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ON THE AVERAGE ORDERS OF A CLASS OF DIVISOR FUNCTIONS

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Abstract

We compute the average orders and study the distribution of values of a class of divisor functions defined by symmetric polynomials on the multi-set of prime factors of a number. These generalize those we have previously defined. The simplest case of these functions is the sum of prime factors with repetition function, whose average order has been computed in various ways by Alladi and Erdős, LeVan,and Kerawala.

1. Introduction

Let us begin by consolidating some notation. Let $\mathbb{N}_0$ denote the set of nonnegative integers. In [13] we considered the functions s_k defined as follows:

Definition 1 Let $k \in \mathbb{N}_0$. Define $s_k : \mathbb{N}_0 \to \mathbb{N}_0$ as follows: if $n = 0$, then $s_k(0) = 0$, for all k. For $n > 0$, if $k = 0$, $s_k(n) = 1$. If $k > 0$, and $n = p_1 \cdots p_r$, where $r = \Omega(n)$ is the number of prime factors (with multiplicity) of n, then

$$s_k(n) = \sum p_{i_1} \cdots p_{i_k},$$

where the sum is taken over all products of k prime factors from the multi-set $\{p_1, \ldots, p_r\}$.

These functions are mere special cases of a larger class of functions:

Definition 2 Let $k, \ell \in \mathbb{N}_0$. We define $s_{k,\ell}(0) = 0$. If $n = p_1 \cdots p_r \in \mathbb{N}$, where the p_i are primes, not necessarily distinct, then

$$s_{k,\ell}(n) = \sum_{1 \le i_1 < \ldots < i_k \le k} (p_{i_1} \cdots p_{i_k})^{\ell}.$$

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We write the function $s_{1,k}$ as e_k. The function $s_{0,k} \equiv 1$.

The special case when $k = \ell = 1$, i.e. the sum of prime factors with repetition function, has been studied in other contexts. Denote this function by s. Lal [10] observes that the sequence of iterates $n, s(n), s(s(n)), \ldots$, for $n \geq 5$ always terminates in a prime number $p \geq 5$. Based on empirical evidence, he conjectures that corresponding to a fixed such prime, the set

$$\{n \in \mathbb{N} : s^{(i)}(n) = p \text{ for some } i \geq 0\}$$

has a positive asymptotic density. Here we are writing $s^{(i)}(n)$ for the ith iterate of n by s.

Alladi and Erdős [1] show that $s(n)$ is uniformly distributed modulo 2 by proving that

$$\sum_{n \leq x} (-1)^{s(n)} = o(x).$$

Since $s(n) \equiv e_k(n) \pmod 2$ for any $k \in \mathbb{N}$, this result clearly extends to $e_k(n)$.

Gupta [4] shows that for a given m,

$$\max\{n \in \mathbb{N} : s(n) = m\} = t \cdot 3^{\lfloor m/3 \rfloor},$$

where $t = 1, 4/3$, or 2; according as $m \equiv 0, 1$, or $2 \pmod 3$.

The size of the set $e_k^{(-1)}(n)$ is equal to the number of partitions of n into kth powers of primes, $p_{\mathbb{P}(k)}(n)$. Bateman and Erdős [2] have shown that the number of partitions into primes $p_{\mathbb{P}}(n)$ is strictly increasing, and Hardy and Ramanujan [5] first demonstrated the asymptotic formula

$$\log p_{\mathbb{P}(k)}(n) \sim (k+1)\left[\Gamma\left(\frac{1}{k}+2\right)\zeta\left(\frac{1}{k}+1\right)\right]^{k/(k+1)}\left[\frac{n}{\log^k n}\right]^{1/(k+1)}.$$

From this it is easily seen that the functions $e_k(n)$ attain all sufficiently high values, for any $k \in \mathbb{N}$. We have shown [13] that $s_2(n)$ does as well.

In Section 3 of the present paper, we shall prove the asymptotic formula

$$\sum_{n \leq x} s_{k,\ell}(n) \sim \frac{\zeta(\ell+1)x^{\ell+1}(\log\log x)^{k-1}}{(\ell+1)(k-1)!\log x}, \tag{1}$$

for $k, \ell \geq 1$ fixed, with a precise error term included. Section 2 is devoted to $e_k(n)$, and, in addition to proving (1) in this case, we study the distribution of values of these functions.

2. Statistical Results for e_k

2.1 The Average Order of e_k

Theorem 2 For $k \in \mathbb{N}$, $\sum_{n \le x} e_k(n) = \frac{\zeta(k+1)x^{k+1}}{(k+1)\log x} + O\left(\frac{x^{k+1}\log\log x}{\log^2 x}\right)$.

The average order of s itself is studied by Alladi and Erdős [1] by comparing it to the average order of the largest prime factor dividing n. The asymptotic in Theorem 2 (without the error term included) has been advanced by Kerawala [9] and LeVan [11]. We shall make precise LeVan's sketch of the proof. First we require a lemma.

