j = c_j(\alpha)$ are some constants. Exchanging the order of summation and estimating the sum over l trivially, we find, for some $l \asymp X(MN)^{-1}Q_1$, the inequality

$$
\begin{aligned}
(2.4)\qquad S(Q_1) \ll (XM^{-1}N^{-1}Q_1)^{1/2}\Big|&\sum\sum_{(n,q_1)\in\mathbf{D}_1(l)} \psi_{n+q_1}\overline{\psi}_n e(\widetilde{\alpha}(At)^\gamma l^{1-\gamma})\Big|\\
&+\{(XM^{-1}N^{-1}Q_1^3)^{1/2}+M^{-1/2}NQ_1\\
&+(X^{-1}MNQ_1)^{1/2}+(X^{-2}MN^4)^{1/2}\}\mathcal{L},
\end{aligned}
$$

where $\mathbf{D}_1(l)$ is a suitable subregion of $\{(n,q_1) : n\sim N,\ q_1\sim Q_1\}$. Let $S_1(Q_1)$ be the double sums on the right-hand side of (2.4). Using Lemma 2.6 of [12] to relax the range of q_1, we see that there exists a real number θ independent of (n,q_1) such that

$$
S_1(Q_1)\ll \mathcal{L}\sum_{n\sim N}\Big|\sum_{q_1\sim Q_1}\psi_{n+q_1}e(\theta q_1)e(\widetilde{\alpha}(At)^\gamma l^{1-\gamma})\Big|.
$$

If $\mathcal{L}\le Q_1\le Q$, using again Cauchy's inequality and a "Weyl shift" with $Q_2\le\varepsilon_0\sqrt{Q_1}$ yields

$$
|S_1(Q_1)/\mathcal{L}|^2 \ll (NQ_1)^2Q_2^{-1}+NQ_1Q_2^{-1}\sum_{1\le q_2\le Q_2}|S_2(q_1,q_2)|,
$$

where

$$
S_2(q_1,q_2):=\sum_{n\sim N}\sum_{q_1+q_2,q_1\sim Q_1}\psi_{n+q_1+q_2}\overline{\psi}_{n+q_1}e(t_1(n,q_1,q_2))
$$

and $t_1(n,q_1,q_2):=\widetilde{\alpha}A^\gamma l^{1-\gamma}\{t(n,q_1+q_2)^\gamma - t(n,q_1)^\gamma\}$. Writing $n':=n+q_1$, exchanging the order of summation and using Lemma 2.6 of [12], we can deduce

$$
\begin{aligned}
S_2(q_1,q_2) &= \sum\sum_{(n',q_1)\in\mathbf{D}_2}\psi_{n'+q_2}\overline{\psi}_{n'}e(t_1(n'-q_1,q_1,q_2))\\
&\ll \mathcal{L}\sum_{n'\sim N}\Big|\sum_{q_1\sim Q_1}e(\theta' q_1)e(T(n',q_1,q_2))\Big|,
\end{aligned}
$$

where $T(n',q_1,q_2):=t_1(n'-q_1,q_1,q_2)$, $\mathbf{D}_2$ is a suitable subregion of $\{(n',q_1): n'\sim N,\ q_1\sim Q_1\}$ and θ' is a real number independent of (n',q_1). A final application of Cauchy's inequality and a "Weyl shift" with $Q_3=Q_2^2$ gives

$$
|S_2(q_1,q_2)/\mathcal{L}|^2\ll (NQ_1)^2Q_3^{-1}+NQ_1Q_3^{-1}\sum_{1\le q_3\le Q_3}\sum_{q_1\sim Q_1}|S_3(q_1,q_2,q_3)|,
$$

where $S_3(q_1,q_2,q_3):=\sum_{n'\sim N}e(f(n'))$ and $f(n'):=T(n',q_1,q_2)-T(n',q_1+q_3,q_2)$. It is easy to show that $f(n')$ satisfies the conditions of exponent pair and $f'(n')\asymp XN^{-2}Q_1^{-1}q_2q_3$ $(n'\sim N)$. Hence we have

$$
S_3(q_1,q_2,q_3)\ll (XN^{-2}Q_1^{-1}q_2q_3)^\kappa N^\lambda + (XN^{-2}Q_1^{-1}q_2q_3)^{-1},
$$

which implies

$$S_1(Q_1) \ll \{(X^{\kappa} N^{3-2\kappa+\lambda} Q_1^{4-\kappa} Q_2^{3\kappa})^{1/4} \\ + N Q_1 Q_2^{-1/2} + (X^{-1} N^5 Q_1^5 Q_2^{-3})^{1/4}\} \mathcal{L}^{7/4}$$

provided $Q_1 \geq \mathcal{L}$, $Q_2 \leq \varepsilon_0 \sqrt{Q_1}$. By Lemma 2.4(ii) of [12] optimizing Q_2 over $(0, \varepsilon_0 \sqrt{Q_1}]$ yields

$$S_1(Q_1) \ll \{(X^{\kappa} N^{3+4\kappa+\lambda} Q_1^{4+5\kappa})^{1/(4+6\kappa)} + N^{3/4+\lambda/(4+4\kappa)} Q_1 \\ + N Q_1^{3/4} + (X^{-2} N^{10} Q_1^7)^{1/8}\} \mathcal{L}^{7/4}$$

