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ON THE AVERAGED DYNAMICS OF THE RANDOM FIELD CURIE–WEISS MODEL

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We describe the averaged over the disordered dynamics for the random field Curie–Weiss model. We consider both the magnetization and the full spin dynamics. Our approach is based on spectral asymptotics and includes results on the random fluctuations of eigenvalues and eigenvectors.

1. Introduction. In spite of its lack of physical significance, the Curie-Weiss model [9], is considered a useful toy model for testing ideas in statistical mechanics. In particular the rigorous formulation of the notion of "metastability," the so-called pathwise approach, was first introduced to describe the time evolution of the magnetization of the Curie–Weiss model in [8]. Metastability was later proved for various systems in statistical mechanics and it became one of the most powerful tools to describe the evolution toward equilibrium of a Markov process.

An important field of interest in physics during the last three decades is the study of disordered systems. As a first step in understanding the phenomena that could occur, mean field models are very important. There are many models for disordered mean field. The simplest one, from the static point of view, is the random field Curie–Weiss model [1, 2, 3, 5, 14, 15, 19, 20]. In some sense it is the less disordered. One of the most complicated, and less understood from a rigorous point of view is the Sherrington and Kirkpatrick (SK) model for spin glass [21]. There is another model called the REM [6, 7] which is in some sense the most disordered mean field model and is considered, however, as the simplest spin glass mean field model. It is solvable and its static properties are well known [9, 13, 17] from a rigorous point of view. We refer to [4] for a general overview of the field. However the name "spin glass" comes from dynamical properties of alloys that present very long relaxation times which have some analogies with dymamics of glasses. The presence of many "metastable" states is frequently invoked as being responsible for this behavior. Therefore it seems important to clarify all the notions of metastable behaviors and convergence to equilibrium in a simple disordered mean field system.

From a rigorous point of view, very little is known on the long-time behavior of disordered mean field models. Even for the simplest model of spin glasses, the REM, the dynamical results are rather surprising; see [11]. They are very nice examples of finite Markov chains in random environment. The infinite temperature case is the usual homogeneous random walk on the hypercube $\{-1, +1\}^N$, which has been studied a lot and is a toy model for a finite

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Markov chain (see [18]). In our case the corresponding nondisordered model is the usual Curie–Weiss model. (See [10] for the statics and [8] for the dynamics.)

Let us recall some facts about the static of the random field Curie–Weiss model. An important static property of the random field Curie–Weiss model, say, at low temperature with a random magnetic field with zero mean and small variance, is that, typically with respect to the Gibbs measure, asymptotically when the volume goes to infinity, the magnetization can take two different values that do not depend on the sample of the random field. This follows from the fact that the associated canonical free energy, which is nothing but the large deviation functional for the empirical magnetization, has two absolute quadratic minima that do not depend on the chosen sample of the random magnetic field. The two corresponding minimizers, m_1 and m_2, depend only on the temperature and the variance of the magnetic fields. In short, we say in this case that these values are "disorder independent." The fact that the canonical free energy has two quadratic minima suggests that the measure induced on the magnetization by the Gibbs measure could be asymptotically the convex mean of two Dirac measures concentrated on these two values m_1 and m_2. To study the relative weight of this convex mean, corrections to the exponential behavior have to be found. These corrections appear to be *dependent* on the disorder and therefore the relative weights in the previous convex mean depend on the disorder. In fact the relative weights converge, in law with respect to the disorder, to a dichotomous random variable that takes values 0 or 1 with probability 1/2 ([3]; see also [2]). The almost sure behavior with respect to the disorder is rather different: since we have a genuine convergence in law, we get that all possible weights in the previous convex mean can be reached asymptotically by well-chosen subsequences of volumes. That is, the cluster set of possible weights is just [0, 1] almost surely. However, it is proved in [15] that the difference between the finite volume Gibbs measure and a random convex mean of two approximate Dirac measures converges almost surely to zero. This means that we were able to find a very good almost sure approximation of the finite volume Gibbs measure, even if it does not converge almost surely. All that is related to chaotic dependence in the volume of a disordered system, an important fact pointed out by Newman and Stein in [4]. Other important static properties of this model related to this chaotic size dependence are studied by Külske [14].

Now let us recall some facts about the dynamics of the random field Curie–Weiss model. The first work was done for short time by Dai Pra and Den Hollander [5]: they fixed the time and considered the infinite volume limit. The long-time behavior, that is, when the time is allowed to diverge with the volume, was considered in [15] and the present work completes that study.

Note first that there are at least two dynamics to consider. The first one is the single spin flip dynamics and the second one is the dynamics induced on empirical magnetization by spin flip dynamics. This latter dynamics is natural when the mean field models considered can be expressed as a function of the sample magnetization. For the usual Curie–Weiss model, it was this latter dynamics that was considered by [8]. In the random field Curie–Weiss model we are considering, the disorder comes from a magnetic field which is ± 1; the induced dynamics is Markovian on a bidimensional magnetization. The first component is the average of the spins on the disorder-dependent subset of the space where the magnetic field is positive and the second component is where the magnetic field is negative. To simplify, we will continue to call this bidimensional quantity "magnetization."

The second point is related to the fact that we consider disordered systems; therefore, all the quantities are random variables defined on the same probability space as the random magnetic field. In particular, the semigroup describing our dynamics is a semigroup-valued random variable. Since we are interested in an asymptotic behavior when the volume and the time diverge, the probabilistic sense of the limiting behaviors of these random variables has to be precised. It is described in the physics literature in two different ways. The terms used come from metallurgy: "quenched" and "annealed." The first one, quenched, is when the sample of the disorder is fixed and the study is done for almost all samples or in probability with respect to the sample. The second one, annealed or averaged, is, roughly speaking, when an average over the disorder is done somewhere. Since there are usually various places where this average can be done, it could be equivalent or not to the quenched description.

We consider first the dynamics induced on the magnetization. We will call "process" the stochastic process defining our dynamics, which is considered for a given realization of the random magnetic field. Therefore, the law of this process is a disorder-dependent random variable. Note that we are considering merely a family of processes indexed by the volume N but we continue calling this family the "process."

The first problem is to find a rough asymptotic for the time scale to reach equilibrium. Since we are in a mean field situation with not too much disorder, and the system is in a low temperature regime, roughly speaking, we can expect that this time scale is of order $e^{\alpha\beta N}$ where N is the volume of our system, β the inverse temperature and the first problem is to find α. However, we have to give a precise sense to "to be of the order of"; that is, the limit $N \uparrow \infty$ has to be taken somewhere. Moreover, the probabilistic sense, with respect to the disorder, in which we have convergence has to be specified. At last, we can consider time scales that are dependent or not on the disorder. As we will see immediately, the results could be completely different.