Lemma 1 For $x \ge 2$, and $k, \ell \in \mathbb{N}_0$ we have $\sum_{p \le x} \frac{p^k}{\log^\ell p} = \frac{x^{k+1}}{(k+1)\log^{\ell+1} x} + O\left(\frac{x^{k+1}}{\log^{\ell+2} x}\right)$.

The proof of Lemma 1 follows from a simple application of Riemann-Stieltjes integration and integration by parts.

Proof of Theorem 2. We have

$$\sum_{n \le x} e_k(n) = \sum_{p \le x} \sum_{i=1}^{\infty} p^k \left\lfloor \frac{x}{p^i} \right\rfloor = \sum_{p \le x} p^k \left\lfloor \frac{x}{p} \right\rfloor + \sum_{i=2}^{\infty} \sum_{p \le x^{1/i}} p^k \left\lfloor \frac{x}{p^i} \right\rfloor. \tag{2}$$

As we shall see, the first term contributes the greater portion to the sum:

$$\begin{aligned} \sum_{p \le x} p^k \left\lfloor \frac{x}{p} \right\rfloor &= \sum_{i \le x/2} \sum_{\frac{x}{i+1} < p \le \frac{x}{i}} i p^k = \sum_{i \le x/2} \sum_{p \le x/i} p^k \\ &= \sum_{i \le x/2} \left(\frac{(x/i)^{k+1}}{(k+1)\log(x/i)} + O\left(\frac{(x/i)^{k+1}}{\log^2(x/i)}\right)\right). \end{aligned} \tag{3}$$

Let $\Sigma_1 = \sum_{i \le \log^2 x} \frac{1}{i^{k+1}\log(x/i)}$ and $\Sigma_2 = \sum_{\log^2 x < i \le x/2} \frac{1}{i^{k+1}\log(x/i)}$ so that

$$\sum_{i \le x/2} \frac{1}{i^{k+1}\log(x/i)} = \Sigma_1 + \Sigma_2.$$

We have that

$$\begin{aligned} \Sigma_1 &\ge \frac{1}{\log x} \sum_{i \le \log^2 x} \frac{1}{i^{k+1}} = \frac{1}{\log x}\left(\zeta(k+1) - \sum_{i > \log^2 x} \frac{1}{i^{k+1}}\right) \\ &= \frac{1}{\log x}\left(\zeta(k+1) + O\left(\frac{1}{\log^{2k} x}\right)\right) = \frac{\zeta(k+1)}{\log x} + O\left(\frac{1}{\log^{2k+1} x}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned}\Sigma_1 &\leq \sum_{i\leq \log^2 x} \frac{1}{i^{k+1}(\log x - 2\log\log x)} = \frac{1}{\log x}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right)\sum_{i\leq\log^2 x}\frac{1}{i^{k+1}}\\ &= \frac{1}{\log x}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right)\left(\zeta(k+1)+O\left(\frac{1}{\log^{2k} x}\right)\right)\\ &= \frac{\zeta(k+1)}{\log x}+O\left(\frac{\log\log x}{\log^2 x}\right).\end{aligned}$$

Combining these results we have that

$$\Sigma_1 = \frac{\zeta(k+1)}{\log x}+O\left(\frac{\log\log x}{\log^2 x}\right). \tag{4}$$

The sum Σ_2 is negligible by comparison:

$$\Sigma_2 = \sum_{\log^2 x< i\leq x/2}\frac{1}{i^{k+1}\log(x/i)} \ll \sum_{i>\log^2 x}\frac{1}{i^{k+1}} \ll \int_{\log^2 x}^{\infty}\frac{1}{t^{k+1}}\,dt \ll \frac{1}{\log^{2k} x}. \tag{5}$$

Now we need to bound the error term in (3):

$$\begin{aligned}\sum_{i\leq x/2}\frac{1}{i^{k+1}\log^2(x/i)} &= O\left(\int_1^{x/2}\frac{dt}{t^2\log^2 x/t}\right)\\ &= O\left(\int_1^{\sqrt{x}}\frac{dt}{t^2\log^2 x/t}+\int_{\sqrt{x}}^{x/2}\frac{dt}{t^2\log^2 x/t}\right)\\ &= O\left(\frac{4}{\log^2 x}\int_1^{\sqrt{x}}\frac{dt}{t^2}+\int_{\sqrt{x}}^{x/2}\frac{dt}{t^2}\right)\\ &= O\left(\frac{1}{\log^2 x}\right)\end{aligned} \tag{6}$$

Hence by (4), (5), and (6), we have that

$$\sum_{p\leq x} p^k\left\lfloor\frac{x}{p}\right\rfloor = \frac{\zeta(k+1)x^{k+1}}{(k+1)\log x}+O\left(\frac{x^{k+1}\log\log x}{\log^2 x}\right).$$