provided $Q_1 \geq \mathcal{L}$. In view of the term $N Q_1^{3/4} \mathcal{L}^{7/4}$, this inequality holds trivially when $Q_1 \leq \mathcal{L}$. Inserting the preceding estimate in (2.4) yields, for any $Q_1 \in [1, Q]$,

$$\begin{aligned} S(Q_1) \ll \{ & (X^{2+4\kappa} M^{-2-3\kappa} N^{1+\kappa+\lambda} Q_1^{6+8\kappa})^{1/(4+6\kappa)} \\ & + X^{1/2} M^{-1/2} N^{1/4+\lambda/(4+4\kappa)} Q_1^{3/2} \\ & + (X^2 M^{-2} N^2 Q_1^5)^{1/4} + (X^2 M^{-4} N^6 Q_1^{11})^{1/8} + (X M^{-1} N^{-1} Q_1^3)^{1/2} \\ & + M^{-1/2} N Q_1 + (X^{-1} M N Q_1)^{1/2} + (X^{-2} M N^4)^{1/2}\} \mathcal{L}^{7/4} \\ =: {} & (E_1 + E_2 + \ldots + E_8) \mathcal{L}^{7/4}. \end{aligned}$$

Since $E_5 \leq E_3$ and $E_6 = (E_4^4 E_8)^{1/5} (M^2 Q_1)^{-1/10}$, both E_5 and E_6 are superfluous. Replacing Q_1 by Q and inserting the bound obtained in (2.3), we find, for any $Q \in [\mathcal{L},\, N/\mathcal{L}]$,

$$\begin{aligned} S \ll \{ & (X^{2+4\kappa} M^{4+6\kappa} N^{5+7\kappa+\lambda} Q^{2+2\kappa})^{1/(8+12\kappa)} \\ & + X^{1/4} M^{1/2} N^{5/8+\lambda/(8+8\kappa)} Q^{1/4} \\ & + (X^2 M^4 N^6 Q)^{1/8} + (X^2 M^8 N^{14} Q^3)^{1/16} \\ & + M N Q^{-1/2} + (X^{-1} M^2 N^3 Q^{-1})^{1/2}\} \mathcal{L}^{11/8}, \end{aligned} \tag{2.5}$$

where we have used the fact that $(X^{-1} M^4 N^3 Q^{-1})^{1/4}$ can be absorbed by $M N Q^{-1/2}$. In view of $M N Q^{-1/2}$, the preceding estimate holds trivially when $Q \in (0,\, \mathcal{L}]$.

If $X(MN)^{-1} Q_1 \leq \varepsilon_0$, we can remove $m^{-1/2}$ by partial summation and then estimate the sum over m by Kuz'min–Landau's inequality ([4], Theorem 2.1). Hence we see that (2.5) always holds for $0 < Q \leq N/\mathcal{L}$. Using Lemma 2.4(ii) of [12] to optimize Q over $(0,\, N/\mathcal{L}]$ yields

$$\begin{aligned} S \ll \{ & (X^{2+4\kappa} M^{8+10\kappa} N^{9+11\kappa+\lambda})^{1/(12+16\kappa)} \\ & + (X^{2\kappa} M^{8+10\kappa} N^{11+13\kappa+\lambda})^{1/(12+16\kappa)} \\ & + X^{1/6} M^{2/3} N^{3/4+\lambda/(12+12\kappa)} \\ & + M^{2/3} N^{11/12+\lambda/(12+12\kappa)} + (X M^3 N^4)^{1/5} \\ & + (X M^6 N^9)^{1/10} + (X M^7 N^{10})^{1/11} \end{aligned}$$

$$\begin{aligned}&+(X^{-1}M^{14}N^{23})^{1/22}+MN^{1/2}+X^{-1/2}MN\}\mathcal{L}^2\\ &=:(F_1+F_2+\ldots+F_{10})\mathcal{L}^2.\end{aligned}$$

Since

$$F_2=(F_4^{6+3\kappa}F_5^{5\kappa})^{1/(6+8\kappa)}N^{-\kappa(1+\kappa-\lambda)/((4+4\kappa)(6+8\kappa))}$$

and $F_6=(F_5^{16}F_8^{11})^{1/27}M^{-2/135}$, they are both superfluous. This proves (2.1).

To prove (2.2), we take $Q_2=\varepsilon_0\min\{\sqrt{Q_1},\,(X^{-1}N^2Q_1)^{1/3}\}$ such that $|f'(n')|\le 1/2$ for $n'\sim N$. Thus Kuz'min–Landau's inequality gives $S_3(q_1,q_2,q_3)\ll(XN^{-2}Q_1^{-1}q_2q_3)^{-1}$, from which we can deduce, as before, the following inequality:

$$\begin{aligned}S\ll\{&(XM^3N^4)^{1/5}+(XM^6N^9)^{1/10}+(X^4M^{10}N^{11})^{1/16}\\ &+(X^2M^{10}N^{13})^{1/16}+(XM^7N^{10})^{1/11}+(X^{-1}M^{14}N^{23})^{1/22}\\ &+MN^{1/2}+X^{-1/2}MN+M^{1/2}N\}\mathcal{L}^2\\ =:{}&(G_1+G_2+\ldots+G_9)\mathcal{L}^2.\end{aligned}$$

It is not difficult to verify that $G_2=(G_1^{16}G_6^{11})^{1/27}M^{-2/135}$, $G_4=(G_3^{15}G_6^{11})^{1/26}(M^2N^{11})^{-1/416}$, $G_9=(G_5G_6^2)^{1/3}M^{-3/22}$. Thus G_2,G_4,G_9 are superfluous. This completes the proof. ■

For $T=T(M,N)$ defined as in Section 1, we have the following result.