In [15], Theorems 2.4, and 2.5 we have exhibited a disorder-independent quantity, denoted $\Delta\mathscr{F}$, such that on a time scale $t_N(\alpha) = \exp \alpha\beta N$, the process, starting not too far from one of the two possible magnetizations, say m_1, has not enough time to reach equilibrium if $\alpha < \Delta\mathscr{F}$. Here the equilibrium is the Gibbs measure in the volume N. In fact, the process is, roughly speaking, near m_1 most of the time. If $\alpha > \Delta\mathscr{F}$, we are already at equilibrium. This result holds for almost all realizations of the disorder. That is, we have found a time scale where the convergence to equilibrium is sharp. This is an example of an almost sure behavior on a time scale which is independent of the disorder. The fact that $\Delta\mathscr{F}$ is independent of the disorder comes from a strong law of large number for the canonical free energy.

At this point the natural problem that comes to mind is to understand what happens if we take $\alpha = \Delta\mathscr{F}$. Here, sample-to-sample fluctuations of the magnetic field start coming into play. We find (see Theorem 2.1) that the right order of magnitude of the time to reach equilibrium is up to polynomial terms, $\tilde{t}_N \approx \exp(\beta N \Delta\mathscr{F} - \beta a|S_N|)$ where a is an explicit constant depending only on the temperature and the variance of the magnetic field and S_N is the sum over the volume of the magnetic fields. The result holds for almost all realizations of the disorder. This is an almost sure result on a disorder-dependent time scale. To conclude questions about convergence to equilibrium, we consider the dynamics of the spins. In Theorem 2.1, we prove that the very same $\tilde{t}_N$ is the right order of magnitude for the time to reach equilibrium for the dynamics of the spins. This is a general fact that has not been pointed out before. There is an analog result in the standard Curie–Weiss model that can be easily proved by using our method.

One can expect to have metastable phenomena related to the question of convergence to equilibrium, since there we have two possible typical values for the magnetization. A metastable phenomenon is a dynamical effect, where a system stays a very long time in a given state which is not the equilibrium state; see [8]. There are various ways to study it. One is to consider the one-dimensional marginal of our two processes, one corresponding to the dynamics of the spins and one on the magnetization. The results for the magnetizations have been given in [15] (See Theorem 2.6 there). In this work we consider the dynamics of the spins. In Theorem 2.2, we show that on a well-chosen disorder-dependent time scale, the one-dimensional marginals of each of the two processes converge, almost surely with respect to the realizations of the magnetic fields, to the marginals of a Markov measure-valued jump process. This process stays a random time with an exponential distribution on a Bernoulli measure with a mean corresponding to the minima nearest from the starting point and then jumps to the equilibrium.

Another fact related to metastability is the exponentiality of the exit time of some domain for the stochastic process describing the dynamics. In [15], we get exponentiality of some exit times, for almost all realizations of the random magnetic fields by normalizing the exit time by a disorder-dependent quantity. However, choosing a disorder-independent time scale that normalizes the exit time by something which does not depend on the disorder but with the right order of magnitude, we do not get almost sure convergence with respect to the disorder nor do we get an exponential random variable. What we were able to do (see Theorem 2.3) is that averaging the law of the process with respect to the disorder, we find the correct normalization for the exit time to get convergence; the limiting distribution shifts from an exponential to a lognormal one. This is to our knowledge a new fact in the study of the dynamics of disordered systems. We call this average over the disorder of the law of the process "averaged dynamics."

With the concept of averaging the law of the process, we can come back to the problem of convergence to equilibrium for the averaged dynamics. Since we have already convergence to equilibrium on the disorder-dependent time scale $\tilde{t}_N$ given above, it is natural to consider the disorder-independent time scale $t_N(\tilde{\alpha}) = \exp(\beta N \Delta \mathscr{F} + \tilde{\alpha}\sqrt{N})$ and the average of the one-dimensional marginal of our process at that time. The result we get (see Theorem 2.4) for the dynamics of the spins and for the dynamics on the magnetization is, roughly speaking, the following: if $\tilde{\alpha}$ is positive, we get convergence, when $N \uparrow \infty$, to the limit of the average, with respect to the disorder, of the Gibbs measure which is a symmetric (1/2, 1/2) convex mean of two measures. It is not really a surprise; we could have expected such a result, at least when $\tilde{\alpha} = \infty$ from the result above with the time scale $t_N(\alpha) = \exp(\alpha\beta N)$ when $\alpha \geq \Delta\mathscr{F}$. What is really interesting is that if $\tilde{\alpha}$ is negative, we get convergence to a convex mean of the same two measures; however if we start, say, near m_1, the weight of the measure that charges m_1 is bigger than 1/2. The excess of mass on that weight is related to the limiting Gaussian asymptotic behavior of the magnetic field. That is, we have exhibited precisely a time scale where a dynamical fluctuation induced by the disorder appears. Namely, when $\tilde{\alpha} \geq 0$, we do not see this dynamical fluctuation; when $\tilde{\alpha} < 0$ we are able to see it. To our knowledge, this fact has not been mentioned before.

To understand these dynamical fluctuations and to be able to isolate them from the disorder-dependent fluctuations of the static mentioned at the beginning, we make a spectral decomposition of the semigroup as in [15]. However, a more precise analysis of the eigenvalues and eigenvectors is needed here.

The paper is organized as follows: in Section 2, we define our model and give the main results. In Section 3, we give results on the statics of the random field Curie–Weiss model. In Section 4, we give results on the spectral properties of the semigroup. In Section 5, we prove the main results.

2. The model and main results. Let $h = (h_i)_{i\in\mathbb{N}}$ be a sequence of independent symmetric Bernoulli random variables defined on some probability space, say $(\Omega, \mathscr{A}, Q)$. That is, $Q[h_i = 1] = Q[h_i = -1] = 1/2$, for any i. Let us denote $S_N = \sum_{i=1}^N h_i$. Let $\beta > 0$ be the inverse temperature and $\theta > 0$. Most of the quantities that we are going to define depend on the realization of h. Usually we shall drop this dependence in the computations. In the sequel, we denote by C a constant which depends on β and θ only. Its value may change from line to line. N_0 is an integer that depends also on β and θ only. Its value may change from line to line. In particular C and N_0 do not depend on h.

Let $\mathscr{S}_N = \{-1, +1\}^N$. Given $\sigma \in \mathscr{S}_N$ and h, let us define the random Hamiltonian,

$$H_N(\sigma) = H_N^h(\sigma) \equiv -\frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right)^2 - \theta\sum_{i=1}^N h_i\sigma_i. \tag{2.1}$$

We denote by $\mu_N = \mu_N^h$ the Gibbs measure on $\mathscr{S}_N$ defined by

$$\mu_N(\sigma) = \frac{\exp(-\beta H_N(\sigma))}{Z_N},$$

where

$$Z_N = Z_N^h \equiv \sum_{\sigma \in \mathscr{S}_N} e^{-\beta H_N(\sigma)} \tag{2.2}$$

is a normalizing constant.

For any $\sigma \in \mathscr{S}_N$, let $\bar{m}_N(\sigma) = (1/N)\sum_{i=1}^N \sigma_i$ be the empirical mean or magnetization. We also define

$$m_N^+(\sigma) = m_N^{h,+}(\sigma) \equiv \frac{1}{N} \sum_{i;\, h_i=+1} \sigma_i$$

and

$$m_N^-(\sigma) = m_N^{h,-}(\sigma) \equiv \frac{1}{N} \sum_{i;\, h_i=-1} \sigma_i$$

and $m_N(\sigma) = (m_N^+(\sigma), m_N^-(\sigma))$. Note that $\bar{m}_N(\sigma) = m_N^+(\sigma) + m_N^-(\sigma)$.