To conclude the proof, we need to bound the second term in (2):

$$\begin{aligned}\sum_{i=2}^{\infty}\sum_{p\leq x^{1/i}} p^k\left\lfloor\frac{x}{p^i}\right\rfloor &\leq x\sum_{i=2}^{\infty}\sum_{p\leq x^{1/i}}\frac{p^k}{p^i} \leq x\sum_{p\leq\sqrt{x}}\frac{p^{k-1}}{p-1} \leq 2x\sum_{p\leq\sqrt{x}} p^{k-2}\\ &= \begin{cases} O(x\log\log x), & \text{if } k=1;\\ O\left(\frac{x^{\frac{k+1}{2}}}{\log x}\right), & \text{if } k>1.\end{cases}\end{aligned}$$

Note that for the $k = 1$ case, $x \log\log x = O\left(\frac{x^2 \log\log x}{\log^2 x}\right)$, and for the $k > 1$ case, $\frac{x^{\frac{k+1}{2}}}{\log x} = O\left(\frac{x^{k+1}\log\log x}{\log^2 x}\right)$. □

2.2 The Distribution of Values of e_k

In this section we shall relate $e_k(n)$ to the largest prime factor dividing n, which we denote by $P(n)$. The following trivial identity turns out to be rather useful in deriving some statistical properties for e_k:

$$P(n)^k \le e_k(n) \le P(n)^k \Omega(n). \tag{7}$$

Definition 3 Let

$$b_k(x, y) = \#\{n \le x : e_k(n) \le y\}, \tag{8}$$

$$\Psi(x, y) = \#\{n \le x : P(n) \le y\}. \tag{9}$$

So $b_k(x, y)$ is the number of partitions into k-th powers of primes whose sum is less than or equal to y, and whose product of parts is less than or equal to x. We wish to relate the two quantities defined above. To do so, we state the following well-known fact concerning the function $\Omega(n)$, which counts the number of prime factors with repetition of n. The proof is by analogy with [8], p.30.

$$\#\{n \le x : |\Omega(n) - \log\log n| > (\log\log n)^{3/4}\} = O\left(\frac{x}{\sqrt{\log\log x}}\right). \tag{10}$$

Theorem 3 There is an absolute constant $c_1 > 0$ such that the following inequalities hold:

$$\Psi\left(x, \left(\frac{y}{\log\log x + (\log\log x)^{3/4}}\right)^{1/k}\right) - \frac{c_1 x}{\sqrt{\log\log x}} \le b_k(x, y) \le \Psi(x, y^{1/k}). \tag{11}$$

Proof. By equation (7), we have that

$$\#\{n \le x : P(n)^k \Omega(n) \le y\} \le b_k(x, y) \le \#\{n \le x : P(n)^k \le y\}.$$

This implies the second inequality. By (10), there is an absolute positive constant c_1 such that $\#\{n \le x : |\Omega(n) - \log\log n| \le (\log\log n)^{3/4}\} \ge x\left(1 - \frac{c_1}{\sqrt{\log\log x}}\right)$. Given this, we have the following chain of inequalities:

$$\begin{aligned}\#\{n \le x : P(n)^k\Omega(n) \le y\} \ge &\#\{n \le x : P(n)^k\Omega(n) \le y, \text{ and} \\ &\quad \Omega(n) \le \log\log n + (\log\log n)^{3/4}\} \\ \ge &\#[\{n \le x : P(n)^k(\log\log n + (\log\log n)^{3/4}) \le y\} \\ &\quad \cap \{n \le x : \Omega(n) \le \log\log n + (\log\log n)^{3/4}\}] \\ \ge &\#\{n \le x : P(n)^k(\log\log x + (\log\log x)^{3/4}) \le y\} \\ &\quad + \#\{n \le x : |\Omega(n) - \log\log n| \le (\log\log n)^{3/4}\} - x \\ \ge &\Psi\left(x, \left(\frac{y}{\log\log x + (\log\log x)^{3/4}}\right)^{1/k}\right) - \frac{c_1 x}{\sqrt{\log\log x}},\end{aligned}$$

and the theorem is proved. □

The "Dickman function" $\rho(u)$ is defined to be the unique continuous solution to the differential-difference equation

$$u\rho'(u) = -\rho(u-1) \quad (u > 1),$$

satisfying the initial condition $\rho(u) = 1 \ \ (0 \le u \le 1)$. The Dickman function is nonnegative for $u > 0$, and decreasing for $u > 1$. This definition and description is taken from [7], an extensive survey of work done on the function $\Psi(x, y)$.