THEOREM 3. *Let* $\alpha,\beta\in\mathbb{R}$ *with* $\alpha\beta(\alpha-1)(\beta-1)(\alpha+\beta-1)(2\alpha+\beta-2)\neq 0$, $X>0$, $M\ge 1$, $N\ge 1$, $\mathcal{L}:=\log(2+XMN)$, $|\varphi_m|\le 1$ *and* $I(m)$ *be a subinterval of* $[N,2N]$. *Then*

$$\begin{aligned}T(M,N)\ll\{&(X^5M^{10}N^8)^{1/16}+(X^3M^{10}N^{12})^{1/16}+(XM^2N^3)^{1/4}\\ &+(X^3M^{14}N^{18})^{1/22}+(XM^6N^9)^{1/10}+(X^7M^{30}N^{24})^{1/40}\\ &+(XM^5N^5)^{1/7}+MN^{1/2}+X^{-1}MN\}\mathcal{L}^3.\end{aligned}$$

Proof. If $X\le\varepsilon_0N$, then $T\ll X^{-1}MN$ by Kuz'min–Landau's inequality. When $X\ge\varepsilon_0N$, using (2.3) with $\psi_n=1$, we have, for any $1\le Q\le\varepsilon_0N$,

$$|T|^2\ll(MN)^2Q^{-1}+\mathcal{L}M^{3/2}NQ^{-1}\max_{1\le Q_1\le Q}|T(Q_1)|,\tag{2.6}$$

where

$$T(Q_1):=\sum_{q\sim Q_1}\sum_{n\in I_1(q)}\sum_{m\in J(n,q)}m^{-1/2}e\biggl(X\frac{m^\alpha t(n,q)}{M^\alpha N^\beta}\biggr),$$

$t(n,q):=(n+q)^\beta-n^\beta$, $I_1(q)$ is a subinterval of $[N,2N]$ and $J(n,q)$ a subinterval of $[M,2M]$.

If $L := X(MN)^{-1}Q_1 \geq \varepsilon_0$, similarly to (2.4), we can prove, for some $l \asymp L$,

$$\begin{aligned} T(Q_1) \ll (XM^{-1}N^{-1}Q_1)^{1/2}\Big| \sum\sum_{(n,q)\in\mathbf{D}(l)} e(f(n,q))\Big| \\ + \{M^{-1/2}NQ_1 + (XM^{-1}N^{-1}Q_1^3)^{1/2} \\ + (X^{-2}MN^4)^{1/2} + (X^{-1}MNQ_1)^{1/2}\}\mathcal{L}, \end{aligned}$$

where $f(n,q) := \widetilde{\alpha}(XQ_1/N)(l/L)^{\alpha/(\alpha-1)}\{t(n,q)/(N^{\beta-1}Q_1)\}^{1/(1-\alpha)}$ and $\mathbf{D}(l)$ is a suitable subregion of $\{(n,q) : n \sim N,\ q \sim Q_1\}$. It is easy to show that $f(n,q)$ satisfies the condition of Lemma 1.5 of [8] (which is a revised form of Theorem 1 of Kolesnik [6]) with $A = XN^{-1}Q_1/(N^{\beta-1}Q_1)^{1/(1-\alpha)}$, $\Delta = Q_1/N$. By this lemma with $(F,X,Y) = (XN^{-1}Q_1, N, Q_1)$, we obtain the estimate

$$\begin{aligned} (2.7)\quad T(Q_1) \ll \{(X^5M^{-3}N^{-2}Q_1^8)^{1/6} + (X^3M^{-3}N^2Q_1^8)^{1/6} \\ + (XM^{-1}NQ_1^2)^{1/2} + (X^3M^{-4}N^4Q_1^{11})^{1/8} \\ + (XM^{-2}N^3Q_1^5)^{1/4} + (X^7M^{-5}N^{-6}Q_1^{20})^{1/10} \\ + (X^2M^{-2}Q_1^7)^{1/4} + (X^{-2}MN^4)^{1/2} + (X^{-1}MNQ_1)^{1/2}\}\mathcal{L}^4, \end{aligned}$$

where we have used the fact that $M^{-1/2}NQ_1 + (XM^{-1}N^{-1}Q_1^3)^{1/2}$ can be absorbed by $(XM^{-2}N^3Q_1^5)^{1/4} + (X^2M^{-2}Q_1^7)^{1/4}$ (in view of the hypothesis $X \geq \varepsilon_0 N$).