Here m_N clearly takes its values in $[-1,+1]^2$. We denote by $\mathscr{M}_N$ the image of $\mathscr{S}_N$ by m_N. Calling $N^+ = N^{h,+} \equiv \#\{i\colon h_i = +1\}$ and $N^- = N^{h,-} \equiv \#\{i\colon h_i = -1\}$, we have

$$\mathscr{M}_N = \left(-\frac{N^+}{N}, -\frac{N^+ + 2}{N}, \dots, \frac{N^+}{N}\right) \times \left(-\frac{N^-}{N}, -\frac{N^- + 2}{N}, \dots, \frac{N^-}{N}\right) \tag{2.3}$$

The point is that the Hamiltonian can be written in terms of m_N:

$$H_N(\sigma) = -N\big(\tfrac{1}{2}(m_N^+(\sigma) + m_N^-(\sigma))^2 + \theta(m_N^+(\sigma) - m_N^-(\sigma))\big). \tag{2.4}$$

With a little abuse of notation, we shall also denote by H_N the function defined on $\mathscr{M}_N$ by

$$H_N(m) = -N\big(\tfrac{1}{2}(m^+ + m^-)^2 + \theta(m^+ - m^-)\big). \tag{2.5}$$

Since we are also interested in the behavior of the magnetization under the law μ_N, let $\mathscr{G}_N \equiv \mathscr{G}_N^h$ be the image of μ_N by m_N. $\mathscr{G}_N$ is a probability measure on $\mathscr{M}_N$. We have, for $m = (m^+, m^-) \in \mathscr{M}_N$,

$$\mathscr{G}_N(m) = \frac{\exp(-\beta N \mathscr{F}_N(m))}{Z_N}, \tag{2.6}$$

where

$$Z_N \equiv Z_N^h = \sum_{m \in \mathscr{M}_N} e^{-\beta N \mathscr{F}_N(m)} \tag{2.7}$$

is a normalizing constant and

$$\mathscr{F}_N(m) \equiv \mathscr{F}_N^h(m) = -\frac{1}{2}(m^+ + m^-)^2 - \theta(m^+ - m^-) - \frac{1}{\beta N}\log\binom{N^+}{\frac{N^+}{2} + m^+\frac{N}{2}}\binom{N^-}{\frac{N^-}{2} + m^-\frac{N}{2}} \tag{2.8}$$

satisfies

$$e^{-\beta N \mathscr{F}_N(m)} = \sum_{\sigma;\, m_N(\sigma)=m} e^{-\beta H_N(\sigma)}. \tag{2.9}$$

As a consequence of the strong law of large numbers, $\mathscr{F}_N$ converges Q-almost surely, as $N \uparrow +\infty$, to the function

$$\mathscr{F}(m) = -\tfrac{1}{2}(m^+ + m^-)^2 - \theta(m^+ - m^-) + \tfrac{1}{2\beta}(I(2m^+) + I(2m^-)), \tag{2.10}$$

which is called the canonical free energy in statistical physics. Here, for $x \in [-1, +1]$, $I(x) = (1+x)/2 \log(1+x)/2 + (1-x)/2 \log(1-x)/2$, and for $|x| \geq 1$, $I(x) = 0$ is the entropy of Bernoulli random variables.

The function $\mathscr{F}$ is symmetric with respect to the diagonal. It has three critical points when $\beta > \cosh^2(\beta\theta)$. They can be found as follows. Let m_* be the unique positive solution of the equation

$$m_* = \tfrac{1}{2}[\tanh(\beta m_* + \beta\theta) + \tanh(\beta m_* - \beta\theta)]. \tag{2.11}$$

Then the critical points of $\mathscr{F}$ are

$$\begin{aligned} m_0 &= \big(\tfrac{1}{2}\tanh(\beta\theta), -\tfrac{1}{2}\tanh(\beta\theta)\big),\\ m_1 &= \big(\tfrac{1}{2}\tanh(\beta m_* + \beta\theta), \tfrac{1}{2}\tanh(\beta m_* - \beta\theta)\big),\\ m_2 &= \big(\tfrac{1}{2}\tanh(-\beta m_* + \beta\theta), -\tfrac{1}{2}\tanh(\beta m_* + \beta\theta)\big). \end{aligned} \tag{2.12}$$

It is not difficult to check that m_0 is a saddle point and m_1 and m_2 are two minima. We define the activation energy as $\Delta\mathscr{F} = \mathscr{F}(m_0) - \mathscr{F}(m_1) = \mathscr{F}(m_0) - \mathscr{F}(m_2)$.

Let us define

$$\begin{aligned} T_1^N &= \mathscr{M}_N \cap \{m^+ + m^- > 0\},\\ \overline{T}_1^N &= \mathscr{M}_N \cap \left\{m^+ + m^- \geq -\frac{3}{N}\right\},\\ \partial T_1^N &= \mathscr{M}_N \cap \left\{0 \geq m^+ + m^- \geq -\frac{3}{N}\right\}. \end{aligned}$$

These are discrete approximations of the basin of attraction of m_1, its closure and its boundary. T_2^N and $\overline{T}_2^N$ are defined analogously. Also let

$$T_0^N = \mathscr{M}_N \cap \{m^+ + m^- = 0\}.$$

We consider two dynamics, the first one on the spins σ, the second one on the magnetization m_N. Given $N \in \mathbb{N}$ and $1 \leq i \leq N$, let T^i be the map from $\mathscr{S}_N$ to $\mathscr{S}_N$ defined by $T^i(\sigma)_j = \sigma_j$ for $j \neq i$, $T^i(\sigma)_i = -\sigma_i$. Consider the following operator acting on real-valued functions ϕ on $\mathscr{S}_N$,

$$
\begin{aligned}
L_N \phi(\sigma) &= L_N^h \phi(\sigma) \\
&\equiv \frac{1}{N} \sum_{i=1}^{N} \big(\phi(T^i(\sigma)) - \phi(\sigma)\big)\, e^{-(\beta/2)[H_N(T^i(\sigma)) - H_N(\sigma)]}.
\end{aligned} \tag{2.13}
$$

L_N is the infinitesimal generator of a continuous time Markov process which we denote by $\sigma_{(N)}(t) = \sigma^h_{(N)}(t)$. We denote by P_σ the law of the Markov process $\sigma_{(N)}(.)$ when $\sigma_{(N)}(0) = \sigma$ and by E_σ, the expectation with respect to P_σ. It is easy to check that μ_N is the unique invariant probability measure for σ_N. It is also reversible; that is, L_N is symmetric in $L^2(\mathscr{S}_N, \mu_N)$.