It is also true that $\rho(u)$ is convex on $(1, \infty)$. By the functional equation, it is apparent that $\lim_{u\to 1^+} \rho'(u) = -1$. It follows that

$$\rho(a + b) \ge \rho(a) - b, \text{ for } a \ge 1, \ b \ge 0. \tag{12}$$

De Bruijin [3] proved that

$$\Psi(x, y) = x\rho(u)\left[1 + O\left(\frac{\log(u+1)}{\log y}\right)\right] \tag{13}$$

holds uniformly for $u = \log x/\log y$ in the range

$$y \ge 2, \ 1 \le u \le (\log y)^{3/5-\epsilon}. \tag{14}$$

This has since been improved by Hildebrand [6] to the range

$$y \ge 2, \ 1 \le u \le \exp\left((\log y)^{3/5-\epsilon}\right). \tag{15}$$

For the next theorem, we are interested in the range $y = x^{\alpha/k}$, and hence $u = k/\alpha$.

Theorem 4 Let $0 < \alpha < 1$, and let $k \in \mathbb{N}$ be fixed constants. Then

$$b_k(x, x^\alpha) \sim \Psi(x, x^{\alpha/k}) \sim \rho\left(\frac{k}{\alpha}\right)x.$$

In fact, we have

$$b_k(x, x^\alpha) = \rho\left(\frac{k}{\alpha}\right)x + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Proof. Note that for $x \geq 2^{k/\alpha}$, we are within the range (15). Consequently (13) applies:

$$\Psi(x, x^{\alpha/k}) = x\rho\left(\frac{k}{\alpha}\right)\left[1 + O\left(\frac{1}{\log x}\right)\right].$$

Furthermore, the second inequality of Theorem 3 gives us $b_k(x, x^\alpha) \leq \Psi(x, x^{\alpha/k})$. Similarly, the first inequality from Theorem 3 yields:

$$\Psi\left(x, \left(\frac{x^\alpha}{2\log\log x}\right)^{1/k}\right) - \frac{c_1 x}{\sqrt{\log\log x}} \leq b_k(x, x^\alpha),$$

since $\Psi(x, y)$ is increasing in y.

For x sufficiently large, (13) gives

$$\Psi\left(x, \left(\frac{x^\alpha}{2\log\log x}\right)^{1/k}\right) = x\rho\left(\frac{\log x}{(\alpha/k)\log x - (1/k)\log(2\log\log x)}\right)\left[1 + O\left(\frac{\log(u+1)}{\log y}\right)\right],$$

where, in this instance,

$$y = \left(\frac{x^\alpha}{2\log\log x}\right)^{1/k}$$

and

$$\begin{aligned} u = \frac{\log x}{\log y} &= \frac{\log x}{(\alpha/k)\log x - (1/k)\log(2\log\log x)} \\ &= \left(\frac{k}{\alpha}\right)\frac{1}{1 - (1/\alpha)\log(2\log\log x)/\log x}. \end{aligned}$$

The identity $\frac{1}{1-a} < 1 + 2a$ holds for $0 < a < 1/2$. So if x is chosen sufficiently large such that $0 < \dfrac{\log(2\log\log x)}{\alpha\log x} < \dfrac{1}{2}$, then $u < \dfrac{k}{\alpha} + \dfrac{2k\log(2\log\log x)}{\alpha^2\log x}$. Hence, since $\rho(u)$ decreases on $(1, \infty)$, we have

$$\begin{aligned} \Psi\left(x, \left(\frac{x^\alpha}{2\log\log x}\right)^{1/k}\right) &\geq x\rho\left(\frac{k}{\alpha} + \frac{2k\log(2\log\log x)}{\alpha^2\log x}\right)\left[1 + O\left(\frac{1}{\log x}\right)\right] \\ &= x\rho\left(\frac{k}{\alpha} + \frac{2k\log(2\log\log x)}{\alpha^2\log x}\right) + O\left(\frac{x}{\log x}\right) \\ &\geq x\rho\left(\frac{k}{\alpha}\right) - \frac{2kx\log(2\log\log x)}{\alpha^2\log x} + O\left(\frac{x}{\log x}\right) \\ &= x\rho\left(\frac{k}{\alpha}\right) + O\left(\frac{x\log\log\log x}{\log x}\right), \end{aligned}$$

using equation (12). Combining this information with Theorem 3 we obtain the following inequalities:

$$x\rho\left(\frac{k}{\alpha}\right)+O\left(\frac{x}{\sqrt{\log\log x}}\right)\leq\Psi\left(x,\left(\frac{x^{\alpha}}{2\log\log x}\right)^{1/k}\right)\leq b_k(x,x^{\alpha})$$

$$\leq\Psi(x,x^{\alpha/k})=x\rho\left(\frac{k}{\alpha}\right)+O\left(\frac{x}{\log x}\right),$$

which proves the theorem. □

3. Average Order of $s_{k,\ell}$

We now devote our attention to equation (1). We have already proved the $k=1$ case, so henceforth we shall assume that $k\geq 2$.