If $L \leq \varepsilon_0$, the Kuz'min–Landau inequality implies that (2.7) also holds. Replacing Q_1 by Q and inserting into (2.6) yield

$$\begin{aligned} |T|^2 \ll \{(X^5M^6N^4Q^2)^{1/6} + (X^3M^6N^8Q^2)^{1/6} + (XM^2N^3)^{1/2} \\ + (X^3M^8N^{12}Q^3)^{1/8} + (XM^4N^7Q)^{1/4} + (X^7M^{10}N^4Q^{10})^{1/10} \\ + (X^2M^4N^4Q^3)^{1/4} + (MN)^2Q^{-1}\}\mathcal{L}^5, \end{aligned}$$

where we have eliminated two superfluous terms $X^{-1}M^2N^3Q^{-1}$ and $(X^{-1}M^4N^3Q^{-1})^{1/2}$ (which can be absorbed by $(MN)^2Q^{-1}$). Using Lemma 2.4(ii) of [12] to optimize Q over $(0, \varepsilon_0 N]$ gives the required result. This concludes the proof. ∎

Next we shall apply Theorems 2 and 3 to treat

$$S_I := \sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\sum_{m_3\sim M_3} \psi_{m_2} e\bigg(X\frac{m_1^{\alpha_1}m_2^{\alpha_2}m_3^{-\alpha_2}}{M_1^{\alpha_1}M_2^{\alpha_2}M_3^{-\alpha_2}}\bigg),$$

$$S_{II} := \sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\sum_{m_3\sim M_3} \varphi_{m_1}\psi_{m_2} e\bigg(X\frac{m_1^{\alpha_1}m_2^{\alpha_2}m_3^{-\alpha_2}}{M_1^{\alpha_1}M_2^{\alpha_2}M_3^{-\alpha_2}}\bigg),$$

which are general forms of (1.4). The following results will be used in the proof of Theorem 1.

Corollary 1. *Let* $\alpha_j \in \mathbb{R}$ *with* $\alpha_1\alpha_2(\alpha_2+1)(\alpha_1-j\alpha_2-j) \neq 0$ $(j=1,2)$, $X>0$, $M_j \geq 1$, $|\varphi_{m_1}| \leq 1$, $|\psi_{m_2}| \leq 1$ *and let* $Y := 2 + XM_1M_2M_3$. *If* (κ,λ) *is an exponent pair*, *then for any* $\varepsilon > 0$,

$$\begin{aligned} S_{II} \ll \{&(X^{4+6\kappa}M_1^{9+11\kappa+\lambda}M_2^{8+10\kappa}M_3^{4+6\kappa})^{1/(12+16\kappa)} \\ &+ X^{1/3}M_1^{3/4+\lambda/(12+12\kappa)}M_2^{2/3}M_3^{1/3} + (X^3M_1^8M_2^6M_3^4)^{1/10} \\ &+ (X^5M_1^{20}M_2^{14}M_3^8)^{1/22} + X^{1/6}M_1^{11/12+\lambda/(12+12\kappa)}M_2^{2/3}M_3^{1/3} \\ &+ (XM_1M_2^2)^{1/2} + (X^2M_1^{23}M_2^{14}M_3^8)^{1/22} + M_1M_2 \\ &+ X^{-1/2}M_2M_3 + X^{-1}M_1M_2M_3\}Y^{\varepsilon}. \end{aligned}$$

In particular, *if* $X \geq M_3 \geq M_1$, *then*

$$\begin{aligned} S_{II} \ll \{&(X^{186}M_1^{407}M_2^{350}M_3^{186})^{1/536} \\ &+ (X^{164}M_1^{385}M_2^{328}M_3^{164})^{1/492} + (X^3M_1^8M_2^6M_3^4)^{1/10} \\ &+ (X^5M_1^{20}M_2^{14}M_3^8)^{1/22} + (XM_1M_2^2)^{1/2}\}Y^{\varepsilon}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} S_{II} \ll \{&(X^{13}M_1^{15}M_2^{22}M_3^4)^{1/26} + (X^2M_1^2M_2^3M_3)^{1/4} \\ &+ (X^9M_1^{11}M_2^{18})^{1/18} + (XM_1^4M_2^3M_3)^{1/4}\}Y^{\varepsilon}. \end{aligned} \tag{2.9}$$

Proof. If $M_3' := X/M_3 \leq \varepsilon_0$, the Kuz'min–Landau inequality implies

$$S_{II} \ll X^{-1}M_1M_2M_3.$$

Next we suppose $M_3' \geq \varepsilon_0$. As before, using Lemma 2.2 of [12] to the sum over m_3 and estimating the corresponding error term by Lemma 2.3 there with $n = m_1$, we obtain

$$S_{II} \ll X^{-1/2}M_3S + (X^{1/2}M_2 + M_1M_2 + X^{-1/2}M_2M_3 + X^{-1}M_1M_2M_3)\log Y,$$