Now let $m_N(t) \equiv m_N(\sigma_{(N)}(t))$, be the induced dynamics on the magnetization. It turns out that m_N is also a Markov process with invariant probability measure $\mathscr{G}_N$. Let $\mathscr{L}_N \equiv \mathscr{L}_N^h$ be its infinitesimal generator. According to formula (2.24) of [15] we have

$$
\mathscr{L}_N \phi(m) = \sum_{\substack{\tilde m \in \mathscr{M}_N \\ \tilde m \sim m}} [\phi(\tilde m) - \phi(m)] \mathscr{N}_N(\tilde m, m) e^{-(\beta/2)[H_N(\tilde m) - H_N(m)]}, \tag{2.14}
$$

where $\tilde m \sim m$ means that $\tilde m$ and m are neighbors in $\mathscr{M}_N$ and $\mathscr{N}_N(\tilde m, m)$ is some correction factor which is between $2/N$ and 1. Call $P_t^N \equiv P_t^{h,N} = e^{t\mathscr{L}_N}$ the associated semigroup. We shall use the notation $P_m \equiv P_m^h$ to denote the law of the process m_N when $m_N(0) = m$ and E_m to denote the expectation w.r.t. P_m. We emphasize that when h is kept fixed, the measure P_m is Markovian.

The first result is a Q-almost sure precise asymptotic, on a disorder-dependent scale, of the time to reach equilibrium with a bound on the errors. It is stated for the two previous dynamics.

THEOREM 2.1. *There exist deterministic constants N_0, K and K' such that, for any $N \geq N_0$, for almost all realizations of h, for any function ϕ defined on $\mathscr{S}_N$ and bounded by* 1, *for any $\sigma \in \mathscr{S}_N$, if we let $t_N = N^K \exp(\beta N \Delta\mathscr{F} - \beta a |S_N|)$, where*

$$
a = a(\beta, \theta) = \frac{1}{2\beta} \log \frac{\cosh(\beta m_* + \beta\theta)}{\cosh(\beta m_* - \beta\theta)} \tag{2.15}
$$

and m_* is the unique positive solution of (2.11) *then, on the set $|S_N| \leq 2 \times \sqrt{N \log N}$, we have*

$$
|E_\sigma[\phi(\sigma_{(N)}(t_N))] - P_N(\phi)| \leq \exp(-N^{K'}). \tag{2.16}
$$

In particular, for any function ψ defined on $\mathscr{M}_N$ and bounded by 1, *for any $m \in \mathscr{M}_N$,*

$$
\big|E_m[\psi(m_N(t_N))] - \mathscr{G}_N(\phi)\big| \leq \exp(-N^{K'}). \tag{2.17}
$$

The proof can be found in Section 5.

REMARK. This result is a sharper version of Theorem 2.5 in [15], where a similar result was proved for a disorder-independent time scale $t_N(\alpha) = e^{\alpha\beta N}$ with $\alpha > \Delta\mathcal{F}$ without an explicit bound on the errors.

It was also proved in Theorem 2.4 in [15] that for $\alpha < \Delta\mathcal{F}$,

$$\lim_{N\uparrow\infty} \left| E_m[\psi(m_N(t_N(\alpha)))] - \psi(m_1) \right| = 0 \tag{2.18}$$

Q-almost surely, if, roughly speaking, $\mathcal{F}(m) < \mathcal{F}(m_0)$ and $m \in T_1^N$.

From Theorem 2.1, we see that on subsequences N_n such that $\beta|S_{N_n}| \geq K \log N_n$, a time of order $\exp(\beta N \Delta\mathcal{F})$ is more than sufficient to reach equilibrium. That is, the disorder helps to reach equilibrium.

The second result is a Q-almost sure one. It describes the metastable behavior of the system. To state it we need some definitions. For a given realization of h, let μ_∞^1 (resp. μ_∞^2) be the probability measure on $\mathcal{S}_\infty$ under which the variables $(\sigma_i, i \in \mathbb{N})$ are independent Bernoulli random variables and σ_i has mean $\tanh(\beta m_* + \beta\theta h_i)$ [resp. $\tanh(-\beta m_* + \beta\theta h_i)$]. Note that μ_∞^1 and μ_∞^2 *depend* on the realization of h. Let

$$\alpha_N = \frac{e^{\beta a S_N}}{e^{\beta a S_N} + e^{-\beta a S_N}}$$

with $a = a(\beta, \theta)$ given by (2.15). The pair $(\alpha_N, 1 - \alpha_N)$ is a finite volume approximation of the weights in the convex decomposition of the measure $\mathcal{G}_N$ into two Dirac measures concentrated on m_1 and m_2. See [15], Lemma 4.1 and also Proposition 3.1 in the next Section. Let us denote by Λ_1^N the spectral gap of the infinitesimal generator $\mathcal{L}_N$, defined in (2.14), on $L_2(\mathcal{M}_N, \mathcal{G}_N)$. We prove in Lemma 4.5 that it coincides with the spectral gap of the infinitesimal generator of the dynamics of the spins, that is L_N, defined in (2.13), on $L_2(\mathcal{S}_N, \mu_N)$.

THEOREM 2.2. *For almost all realizations of h, for all sequences $\sigma^N \in \widetilde{T}_1^N$ with* $\limsup \mathcal{F}(m_N(\sigma^N)) < \mathcal{F}(m_0)$, *we have, for all $t > 0$, for any continuous function ϕ defined on $\mathcal{S}_\infty$,*

$$\begin{aligned} &E_{\sigma^N}\Big[\phi(\sigma_N(t/\Lambda_1^N))\Big] \\ &\quad - \big(e^{-t}\mu_\infty^1(\phi) + (1 - e^{-t})(\alpha_N \mu_\infty^1(\phi) + (1 - \alpha_N)\mu_\infty^2(\phi))\big) \to 0. \end{aligned} \tag{2.19}$$

The proof can be found in Section 5.

Now we consider the law of averaged dynamics, that is, the law of m_N averaged over the realizations of h. For a given $m = m_N \in \mathcal{M}_N$, let us define the measure $\mathbb{P}_m = P_m^h \times Q$. $\mathbb{E}_m$ is the expectation with respect to $\mathbb{P}_m$. By definition, if Φ is a measurable function on the paths space, $\mathbb{E}_m[\Phi] = \int P_m^h[\Phi] Q(dh)$. Under $\mathbb{P}_m$, m_N is not a Markov process anymore. In a similar way, for a given $\sigma = \sigma_{(N)} \in \mathcal{S}_N$ we define the measure $\mathbb{P}_\sigma = P_\sigma^h \times Q$ and $\mathbb{E}_\sigma$ is the expectation with respect to $\mathbb{P}_\sigma$.

To state the next result, let $\tau_N = \inf\{t > 0\colon\ m_N(t) \in \partial T_1^N\}$ be the hitting time of ∂T_1^N.

THEOREM 2.3. *Let $\mathscr{N}$ be a normalized Gaussian random variable. Let a be the constant defined in* (2.15). *For all sequences $m^N \in T_1^N$ with $\limsup \mathscr{F}(m^N) < \mathscr{F}(m_0)$, and for all $\alpha \in \mathbb{R}$, we have*

$$\mathbb{P}_{m^N}\big[N^{-1/2}(\log \tau_N - \beta N \Delta\mathscr{F}) \geq \alpha\big] \to P[\beta a \mathscr{N} \geq \alpha]. \tag{2.20}$$

The proof can be found in Section 5.