Theorem 5 The average order of $s_{k,\ell}(n)$ has the following expression:

$$\sum_{n\leq x}s_{k,\ell}(n)=\sum_{r=1}^{k}\sum_{p_1<\ldots<p_r}\sum_{\substack{i_1+\cdots+i_r=k\\ i_1,\ldots,i_r>0}}(p_1^{i_1}\cdots p_r^{i_r})^{\ell}\times$$

$$\left(\sum_{j_1,\ldots,j_r=1}^{\infty}\binom{j_1-1}{i_1-1}\cdots\binom{j_r-1}{i_r-1}\left\lfloor\frac{x}{p_1^{j_1}\cdots p_r^{j_r}}\right\rfloor\right).$$

Proof. The terms in the sum $\sum_{n\leq x}s_{k,\ell}(n)$ are products of k ℓ-th powers of primes, not necessarily distinct. In other words, an arbitrary term is of the form $(p_1^{i_1}\cdots p_r^{i_r})^{\ell}$, where $r\leq k$, $p_1<\ldots<p_r$, $i_1,\ldots,i_r>0$, and $i_1+\cdots+i_r=k$. Fix $(p_1^{i_1}\cdots p_r^{i_r})^{\ell}$. We shall count the number of times this expression occurs in the sum.

By the inclusion-exclusion principle, the number of $n\leq x$ such that $p_1^{j_1},\ldots,p_r^{j_r}||n$ is

$$\left\lfloor\frac{x}{p_1^{j_1}\cdots p_r^{j_r}}\right\rfloor-\left(\left\lfloor\frac{x}{p_1^{j_1+1}p_2^{j_2}\cdots p_r^{j_r}}\right\rfloor+\left\lfloor\frac{x}{p_1^{j_1}p_2^{j_2+1}\cdots p_r^{j_r}}\right\rfloor+\cdots+\left\lfloor\frac{x}{p_1^{j_1}p_2^{j_2}\cdots p_r^{j_r+1}}\right\rfloor\right)$$

$$+\cdots+(-1)^r\left\lfloor\frac{x}{p_1^{j_1+1}\cdots p_r^{j_r+1}}\right\rfloor,$$

which we write as $\beta(j_1,\ldots,j_r)$. Each such n contributes $\binom{j_1}{i_1}\cdots\binom{j_r}{i_r}$ copies of $(p_1^{i_1}\cdots p_r^{i_r})^{\ell}$ to the sum $\sum_{n\leq x}s_k(n)$. Thus $(p_1^{i_1}\cdots p_r^{i_r})^{\ell}$ occurs

$$\sum_{j_1,\ldots,j_r=1}^{\infty}\binom{j_1}{i_1}\cdots\binom{j_r}{i_r}\beta(j_1,\ldots,j_r)$$

times. We make the following claim:

$$\sum_{j_1,\dots,j_r=1}^{\infty} \binom{j_1}{i_1}\cdots\binom{j_r}{i_r}\beta(j_1,\dots,j_r) = \sum_{j_1,\dots,j_r=1}^{\infty} \binom{j_1-1}{i_1-1}\cdots\binom{j_r-1}{i_r-1}\left\lfloor \frac{x}{p_1^{j_1}\cdots p_r^{j_r}}\right\rfloor. \tag{16}$$

To prove (16), first note that $\left\lfloor \frac{x}{p_1^{j_1}\cdots p_r^{j_r}}\right\rfloor$ occurs

$$\binom{j_1}{i_1}\cdots\binom{j_r}{i_r} - \left(\binom{j_1-1}{i_1}\binom{j_2}{i_2}\cdots\binom{j_r}{i_r} + \cdots + \binom{j_1}{i_1}\binom{j_2}{i_2}\cdots\binom{j_r-1}{i_r}\right) + \cdots + (-1)^r\binom{j_1-1}{i_1}\binom{j_2-1}{i_2}\cdots\binom{j_r-1}{i_r}$$

times in the left hand side. But an induction on r, with the identity

$$\binom{j}{i} - \binom{j-1}{i} = \binom{j-1}{i-1} \tag{17}$$

being the $r = 1$ case, gives us

$$\binom{j_1}{i_1}\cdots\binom{j_r}{i_r} - \left(\binom{j_1-1}{i_1}\binom{j_2}{i_2}\cdots\binom{j_r}{i_r} + \cdots + \binom{j_1}{i_1}\binom{j_2}{i_2}\cdots\binom{j_r-1}{i_r}\right) + \cdots + (-1)^r\binom{j_1-1}{i_1}\binom{j_2-1}{i_2}\cdots\binom{j_r-1}{i_r} = \binom{j_1-1}{i_1-1}\cdots\binom{j_r-1}{i_r-1}. \tag{18}$$