where

$$\begin{aligned} S &:= \sum_{m_1\sim M_1}\ \sum_{m_2\sim M_2}\ \sum_{m_3'\sim M_3'} \widetilde{\varphi}_{m_1}\widetilde{\psi}_{m_2}\xi_{m_3'}\, e\biggl(\widetilde{\alpha}_2 X \frac{m_1^{\beta_1}m_2^{\beta_2}m_3'^{\beta_2}}{M_1^{\beta_1}M_2^{\beta_2}M_3'^{\beta_2}}\biggr) \\ &= \sum_{m_1\sim M_1}\ \sum_{m_2'\sim M_2'} \widetilde{\varphi}_{m_1}\widetilde{\xi}_{m_2'}\, e\biggl(\widetilde{\alpha}_2 X \frac{m_1^{\beta_1}m_2'^{\beta_2}}{M_1^{\beta_1}M_2'^{\beta_2}}\biggr), \end{aligned}$$

and $\beta_j := \alpha_j/(1+\alpha_2)$ $(j=1,2)$, $\widetilde{\alpha}_2 := |1+\alpha_2|\cdot|\alpha_2|^{-\beta_2}$, $|\widetilde{\varphi}_{m_1}| \leq 1$, $|\widetilde{\psi}_{m_2}| \leq 1$, $|\xi_{m_3'}| \leq 1$, $M_2' := M_2M_3'$, $\widetilde{\xi}_{m_2'} := \sum\sum_{m_2m_3'=m_2'} \widetilde{\psi}_{m_2}\xi_{m_3'}$. By Theorem 2 with $(M,N) = (M_2', M_1)$ we estimate S to get the first assertion.

In particular taking $(\kappa,\lambda) = BA^2(\frac{1}{6},\frac{4}{6}) = (\frac{11}{30},\frac{16}{30})$ yields

$$\begin{aligned} S_{II} \ll \{&(X^{186}M_1^{407}M_2^{350}M_3^{186})^{1/536} + (X^{164}M_1^{385}M_2^{328}M_3^{164})^{1/492} \\ &+ (X^3M_1^8M_2^6M_3^4)^{1/10} + (X^5M_1^{20}M_2^{14}M_3^8)^{1/22} \\ &+ (X^{82}M_1^{467}M_2^{328}M_3^{164})^{1/492} + (XM_1M_2^2)^{1/2} \end{aligned}$$

$$\begin{aligned}&+(X^2M_1^{23}M_2^{14}M_3^8)^{1/22}\\&+M_1M_2+X^{-1/2}M_2M_3+X^{-1}M_1M_2M_3\}Y^{\varepsilon}\\=:{}&(H_1+H_2+\ldots+H_{10})Y^{\varepsilon}.\end{aligned}$$

Since $X\ge M_3\ge M_1$, we have $H_5\le H_2$, $H_7\le H_4$, $H_j\le H_6$ $(8\le j\le 10)$ and thus H_5, H_j $(7\le j\le 10)$ are superfluous. This proves (2.8).

The last inequality can be proved similarly by using Theorem 7 of [12] with $(M_1,M_2,M_3)=(M_1,1,M_2')$. This completes the proof. ∎

COROLLARY 2. *Let* $\alpha_j\in\mathbb{R}$ *with* $\alpha_1\alpha_2(\alpha_1-1)(\alpha_1-2)(\alpha_2+1)(\alpha_1-\alpha_2-1)\ne 0$, $X>0$, $M_j\ge 1$, $|\psi_{m_2}|\le 1$ *and let* $Y:=2+XM_1M_2M_3$. *If* $M_3\ge M_1$, *then for any* $\varepsilon>0$ *we have*

$$\begin{aligned}\text{(2.10)}\quad S_I\ll{}&\{(X^7M_1^8M_2^{10}M_3^6)^{1/16}+(X^5M_1^{12}M_2^{10}M_3^6)^{1/16}\\&+(XM_1^3M_2^2M_3^2)^{1/4}+(X^3M_1^9M_2^7M_3^4)^{1/11}\\&+(X^2M_1^9M_2^6M_3^4)^{1/10}+(X^{17}M_1^{24}M_2^{30}M_3^{10})^{1/40}\\&+(X^5M_1^{10}M_2^{10}M_3^4)^{1/14}+(XM_1M_2^2)^{1/2}+X^{-1}M_1M_2M_3\}Y^{\varepsilon},\\\text{(2.11)}\quad S_I\ll{}&\{(X^{15}M_1^{11}M_2^{22}M_3^4)^{1/26}+(X^2M_1^2M_2^3M_3)^{1/4}+(X^3M_2^3M_3)^{1/4}\\&+(X^{11}M_1^7M_2^{18})^{1/18}+M_2M_3+X^{-1}M_1M_2M_3\}(\log 2Y)^4.\end{aligned}$$