REMARK. Note that it is a result on a disorder-independent scale. We recall that $\Delta\mathscr{F}$ does not depend on the realizaton of h. Let us recall that in [15], we have proved that, Q-almost surely, $\Lambda_1^{N,K}\tau_N$ converges in law to an exponential random variable. Here $\Lambda_1^{N,K}$ is the first eigenvalue of minus the infinitesimal generator of the process m_N killed at time τ_N. Note that $\Lambda_1^{N,K}$ depends on the realization of h. The result of Theorem 2.3 is that for the averaged dynamics this hitting time converges to a log normal. This difference comes from the disorder-dependent fluctuations of $\Lambda_1^{N,K}$; it is a dynamical fluctuation induced by the disorder.

The last result is a long-time asymptotic for the averaged dynamics. We consider here a disorder-independent time scale.

THEOREM 2.4. *Let $\mathscr{N}$ be a normalized Gaussian random variable. Let a be the constant defined in* (2.15). *For all sequences $m^N \in T_1^N$ with $\limsup \mathscr{F}(m^N) < \mathscr{F}(m_0)$ and for all $\alpha \in \mathbb{R}$, for any continuous function ϕ on $[-1,+1]^2$, we have*

$$\begin{aligned}\mathbb{E}_{m^N}\Big[\phi\big(m_N(e^{\beta N\Delta\mathscr{F}+\alpha\sqrt{N}})\big)\Big] \to{}& \big(\tfrac{1}{2} + P[0 \geq \beta a\mathscr{N} \geq \alpha]\big)\phi(m_1)\\ &+ \big(\tfrac{1}{2} - P[0 \geq \beta a\mathscr{N} \geq \alpha]\big)\phi(m_2).\end{aligned} \tag{2.21}$$

Moreover, let $\mathscr{B}^1$ (resp. $\mathscr{B}^2$) denote the law of independent Bernoulli variables of mean m_* (resp. $-m_*$), the unique positive solution of (2.11). *Then, for all sequences $\sigma^N \in \widetilde{T}_1^N$ with $\limsup \mathscr{F}(m_N(\sigma^N)) < \mathscr{F}(m_0)$, for all $\alpha \in \mathbb{R}$, and for any continuous function ϕ on $\mathscr{S}_\infty$, we have*

$$\begin{aligned}\mathbb{E}_{\sigma^N}\Big[\phi\Big(\sigma_{(N)}\Big(e^{\beta N\Delta\mathscr{F}+\alpha\sqrt{N}}\Big)\Big)\Big] \to{}& \big(\tfrac{1}{2} + P[0 \geq \beta a\mathscr{N} \geq \alpha]\big)\mathscr{B}^1(\phi)\\ &+ \big(\tfrac{1}{2} - P[0 \geq \beta a\mathscr{N} \geq \alpha]\big)\mathscr{B}^2(\phi).\end{aligned} \tag{2.22}$$

REMARK. It is the disorder-dependent fluctuation of the spectral gap of $\mathscr{L}_N$ (resp. L_N) that is responsible for this behavior. Here also, it is a dynamical fluctuation induced by the disorder. Note that taking the average of the Gibbs measure with respect to Q, we get

$$\lim_{N\uparrow\infty} Q[\mathscr{G}_N(\phi)] = \tfrac{1}{2}\phi(m_1) + \tfrac{1}{2}\phi(m_2), \tag{2.23}$$

which is the same as the right-hand side of (2.21) when $\alpha > 0$. When $\alpha < 0$, with the extreme case $\alpha = -\infty$, the averaged process stays more near m_1, but may make the transition to m_2. As we have already said in the introduction this phenomena seems new, at least to us.

3. Static results. In this section we collect all the results for the static of the random field Curie–Weiss model that we need for proving the theorems. There are precise asymptotics on the behavior of the logarithm of partition functions restrained on various domains. This is done with an explicit bound on the errors. We also study some property of the landscape of the graph of $\mathscr{F}_N$, defined in (2.8).

Recalling (2.3), let

$$T_1^N = \mathscr{M}_N \cap \{m^+ + m^- > 0\},$$
$$\overline{T}_1^N = \mathscr{M}_N \cap \left\{m^+ + m^- \geq -\frac{3}{N}\right\},$$
$$\partial T_1^N = \mathscr{M}_N \cap \left\{0 \geq m^+ + m^- \geq -\frac{3}{N}\right\}$$

and define T_2^N and $\overline{T}_2^N$ analogously.

Let $\mathscr{G}_N^1 \equiv \mathscr{G}_N^{h,1}$ be the restriction of $\mathscr{G}_N$ to T_1^N, that is,

$$\mathscr{G}_N^1(m) = \frac{Z_N}{Z_N^1}\mathscr{G}_N(m)\,\mathbb{1}\left(m \in \overline{T}_1^N\right),$$
$$Z_N^1 = \sum_{m\in\overline{T}_1^N} e^{-\beta N\mathscr{F}_N(m)}. \tag{3.1}$$

Define $\mathscr{G}_N^2$ analogously and

$$Z_N^2 = \sum_{m\in\overline{T}_2^N} e^{-\beta N\mathscr{F}_N(m)}. \tag{3.2}$$

Define also

$$z_N^1 = \sum_{m\in\partial T_1^N} e^{-\beta N\mathscr{F}_N(m)}. \tag{3.3}$$

Clearly, as N tends to $+\infty$, under $\mathscr{G}_N$, the magnetization m_N gets close to one of the two values m_1 or m_2. The asymptotic support of the law of m_N is therefore deterministic. We have the following proposition.

PROPOSITION 3.1 (Static asymptotics). *Define*

$$a = a(\beta, \theta) = \frac{1}{2\beta}\log\frac{\cosh(\beta m_* + \beta\theta)}{\cosh(\beta m_* - \beta\theta)}. \tag{3.4}$$

Then, for $N \geq N_0$, *on the set* $|S_N| \leq 2\sqrt{N \log N}$, *we have*

$$\left|\log Z_N + \beta N \mathscr{F}(m_1) - \beta a |S_N|\right| \leq C \log N, \tag{3.5}$$

$$\left|\log Z_N^1 + \beta N \mathscr{F}(m_1) - \beta a S_N\right| \leq C \log N, \tag{3.6}$$

$$\left|\log z_N^1 + \beta N \mathscr{F}(m_0)\right| \leq C \log N \tag{3.7}$$

for some positive constant C.

Besides, for Q*-almost all realizations of* h, *for any continuous function* ϕ *defined on* $[-1, +1]^2$, *we have*

$$\mathscr{G}_N(\phi) - (\alpha_N \phi(m_1) + (1 - \alpha_N)\phi(m_2)) \to 0, \tag{3.8}$$

where

$$\alpha_N \equiv \alpha_N^h = \frac{e^{\beta a S_N}}{e^{\beta a S_N} + e^{-\beta a S_N}}. \tag{3.9}$$

Equation (3.8) is actually proved in [16], Lemma 4.1.