Indeed, suppose that (18) holds for $r-1$. Denote the left-hand side of (18) by C_r. We need to show that $C_r = \binom{j_1-1}{i_1-1}\cdots\binom{j_r-1}{i_r-1}$. But factoring, the identity (17), and the induction hypothesis give us that

$$C_r = \binom{j_r}{i_r}C_{r-1} - \binom{j_r-1}{i_r}C_{r-1} = C_{r-1}\binom{j_r-1}{i_r-1} = \binom{j_1-1}{i_1-1}\cdots\binom{j_{r-1}-1}{i_{r-1}-1}\binom{j_r-1}{i_r-1}.$$

Thus, the claim (16) is proved. The theorem follows by summing over all values of r from 1 to k, all possible r-tuples of primes, and for each such r-tuple, all r-tuples $(i_1,\dots,i_r)$ satisfying $i_j > 0$ for $j = 1,\dots,r$, and $i_1 + \cdots + i_r = k$. □

We will first investigate the part of the sum in Theorem 5 corresponding to $r = k$, and hence $i_1 = \ldots = i_k = 1$. To do so, we require some generalizations of the prime number theorem. These are taken from Nathanson [12], pp.313-319, however we also include precise error terms.

Let $\pi_k(x) = \#\{n \le x : \Omega(n) = \omega(n) = k\}$ and $\pi_k^*(x) = \#\{n \le x : \Omega(n) = k\}$. That is, $\pi_k(x)$ counts the number of $n \le x$ that are products of exactly k distinct prime factors, and $\pi_k^*(x)$ counts the number of $n \le x$ which have k prime factors with repetition. Note that $\pi_1(x) = \pi_1^*(x) = \pi(x)$. For $k \ge 2$, we have that

$$\pi_k(x) = \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} + O\left(\frac{x(\log\log x)^{k-2}}{\log x}\right), \tag{19}$$

and

$$0 \le \pi_k^*(x) - \pi_k(x) \ll \frac{x(\log\log x)^{k-2}}{\log x}. \tag{20}$$

The first result can be used to prove the next lemma via Riemann-Stieltjes integration.

Lemma 6 Let $u \ge 0$, and $k \ge 2$. Then

$$\sum_{\substack{n\le x\\ \omega(n)=\Omega(n)=k}} n^u = \frac{x^{u+1}(\log\log x)^{k-1}}{(u+1)(k-1)!\log x} + O\left(\frac{x^{u+1}(\log\log x)^{k-2}}{\log x}\right).$$

For $r = k$, we have the following:

$$\sum_{p_1<\ldots<p_k} (p_1\cdots p_k)^\ell \sum_{j_1,\ldots,j_k=1}^{\infty} \left\lfloor \frac{x}{p_1^{j_1}\cdots p_k^{j_k}} \right\rfloor = \sum_{p_1<\ldots<p_k} (p_1\cdots p_k)^\ell \left\lfloor \frac{x}{p_1\cdots p_k} \right\rfloor + \sum_{p_1<\ldots<p_k} (p_1\cdots p_k)^\ell \sum_{\substack{j_1,\ldots,j_k=1\\ j_1\cdots j_k>1}}^{\infty} \left\lfloor \frac{x}{p_1^{j_1}\cdots p_k^{j_k}} \right\rfloor. \tag{21}$$

We further focus by looking at the first term on the right-hand side of (21), that is, the term corresponding to $j_1 = \ldots = j_k = 1$. Making use of Lemma 6, we have

$$\begin{aligned}\sum_{p_1<\ldots<p_k} (p_1\cdots p_k)^\ell \left\lfloor \frac{x}{p_1\cdots p_k} \right\rfloor &= \sum_{m\le x/2^k} \sum_{\substack{p_1<\ldots<p_k\\ \frac{x}{m+1}<p_1\cdots p_k\le\frac{x}{m}}} m(p_1\cdots p_k)^\ell \\ &= \sum_{m\le x/2^k} \sum_{\substack{p_1<\ldots<p_k\\ p_1\cdots p_k\le x/m}} (p_1\cdots p_k)^\ell \\ &= \frac{x^{\ell+1}}{(\ell+1)(k-1)!} \sum_{m\le x/2^k} \frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)} \\ &\quad + O\left(x^{\ell+1} \sum_{m\le x/2^k} \frac{(\log\log(x/m))^{k-2}}{m^{\ell+1}\log(x/m)}\right).\end{aligned} \tag{22}$$

Now

$$\sum_{m\le x/2^k} \frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)} = \sum_{m\le\log^2 x} \frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)} + \sum_{\log^2 x<m\le x/2^k} \frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)}. \tag{23}$$