Proof. As before we may suppose $M_3':=X/M_3\ge\varepsilon_0$ and prove

$$\begin{aligned}\text{(2.12)}\quad S_I\ll{}&X^{-1/2}M_3T\\&+(X^{1/2}M_2+M_1M_2+X^{-1/2}M_2M_3+X^{-1}M_1M_2M_3)\log Y,\end{aligned}$$

where

$$T:=\sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\sum_{m_3'\in I_3}g(m_1)\widetilde{\psi}_{m_2}\xi_{m_3'}e\bigg(\widetilde{\alpha}_2X\frac{m_1^{\beta_1}m_2^{\beta_2}m_3'^{\beta_2}}{M_1^{\beta_1}M_2^{\beta_2}M_3'^{\beta_2}}\bigg),$$

$$I_3:=[c_3(m_1/M_1)^{\alpha_1}(m_2/M_2)^{\alpha_2}M_3',\ c_4(m_1/M_1)^{\alpha_1}(m_2/M_2)^{\alpha_2}M_3'],$$

and $\beta_j,\widetilde{\alpha}_2,\widetilde{\psi}_{m_2},\xi_{m_3'}$ are defined as before, $c_j=c_j(\alpha_2)$ are constants, $g(m_1)$ is a monomial with $|g(m_1)|\le 1$. We define $\widetilde{\xi}_{m_2'}$ and M_2' in the same way as in the proof of Corollary 1. Exchanging the order of summation, we have

$$T=\sum_{m_2'\sim M_2'}\widetilde{\xi}_{m_2'}\sum_{m_1\in I_1(m_2')}g(m_1)e\bigg(\widetilde{\alpha}_2X\frac{m_1^{\beta_1}m_2'^{\beta_2}}{M_1^{\beta_1}M_2'^{\beta_2}}\bigg),$$

where $I_1(m_2')$ is a subinterval of $[M_1,2M_1]$. Removing $g(m_1)$ by partial summation and estimating the double sum obtained by Theorem 3 with $(M,N)=(M_2',M_1)$, we find

$$\begin{aligned} T \ll \{&(X^5M_1^8M_2'^{10})^{1/16} + (X^3M_1^{12}M_2'^{10})^{1/16} + (XM_1^3M_2'^2)^{1/4} \\ &+ (X^3M_1^{18}M_2'^{14})^{1/22} + (XM_1^9M_2'^6)^{1/10} + (X^7M_1^{22}M_2'^{26})^{1/36} \\ &+ (XM_1^5M_2'^5)^{1/7} + M_1^{1/2}M_2'\}Y^{\varepsilon}. \end{aligned}$$

Inserting into (2.12) and noticing that the last four terms on the right-hand side of (2.12) can be absorbed by $(XM_1M_2^2)^{1/2}$, we obtain (2.10). The inequality (2.11) is (2.7) of [13] with $(M_1, M_2, M_3) = (M_2, M_1, M_3)$ and $(\alpha_1, \alpha_2, \alpha_3) = (\alpha_2, \alpha_1, -\alpha_2)$. This concludes the proof. ■

3. Proof of Theorem 1. We shall prove only

$$\Delta(1,1,2;x) \ll_{\varepsilon} x^{47/131+\varepsilon}, \tag{3.1}$$

since this implies $\Delta(x) \ll_{\varepsilon} x^{47/131+\varepsilon}$ by a simple convolution argument. For this we recall some standard notations. Let $\mathbf{u} := (u_1, u_2, u_3)$ be a permutation of $(1,1,2)$ and let $\mathbf{N} := (N_1, N_2) \in \mathbb{N}^2$. We write $\psi(t) := \{t\} - 1/2$ ($\{t\}$ is the fractional part of t) and define

$$S(\mathbf{u}, \mathbf{N}; x) := \sum\nolimits_1 \psi((x/(n_1^{u_1}n_2^{u_2}))^{1/u_3}),$$

where the summation condition of $\sum_1$ is $n_1^{u_1}n_2^{u_2+u_3} \le x$, $n_1(\le)n_2$, $n_1 \sim N_1$, $n_2 \sim N_2$. The notation $n_1(\le)n_2$ means that $n_1 = n_2$ for $u_1 < u_2$, and $n_1 < n_2$ otherwise. It is well known that for proving (3.1) it suffices to verify

$$S(\mathbf{u}, \mathbf{N}; x) \ll x^{47/131+\varepsilon} \quad \text{for } \mathbf{u} = (1,1,2),\ (2,1,1),\ (1,2,1).$$

Since $S(1,1,2,\mathbf{N};x) \ll x^{5/14+\varepsilon}$ (see [10], p. 263), it remains to consider $\mathbf{u} = (2,1,1)$, $(1,2,1)$. We shall prove the desired estimate for $\mathbf{u} = (2,1,1)$ in two cases according to the size of N_1, which we shall formulate as two lemmas. The case of $\mathbf{u} = (1,2,1)$ can be treated similarly (more easily). We recall that we have $N_1 \le N_2 \le G := x/(N_1^2N_2)$, $N_1N_2 \le x^{1/2}$ when $\mathbf{u} = (2,1,1)$. This fact will be used (implicitly) many times in the proofs of Lemmas 3.1 and 3.2.

LEMMA 3.1. *For* $\mathbf{u} = (2,1,1)$, *we have*

$$S(\mathbf{u}, \mathbf{N}; x) \ll_{\varepsilon} \{(x^{186}N_1^{35})^{1/536} + (xN_1^2)^{1/4} + (x^{40}N_1^7)^{1/116} + x^{5/14}\}x^{\varepsilon}.$$

In particular, if $N_1 \le x^{118/655}$, *then* $S(\mathbf{u}, \mathbf{N}; x) \ll_{\varepsilon} x^{47/131+\varepsilon}$.