Note that $Q\big[|S_N| \geq 2\sqrt{N \log N}\big] \leq 2\exp(-2\log N)$. Therefore, the Borel–Cantelli lemma implies that the difference $\mathscr{G}_N - (\alpha_N \delta_{m_1} + (1-\alpha_N)\delta_{m_2})$ weakly converges to 0 for almost all realizations of h.

PROOF OF PROPOSITION 3.1. The proof is inspired by the arguments of [5]. Let us first note that, by symmetry and because the spins are exchangeable, we may assume without loss of generality that $S_N \geq 0$ and that $h_i = +1$, for $i = 1, \dots, (N+S_N)/2$ and $h_i = -1$ for $i = (N+S_N)/2+1, \dots, N$. Let $M = [N/2]$, where $[\cdot]$ denotes the integer part. We introduce a different parametrization of the magnetization: for a configuration $\sigma \in \mathscr{S}_N$, define

$$\tilde{m}_N^+(\sigma) = \frac{1}{N} \sum_{i=1}^{M} \sigma_i,$$

$$\tilde{m}_N^-(\sigma) = \frac{1}{N} \sum_{i=M+1}^{N} \sigma_i.$$

We use the notation $\tilde{m}_N(\sigma) = (\tilde{m}_N^+(\sigma), \tilde{m}_N^-(\sigma))$, and we denote by $\widetilde{\mathscr{M}}_N$ the image of $\mathscr{S}_N$ by the application $\tilde{m}_N$. $\widetilde{\mathscr{M}}_N$ is therefore a deterministic subset of $[-1/2, 1/2]^2$. We have $\bar{m}_N(\sigma) = \tilde{m}_N^+(\sigma) + \tilde{m}_N^-(\sigma)$. Let $D = \{i \geq M+1\colon h_i = 1\}$. Note that the cardinality of D satisfies $|D| = (N + S_N)/2 - M \leq (1 + S_N)/2$. The Hamiltonian H_N can be expressed in these new coordinates as

$$H_N(\sigma) = -\frac{N}{2}(\bar{m}_N(\sigma))^2 - N\theta(\tilde{m}_N^+(\sigma) - \tilde{m}_N^-(\sigma)) - 2\theta \sum_{i \in D} \sigma_i$$

and

$$Z_N^1 = \sum_{\sigma \in \mathcal{S}_N} \mathbb{1}_{\{\bar{m}_N(\sigma) \geq -3/N\}} e^{-\beta H_N(\sigma)},$$

that is,

$$(3.10) \qquad \begin{aligned} Z_N^1 = \sum_{\tilde{m} \in \widetilde{\mathcal{M}}_N} &\mathbb{1}_{\{\bar{m} \geq -3/N\}} \#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\} e^{\beta N((1/2)\bar{m}^2 + \theta(\tilde{m}^+ - \tilde{m}^-))} \\ &\times \frac{\sum_{\sigma; \tilde{m}_N(\sigma) = \tilde{m}} e^{2\beta\theta \sum_{i \in D} \sigma_i}}{\#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\}}. \end{aligned}$$

In this last expression, the only term that depends on h is the set D.

Let $\widetilde{\mathcal{F}}_N(\tilde{m}) = -\frac{1}{2}\bar{m}^2 - \theta(\tilde{m}^+ - \tilde{m}^-) - \frac{1}{\beta N} \log \#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\}$ and note that

$$(3.11) \qquad \left| \widetilde{\mathcal{F}}_N(\tilde{m}^N) - \mathcal{F}(\tilde{m}) \right| \leq C \left(\frac{\log N}{N} + \left\| \tilde{m}^N - \tilde{m} \right\| \right)$$

for any $\tilde{m}^N \in \widetilde{\mathcal{M}}_N$, $\tilde{m} \in [-1/2, 1/2]^2$.

The minimum of $\mathcal{F}$ in the set $\bar{m} \geq 0$ is achieved at only point: m_1. Since we have assumed that $|S_N| \leq 2\sqrt{N \log N}$, we have $|\sum_{i \in D} \sigma_i| \leq \sqrt{N \log N}$. Taking into account the estimate (3.11), one deduces that there exists a small enough ball, B, centered at point m_1 such that, if we define

$$\widetilde{Z}_N^1 = \sum_{\tilde{m} \in \widetilde{\mathcal{M}}_N \cap B} e^{-\widetilde{\mathcal{F}}_N(\tilde{m})} \frac{\sum_{\sigma; \tilde{m}_N(\sigma) = \tilde{m}} e^{2\beta\theta \sum_{i \in D} \sigma_i}}{\#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\}},$$

then, for some deterministic constant C depending on the choice of B, we have $\log \widetilde{Z}_N^1 \leq \log Z_N^1 \leq \log \widetilde{Z}_N^1 - \log(1 - e^{-CN})$. It is therefore enough to prove Proposition 3.1 for $\widetilde{Z}_N^1$ instead of Z_N^1.

For $\nu \in \mathbb{R}$, define the probability measure

$$(3.12) \qquad E_\nu[f] = \frac{\sum_{\sigma \in \mathcal{S}_N} f(\sigma) e^{\nu \sum_{i=M+1}^N \sigma_i}}{\sum_{\sigma \in \mathcal{S}_N} e^{\nu \sum_{i=M+1}^N \sigma_i}}.$$

Note that

$$\sum_{\sigma \in \mathcal{S}_N} e^{\nu \sum_{i=M+1}^N \sigma_i} = 2^N \cosh(\nu)^{N-M}.$$

For any choice of ν_2 and ν_1, $\widetilde{Z}_N^1$ can be rewritten as

$$\begin{aligned}\widetilde{Z}_N^1 &= \sum_{\tilde{m}\in\widetilde{\mathscr{M}}_N\cap B} e^{-\beta N\widetilde{\mathscr{F}}_N(\tilde{m})}\left(\frac{\cosh\nu_2}{\cosh\nu_1}\right)^{N-M} e^{-N\tilde{m}^-(\nu_2-\nu_1)} \\ &\qquad\times \frac{E_{\nu_2}\big[e^{2\beta\theta\sum_{i\in D}\sigma_i}\mathbb{1}_{\tilde{m}_N^-(\sigma)=\tilde{m}^-}\big]}{E_{\nu_1}\big[\mathbb{1}_{\tilde{m}_N^-(\sigma)=\tilde{m}^-}\big]} \\ &= \sum_{\tilde{m}\in\widetilde{\mathscr{M}}_N\cap B} e^{-\beta N\widetilde{\mathscr{F}}_N(\tilde{m})}\left(\frac{\cosh\nu_2}{\cosh\nu_1}\right)^{N-M} e^{-N\tilde{m}^-(\nu_2-\nu_1)} \\ &\qquad\times \left(\frac{\cosh(\nu_2+2\beta\theta)}{\cosh\nu_2}\right)^{|D|}\frac{\Psi(\nu_2,\theta)(\tilde{m}^-)}{\Psi(\nu_1,0)(\tilde{m}^-)},\end{aligned}\tag{3.13}$$

where

$$\Psi(\nu,\theta)(\tilde{m}^-) = \frac{E_\nu\big[e^{2\beta\theta\sum_{i\in D}\sigma_i}\mathbb{1}_{\tilde{m}_N^-(\sigma)=\tilde{m}^-}\big]}{E_\nu\big[e^{2\beta\theta\sum_{i\in D}\sigma_i}\big]}.$$