For $m \in [1, \log^2 x]$,

$$\frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)} \le \frac{(\log\log x)^{k-1}}{m^{\ell+1}(\log x - 2\log\log x)} = \frac{(\log\log x)^{k-1}}{m^{\ell+1}\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right),$$

which implies that

$$\begin{aligned}\sum_{m\le\log^2 x}\frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)} \le&\frac{(\log\log x)^{k-1}}{\log x}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right)\sum_{m\le\log^2 x}\frac{1}{m^{\ell+1}}\\ =&\frac{(\log\log x)^{k-1}}{\log x}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right)\\ &\times\zeta(\ell+1)\left(1+O\left(\frac{1}{\log^{2\ell}x}\right)\right)\\ =&\frac{\zeta(\ell+1)(\log\log x)^{k-1}}{\log x}+O\left(\frac{(\log\log x)^{k}}{\log^2 x}\right).\end{aligned}\tag{24}$$

On the other hand, for $m\in[1,\log^2 x]$ we have

$$\begin{aligned}\frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)}\ge\frac{(\log(\log x-2\log\log x))^{k-1}}{m^{\ell+1}\log x}&=\frac{\left(\log\log x+\log\left(1-\frac{2\log\log x}{\log x}\right)\right)^{k-1}}{m^{\ell+1}\log x}\\ &=\frac{\left(\log\log x+O\left(\frac{\log\log x}{\log x}\right)\right)^{k-1}}{m^{\ell+1}\log x}\\ &=\frac{(\log\log x)^{k-1}+O\left(\frac{(\log\log x)^{k-1}}{\log x}\right)}{m^{\ell+1}\log x},\end{aligned}$$

and so

$$\begin{aligned}\sum_{m\le\log^2 x}\frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)}&\ge\frac{(\log\log x)^{k-1}}{\log x}\left(\zeta(\ell+1)+O\left(\frac{1}{\log^{2\ell}x}\right)\right)+O\left(\frac{(\log\log x)^{k-1}}{\log^2 x}\right)\\ &=\frac{\zeta(\ell+1)(\log\log x)^{k-1}}{\log x}+O\left(\frac{(\log\log x)^{k-1}}{\log^2 x}\right).\end{aligned}\tag{25}$$

Combining (24) and (25) we have that

$$\sum_{m\le\log^2 x}\frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)}=\frac{\zeta(\ell+1)(\log\log x)^{k-1}}{\log x}+O\left(\frac{(\log\log x)^{k}}{\log^2 x}\right).\tag{26}$$

Now we must bound the second term on the right-hand side of (23).

$$\sum_{\log^2 x<m\le x/2^k}\frac{(\log\log(x/m))^{k-1}}{m^{\ell+1}\log(x/m)}\ll\sum_{\log^2 x<m\le x/2^k}\frac{(\log\log x)^{k-1}}{m^{\ell+1}}\ll\frac{(\log\log x)^{k-1}}{\log^2 x}.$$

A similar argument shows that

$$\sum_{m\le x/2^k}\frac{(\log\log(x/m))^{k-2}}{m^{\ell+1}\log(x/m)}\ll\frac{(\log\log x)^{k-2}}{\log x}.\tag{27}$$

This we use to bound the error term in (22).

Applying this information to (22) we have that

$$\begin{aligned}\sum_{p_1<\ldots<p_k}(p_1\cdots p_k)^\ell\left\lfloor\frac{x}{p_1\cdots p_k}\right\rfloor =&\frac{x^{\ell+1}}{(\ell+1)(k-1)!}\times\\ &\left(\frac{\zeta(\ell+1)(\log\log x)^{k-1}}{\log x}+O\left(\frac{(\log\log x)^k}{\log^2 x}\right)\right)\\ &+O\left(\frac{x^{\ell+1}(\log\log x)^{k-2}}{\log x}\right)\\ =&\frac{\zeta(\ell+1)x^{\ell+1}(\log\log x)^{k-1}}{(\ell+1)(k-1)!\log x}+O\left(\frac{x^{\ell+1}(\log\log x)^{k-2}}{\log x}\right). \qquad (28)\end{aligned}$$

This is the main term in $\sum_{n\le x} s_{k,\ell}(n)$. To complete the computation of the sum, we need only bound all that remains. We will first complete the case when $r=k$, by bounding the second term in the right-hand side of (21):