Proof. By Lemma 2.5 of [12], we have, for any $H \ge 1$,

$$S(\mathbf{u}, \mathbf{N}; x) \ll H^{-1}N_1N_2 + (\log x) \max_{1 \le H_0 \le H} H_0^{-1}|S(H_0, \mathbf{N})|, \tag{3.2}$$

where

$$S(H_0, \mathbf{N}) := \sum_{h \sim H_0} a_h \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_2} e(hx/(n_1^2n_2)), \qquad |a_h| \le 1.$$

The inequalities (2.8) and (2.9) with $(X, M_1, M_2, M_3) = (GH_0, N_1, H_0, N_2)$ imply

$$\begin{aligned} S(H_0, \mathbf{N}) &\ll \{(G^{186}N_1^{407}N_2^{186})^{1/536} + (GN_1H_0)^{1/2} + \chi_1 + \chi_2\}H_0x^{\varepsilon}, \\ S(H_0, \mathbf{N}) &\ll \{(G^{13}N_1^{15}N_2^{4}H_0^{9})^{1/26} + (G^2N_1^2N_2H_0)^{1/4} + (G^9N_1^{11}H_0^{9})^{1/18} \\ &\qquad + (xN_1^2)^{1/4}\}H_0x^{\varepsilon} \\ &\qquad =: \{D_1 + D_2 + D_3 + (xN_1^2)^{1/4}\}H_0x^{\varepsilon}, \end{aligned}$$

with $\chi_1 := (G^3N_1^8N_2^4H_0^{-1})^{1/10}$ and $\chi_2 := (G^5N_1^{20}N_2^8H_0^{-3})^{1/22}$, where we have used the fact that $(G^{164}N_1^{385}N_2^{164})^{1/492} \le (G^{186}N_1^{407}N_2^{186})^{1/536}$ (in view of $N_1 \le x^{1/4}$). From these, we deduce that for any $H_0 \ge 1$,

$$\begin{aligned} S(H_0, \mathbf{N}) \ll \Big\{&(G^{186}N_1^{407}N_2^{186})^{1/536} + (GN_1H_0)^{1/2} \\ &+ (xN_1^2)^{1/4} + \sum_{1\le j\le 2}\sum_{1\le k\le 3} R_{j,k}\Big\}H_0x^{\varepsilon}, \end{aligned}$$

where $R_{j,k} := \min\{\chi_j, D_k\}$. Since

$$\begin{cases} R_{1,1} \le (\chi_1^{45}D_1^{13})^{1/58} = (x^{40}N_1^{7})^{1/116}, \\ R_{1,2} \le (\chi_1^{5}D_2^{2})^{1/7} = x^{5/14}, \\ R_{1,3} \le (\chi_1^{5}D_3)^{1/6} = (x^{36}N_1^{11})^{1/108}, \end{cases}$$

$$\begin{cases} R_{2,1} \le (\chi_2^{33}D_1^{13})^{1/46} = (x^{28}N_1^{19})^{1/92} < x^{5/14}, \\ R_{2,2} \le (\chi_2^{11}D_2^{6})^{1/17} = (x^{11}N_1^{4})^{1/34} < x^{5/14}, \\ R_{2,3} \le (\chi_2^{11}D_3^{3})^{1/14} = (x^{24}N_1^{23})^{1/84} < x^{5/14}, \end{cases}$$

and $(x^{36}N_1^{11})^{1/108} \le (x^{40}N_1^{7})^{1/116}$, we have

$$\begin{aligned} S(H_0, \mathbf{N}) \ll \{&(G^{186}N_1^{407}N_2^{186})^{1/536} + (GN_1H_0)^{1/2} \\ &+ (xN_1^2)^{1/4} + (x^{40}N_1^{7})^{1/116} + x^{5/14}\}H_0x^{\varepsilon}. \end{aligned}$$

Inserting into (3.2) and optimizing H by Lemma 2.4(iii) of [12] yield the desired estimate. ∎

LEMMA 3.2. *For* $\mathbf{u} = (2, 1, 1)$, *we have*

$$\begin{aligned} S(\mathbf{u}, \mathbf{N}; x) \ll \{&(x^7/N_1^5)^{1/17} + (x^{17}/N_1^3)^{1/47} + (x^{11}/N_1^4)^{1/29} \\ &+ (x^{13}/N_1^2)^{1/36} + (x^5/N_1^4)^{1/12} + x^{103/294}\}x^{\varepsilon}. \end{aligned}$$

In particular, *if* $N_1 \ge x^{118/655}$, *then* $S(\mathbf{u}, \mathbf{N}; x) \ll_{\varepsilon} x^{47/131+\varepsilon}$.