Let $\alpha = S_N/(N-M)$. We now choose for ν_1 and ν_2 the solutions of the equations

$$\begin{aligned}\tanh(\nu_1) &= 2\tilde{m}^-, \\ \alpha\tanh(\nu_2+2\beta\theta)+(1-\alpha)\tanh(\nu_2) &= 2\tilde{m}^-.\end{aligned}\tag{3.14}$$

Since we are only interested in estimates for $\tilde{m}\in B$, and since $|\alpha|\le 2\times\sqrt{\log N/N}$, then ν_1 and ν_2 are uniformly bounded as S_N and $\tilde{m}$ vary. Besides, we deduce from (3.14) that

$$\begin{aligned}&\bigg|\alpha(\tanh(\nu_1+2\beta\theta)-\tanh(\nu_1)) \\ &\qquad + (\nu_2-\nu_1)\left(\frac{\alpha}{\cosh^2(\nu_1+2\beta\theta)}+\frac{1-\alpha}{\cosh^2(\nu_1)}\right)\bigg| \le C(\nu_2-\nu_1)^2.\end{aligned}$$

Therefore,

$$\big|(\nu_2-\nu_1)-\alpha\cosh^2(\nu_1)(\tanh(\nu_1)-\tanh(\nu_1+2\beta\theta))\big| \le C\frac{\log N}{N}$$

and

$$\begin{aligned}&\bigg| - N\tilde{m}^-(\nu_2-\nu_1)+(N-M)\log\frac{\cosh\nu_2}{\cosh\nu_1} \\ &\qquad + |D|\log\left(\frac{\cosh(\nu_2+2\beta\theta)}{\cosh\nu_2}\right) - \frac{S_N}{2}\log\frac{\cosh(\nu_1+2\beta\theta)}{\cosh\nu_1}\bigg| \le C\log N\end{aligned}$$

and

$$\left|\log \widetilde{Z}_N^1 - \log \sum_{\tilde{m}\in\widetilde{\mathscr{M}}_N\cap B} e^{-\beta N\widetilde{\mathscr{F}}_N(\tilde{m})}\left(\frac{\cosh(\nu_1+2\beta\theta)}{\cosh\nu_1}\right)^{S_N/2}\frac{\Psi(\nu_2,\theta)(\tilde{m}^-)}{\Psi(\nu_1,0)(\tilde{m}^-)}\right| \le C\log N \tag{3.15}$$

It now only remains to estimate Ψ. This can be done through a local central limit theorem just as in [5]. Repeating the arguments of Proposition 3.2 of [5], we get that

$$\Psi(\nu,\theta)(\tilde{m}^-) = \frac{1}{2\pi}\int_0^{2\pi} dk e^{-ikN\tilde{m}^-}\left(\frac{\cosh(2\beta\theta+\nu+ik)}{\cosh(2\beta\theta+\nu)}\right)^{|D|}\left(\frac{\cosh(\nu+ik)}{\cosh(\nu)}\right)^{N-M-|D|}.$$

From this last expression, following the estimates (3.36) to (3.44) in [5], one deduces that $C/\sqrt{N} \le \Psi(\nu,\theta)(\tilde{m}^-) \le 1$ provided that $2\tilde{m}^- = \alpha\tanh(\nu+2\beta\theta)+(1-\alpha)\tanh(\nu)$. The constant C is chosen deterministic and independent of $\tilde{m}\in B$. Therefore,

$$\left|\log \widetilde{Z}_N^1 - \log \sum_{\tilde{m}\in\widetilde{\mathscr{M}}_N\cap B} e^{-\beta N\widetilde{\mathscr{F}}_N(\tilde{m})}\left(\frac{\cosh(\nu_1+2\beta\theta)}{\cosh\nu_1}\right)^{S_N/2}\right| \le C\log N. \tag{3.16}$$

From the estimate (3.11), it is easy to deduce that one can replace $\mathscr{F}_N$ by $\mathscr{F}$ in this expression, that is,

$$\left|\log \widetilde{Z}_N^1 - \log \sum_{\tilde{m}\in\widetilde{\mathscr{M}}_N\cap B} e^{-\beta N\widetilde{\mathscr{F}}(\tilde{m})}\left(\frac{\cosh(\nu_1+2\beta\theta)}{\cosh\nu_1}\right)^{S_N/2}\right| \le C\log N. \tag{3.17}$$

Let us denote by ν_1^* the solution of (3.14) for the value $\tilde{m}^- = m_1^- = \frac{1}{2}\tanh(\beta m_* - \beta\theta)$ that is $\nu_1^* = \beta m_* - \beta\theta$. By standard Laplace arguments, we deduce from (3.16) that

$$\left|\log \widetilde{Z}_N^1 + \beta N\mathscr{F}(m_1) - \frac{S_N}{2}\log\frac{\cosh(\nu_1^*+2\beta\theta)}{\cosh\nu_1^*}\right| \le C\log N. \tag{3.18}$$

Equation (3.6) is proved with

$$a = \frac{1}{2\beta}\log\frac{\cosh(\beta m_*+\beta\theta)}{\cosh(\beta m_*-\beta\theta)}.$$

By symmetry, we also have

$$\left|\log Z_N^2 + \beta N\mathscr{F}(m_1) + \beta a S_N\right| \le C\log N$$

Since $Z_N = Z_N^1 + Z_N^2$, we clearly have $Z_N^1 \vee Z_N^2 \le Z_N \le 2(Z_N^1 \vee Z_N^2)$. It yields (3.5).

Let us prove (3.7). As before we can assume that $S_N \geq 0$. As in the proof of (3.18), one gets that

$$\left| \log z_N^1 + \beta N \mathscr{F}(m_0) - \frac{S_N}{2} \log \frac{\cosh(\nu_1^* + 2\beta\theta)}{\cosh \nu_1^*} \right| \leq C \log N, \tag{3.19}$$

where ν_1^* is now the solution of the equation $\tanh(\nu_1^*) = 2m_0^-$. From (2.12), we therefore have $\nu_1^* = -\beta\theta$ and $\cosh(\nu_1^* + 2\beta\theta) = \cosh(\nu_1^*)$. This entails (3.7). □

Let us conclude this section by the following corollary.