$$\begin{aligned}&\sum_{p_1<\ldots<p_k}(p_1\cdots p_k)^\ell\sum_{\substack{j_1,\ldots,j_k=1\\ j_1\cdots j_k>1}}^{\infty}\left\lfloor\frac{x}{p_1^{j_1}\cdots p_k^{j_k}}\right\rfloor\\ &\ll x\sum_{\substack{p_1<\ldots<p_k\\ p_1^2p_2\cdots p_k\le x}}(p_1\cdots p_k)^\ell\sum_{\substack{j_1,\ldots,j_k=1\\ j_1\cdots j_k>1}}^{\infty}\frac{1}{p_1^{j_1}\cdots p_k^{j_k}}\\ &\ll x\sum_{\substack{p_1<\ldots<p_k\\ p_1^2p_2\cdots p_k\le x}}(p_1\cdots p_k)^\ell\left(\frac{1}{(p_1-1)\cdots(p_k-1)}-\frac{1}{p_1\cdots p_k}\right)\\ &\ll x\sum_{\substack{p_1<\ldots<p_k\\ p_1^2p_2\cdots p_k\le x}}\frac{(p_1\cdots p_k)^\ell}{p_1^2p_2\cdots p_k}\\ &\ll x\sum_{p\le x^{\frac{1}{k+1}}}\left(p^{\ell-2}\sum_{\substack{n\le x/p\\ \omega(n)=\Omega(n)=k-1}}n^{\ell-1}\right)\\ &\ll x\sum_{p\le x^{\frac{1}{k+1}}}\left(p^{\ell-2}\frac{(x/p)^\ell(\log\log(x/p))^{k-2}}{\log(x/p)}\right)\\ &=x^{\ell+1}\sum_{p\le x^{\frac{1}{k+1}}}\frac{(\log\log(x/p))^{k-2}}{p^2\log(x/p)}\\ &\ll\frac{x^{\ell+1}(\log\log x)^{k-2}}{\log x}. \qquad (29)\end{aligned}$$

Let us now bound the terms of Theorem 5 corresponding to $r < k$. We require the following power series identity:

$$\sum_{n=j}^{\infty} \binom{n-1}{j-1} x^n = \left(\frac{x}{1-x}\right)^j,$$

which holds for $|x| < 1$. We have

$$\begin{aligned}
&\sum_{r=1}^{k-1} \sum_{p_1<\ldots<p_r} \sum_{\substack{i_1+\cdots+i_r=k \\ i_1,\ldots,i_r>0}} (p_1^{i_1}\cdots p_r^{i_r})^\ell \left(\sum_{j_1,\ldots,j_r=1}^{\infty} \binom{j_1-1}{i_1-1} \cdots \binom{j_r-1}{i_r-1} \left\lfloor \frac{x}{p_1^{j_1}\cdots p_r^{j_r}} \right\rfloor \right) \\
&\qquad \ll x \sum_{r=1}^{k-1} \sum_{\substack{i_1+\cdots+i_r=k \\ i_1,\ldots,i_r>0}} \sum_{\substack{p_1<\ldots<p_r \\ p_1^{i_1}\cdots p_r^{i_r}\le x}} (p_1^{i_1}\cdots p_r^{i_r})^\ell \prod_{m=1}^{r} \sum_{j_m=i_m}^{\infty} \binom{j_m-1}{i_m-1} \frac{1}{p^{j_m}} \\
&\qquad = x \sum_{r=1}^{k-1} \sum_{\substack{i_1+\cdots+i_r=k \\ i_1,\ldots,i_r>0}} \sum_{\substack{p_1<\ldots<p_r \\ p_1^{i_1}\cdots p_r^{i_r}\le x}} (p_1^{i_1}\cdots p_r^{i_r})^\ell \prod_{m=1}^{r} \left(\frac{1}{p_m-1}\right)^{i_m} \\
&\qquad \ll x \sum_{r=1}^{k-1} \sum_{\substack{i_1+\cdots+i_r=k \\ i_1,\ldots,i_r>0}} \sum_{\substack{p_1<\ldots<p_r \\ p_1^{i_1}\cdots p_r^{i_r}\le x}} (p_1^{i_1}\cdots p_r^{i_r})^{\ell-1} \\
&\qquad = x \sum_{\substack{n\le x \\ \omega(n)<k=\Omega(n)}} n^{\ell-1} \\
&\qquad = x \int_{2^-}^{x} t^{\ell-1}\, d(\pi_k^*(t)-\pi_k(t)). \qquad (30)
\end{aligned}$$

Applying integration by parts to (30), and using the bound (20), we have that

$$\begin{aligned}
&\sum_{r=1}^{k-1} \sum_{p_1<\ldots<p_r} \sum_{\substack{i_1+\cdots+i_r=k \\ i_1,\ldots,i_r>0}} (p_1^{i_1}\cdots p_r^{i_r})^\ell \times \left(\sum_{j_1,\ldots,j_r=1}^{\infty} \binom{j_1-1}{i_1-1} \cdots \binom{j_r-1}{i_r-1} \left\lfloor \frac{x}{p_1^{j_1}\cdots p_r^{j_r}} \right\rfloor \right) \\
&\qquad \ll \frac{x^{\ell+1}(\log\log x)^{k-2}}{\log x}. \qquad (31)
\end{aligned}$$

Combining Theorem 5 with (28), (29), and (31), we have proved the following theorem:

Theorem 7 Let $k \ge 2$, and let $\ell \ge 1$. Then

$$\sum_{n\le x} s_{k,\ell}(n) = \frac{\zeta(\ell+1)x^{\ell+1}(\log\log x)^{k-1}}{(\ell+1)(k-1)!\log x} + O\left(\frac{x^{\ell+1}(\log\log x)^{k-2}}{\log x}\right).$$

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