Proof. Corollary 2 with $(X, M_1, M_2, M_3) = (GH_0, N_1, H_0, N_2)$ gives

$$\begin{aligned} S(H_0, \mathbf{N}) \ll \{&L(H_0) + (G^5N_1^{12}N_2^{6}H_0^{-1})^{1/16} + (GN_1^3N_2^2H_0^{-1})^{1/4} \\ &+ (G^3N_1^9N_2^4H_0^{-1})^{1/11} + (G^2N_1^9N_2^4H_0^{-2})^{1/10}\}H_0x^{\varepsilon} \\ =: \{&L(H_0) + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4\}H_0x^{\varepsilon}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} S(H_0,\mathbf{N}) &\ll \{(G^{15}N_1^{11}N_2^4H_0^{11})^{1/26}+(G^2N_1^2N_2H_0)^{1/4} \\ &\qquad + (G^3N_2H_0^2)^{1/4}+(G^{11}N_1^7H_0^{11})^{1/18}\}H_0x^{\varepsilon} \\ &=: \{E_1+E_2+E_3+(G^{11}N_1^7H_0^{11})^{1/18}\}H_0x^{\varepsilon}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} L(H_0) := (G^7N_1^8N_2^6H_0)^{1/16} &+ (G^{17}N_1^{24}N_2^{10}H_0^7)^{1/40} \\ &+ (G^5N_1^{10}N_2^4H_0)^{1/14}+(GN_1H_0)^{1/2}, \end{aligned}$$

and we have used the fact that $G^{-1}N_1N_2H_0^{-1}$ can be absorbed by $(GN_1^3N_2^2H_0^{-1})^{1/4}$ in (3.3), both N_2 and $G^{-1}N_1N_2H_0^{-1}$ by $(G^3N_2H_0^2)^{1/4}$ in (3.4). From (3.3) and (3.4), we deduce that for any $H_0\geq 1$,

$$S(H_0,\mathbf{N}) \ll \Big\{L(H_0)+(G^{11}N_1^7H_0^{11})^{1/18}+\sum_{1\leq j\leq 4}\sum_{1\leq k\leq 3}S_{j,k}\Big\}H_0x^{\varepsilon},$$

where $S_{j,k}:=\min\{\sigma_j,E_k\}$. It is easy to verify that

$$\begin{cases} S_{1,1}\leq(\sigma_1^{88}E_1^{13})^{1/101}=(x^{70}N_1^3)^{1/202}<x^{103/294}, \\ S_{1,2}\leq(\sigma_1^4E_2)^{1/5}=x^{7/20}<x^{103/294}, \\ S_{1,3}\leq(\sigma_1^8E_3)^{1/9}=(x^{13}/N_1^2)^{1/36}, \end{cases}$$

$$\begin{cases} S_{2,1}\leq(\sigma_2^{22}E_1^{13})^{1/35}=(x^{13}/N_1^4)^{1/35}, \\ S_{2,2}\leq(\sigma_2E_2)^{1/2}=(x^3/N_1)^{1/8}, \\ S_{2,3}\leq(\sigma_2^2E_3)^{1/3}=(x^5/N_1^4)^{1/12}, \end{cases}$$

$$\begin{cases} S_{3,1}\leq(\sigma_3^{121}E_1^{26})^{1/147}=(x^{48}N_1^{14})^{1/147}\leq x^{103/294}, \\ S_{3,2}\leq(\sigma_3^{11}E_2^4)^{1/15}=(x^5N_1)^{1/15}<x^{103/294}, \\ S_{3,3}\leq(\sigma_3^{11}E_3^2)^{1/13}=x^{9/26}<x^{103/294}, \end{cases}$$

$$\begin{cases} S_{4,1}\leq(\sigma_4^{55}E_1^{26})^{1/81}=(x^{52}N_1^{17})^{1/162}<x^{103/294}, \\ S_{4,2}\leq(\sigma_4^5E_2^4)^{1/9}=(x^6N_1)^{1/18}<x^{103/294}, \\ S_{4,3}\leq(\sigma_4^5E_3^2)^{1/7}=(x^5/N_1)^{1/14}, \end{cases}$$

and $(x^{13}/N_1^4)^{1/35}\leq(x^3/N_1)^{1/8}$, $(x^5/N_1)^{1/14}\leq(x^3/N_1)^{1/8}$. Consequently, we obtain, for any $H_0\geq 1$, the inequality

$$\begin{aligned} S(H_0,\mathbf{N}) \ll \{L(H_0)&+(G^{11}N_1^7H_0^{11})^{1/18}+(x^{13}/N_1^2)^{1/36} \\ &+(x^3/N_1)^{1/8}+(x^5/N_1^4)^{1/12}+x^{103/294}\}H_0x^{\varepsilon}. \end{aligned}$$

Inserting this estimate in (3.2) and using Lemma 2.4(iii) of [12] to optimize H, we find

$$\begin{aligned} S(\mathbf{u},\mathbf{N};x) \ll \{&(x^7/N_1^5)^{1/17}+(x^{17}/N_1^3)^{1/47}+(x^{11}/N_1^4)^{1/29} \\ &+(x^{13}/N_1^2)^{1/36}+(x^3/N_1)^{1/8}+(x^5/N_1^4)^{1/12}+x^{103/294}\}x^{\varepsilon}. \end{aligned}$$

Observing that $(x^3/N_1)^{1/8}\leq(x^{11}/N_1^4)^{1/29}$, we get the required estimate. ∎

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Laboratoire de Mathématiques
Institut Élie Cartan – CNRS UMR 9973
Université Henri Poincaré (Nancy 1)
54506 Vandœuvre-lès-Nancy, France
E-mail: wujie@iecn.u-nancy.fr

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