LEMMA 3.2. *On the set* $|S_N| \leq 2\sqrt{N \log N}$, *we have*

$$\left| \inf_{m \in \overline{T}_1^N} \mathscr{F}_N(m) - \mathscr{F}(m_1) + \frac{a}{N} S_N \right| \leq C \frac{\log N}{N}, \tag{3.20}$$

$$\left| \inf_{m \in \mathscr{M}_N} \mathscr{F}_N(m) - \mathscr{F}(m_1) + \frac{a}{N} \right| S_N|| \leq C \frac{\log N}{N}, \tag{3.21}$$

$$\left| \inf_{m \in \partial T_1^N} \mathscr{F}_N(m) - \mathscr{F}(m_0) \right| \leq C \frac{\log N}{N}. \tag{3.22}$$

PROOF. The number of points in $\mathscr{M}_N$ being bounded by $(N+1)^2$, we have

$$e^{-\beta N \inf_{m \in \overline{T}_1^N} \mathscr{F}_N(m)} \leq Z_N^1 \leq (N+1)^2 e^{-\beta N \inf_{m \in \overline{T}_1^N} \mathscr{F}_N(m)}.$$

Combining this inequality with (3.6) yields (3.21). The proof of (3.20) and (3.22) is identical. □

We now derive some a priori estimates on the landscape of the graph of $\mathscr{F}_N$ that will be used in the proof of Theorem 4.3. Let A be a subset of $\mathscr{M}_N$. By definition, a path, γ in A is a sequence $(x_0, x_1, \dots, x_k)$ of points belonging to A such that x_i and x_{i+1} are neighbors and $x_i \neq x_j$ for $i \neq j$. The length of a path is therefore always bounded by N^2.

Since m_1 is an absolute minimum of $\mathscr{F}$ and m_0 is the unique saddle point, we know that there exists a continuous function γ: $[0,1] \to [-1/2, 1/2]^2$ s.t. $\gamma(0) = m_1$, $\gamma(1) = m_0$ and the function $t \to \mathscr{F}(\gamma(t))$ is increasing. We further assume that the curve $\gamma([0,1])$ lies in $]-1/2, 1/2[^2$. Let m_1^N (resp. m_0^N) be a point in $\mathscr{M}_N$ s.t. the distance $\|m_1^N - m_1\|$ (resp. $\|m_0^N - m_0\|$) is minimal. There exists a path in $\overline{T}_1^N$, say, $\gamma_1^N = (x_0, \dots, x_k)$, such that $x_0 = m_1^N$, $x_k = m_0^N$ and the distance between x_i and the curve $\gamma([0,1])$ is less than $\sqrt{2}/N$. Furthermore, we have the following lemma.

LEMMA 3.3. *For* $N \geq N_0$, *on the set* $|S_N| \leq 2\sqrt{N \log N}$, *we have*

$$\sup_{x \in \gamma_1^N} \mathscr{F}_N(x) \leq \mathscr{F}(m_0) + C \frac{\log N}{N}. \tag{3.23}$$

PROOF. Let K be a compact subset of $]-1/2, 1/2[^2$ that contains the paths γ_1^N for all $N \geq N_0$ and all realizations of h. Here m_0 is a critical point of $\mathscr{F}$. Therefore,

$$|\mathscr{F}(m) - \mathscr{F}(m_0)| \leq C\|m - m_0\|^2. \tag{3.24}$$

Using Taylor expansions and the Stirling formula, one immediately gets that, for $m \in \mathscr{M}_N \cap K$,

$$\left|\mathscr{F}_N(m) - \mathscr{F}(m) - \frac{1}{\beta}\log\frac{(\frac{1}{2}+m^+)(\frac{1}{2}-m^+)}{(\frac{1}{2}+m^-)(\frac{1}{2}-m^-)}\frac{S_N}{N}\right| \leq C\frac{\log N}{N}. \tag{3.25}$$

Let $A > 0$. Let $x \in \gamma_1^N$. First assume that $\|x - m_0\| \leq A\sqrt{\log N/N}$. Since $m_0^+ = -m_0^-$, (3.25) implies that

$$\begin{aligned}|\mathscr{F}_N(x) - \mathscr{F}(x)| &\leq C\|x - m_0\|\frac{|S_N|}{N} + C\frac{\log N}{N}\\ &\leq C(1+A)\frac{\log N}{N}\end{aligned}$$

on the set $|S_N| \leq 2\sqrt{N\log N}$.

Then, from (3.24), we deduce that

$$\begin{aligned}\mathscr{F}_N(x) &= \mathscr{F}(m_0) + \mathscr{F}_N(x) - \mathscr{F}(x) + \mathscr{F}(x) - \mathscr{F}(m_0)\\ &\leq \mathscr{F}(m_0) + C(1+A)\frac{\log N}{N} + C\left(\frac{\sqrt{\log N}}{\sqrt{N}}\right)^2\\ &\leq \mathscr{F}(m_0) + C(2+A)\frac{\log N}{N}.\end{aligned}$$

Assume now that $\|x-m_0\| \geq A\sqrt{\log N/N}$. Using (3.25) and the fact that since m_0 is nondegenerate, there exists a constant $C' > 0$ such that

$$\begin{aligned}\mathscr{F}_N(x) &= \mathscr{F}(m_0) + \mathscr{F}_N(x) - \mathscr{F}(x) + \mathscr{F}(x) - \mathscr{F}(m_0)\\ &\leq \mathscr{F}(m_0) + C\|x - m_0\|\sqrt{\frac{\log N}{N}} + C\frac{\log N}{N} - C'\|x - m_0\|^2\\ &\leq \mathscr{F}(m_0) + \|x - m_0\|(C - AC')\sqrt{\frac{\log N}{N}} + C\frac{\log N}{N}\\ &\leq \mathscr{F}(m_0) + C\frac{\log N}{N}\end{aligned}$$

provided that we choose $A > C/C'$. □

4. Spectral estimates. We will need some estimates of the eigenvalues of $\mathscr{L}_N$ and L_N defined in (2.13) and (2.14). Since the operator $\mathscr{L}_N$ is symmetric in $L_2(\mathscr{M}_N, \mathscr{G}_N)$, we can consider its spectral decomposition: let $(\Lambda_i^N \equiv \Lambda_i^{h,N})_{i=0\cdots}$ denote the eigenvalues of $-\mathscr{L}_N$ in increasing order, with $\Lambda_0^N = 0$. Let $\psi_i^N \equiv \psi_i^{h,N}$ be the corresponding eigenvectors. We have $\psi_0^N \equiv 1$. We assume that the ψ_i^N form an orthonormal basis of $L_2(\mathscr{M}_N, \mathscr{G}_N)$. We can now express the law of m_N at time t on this basis:

$$E_m[\phi(m_N(t))] = \sum_i \psi_i^N(m)\mathscr{G}_N(\phi\psi_i^N)e^{-\Lambda_i^N t}. \tag{4.1}$$

Similarly, let $\mathscr{L}_N^K$ be the generator of the process m_N killed at time τ_N. In other words, $\mathscr{L}_N^K$ is the restriction of $\mathscr{L}_N$ to functions $\phi \in L_2(\overline{T}_1^N, \mathscr{G}_N^1)$ with $\phi(m) = 0$ for $m \in \partial T_1^N$. Then $-\mathscr{L}_N^K$ is a symmetric operator on $L_2(\overline{T}_1^N, \mathscr{G}_N^1)$. We denote by L_i^N its eigenvalues and ϕ_i^N the corresponding normalized eigenfunctions. We then have

$$P_m[\tau_N > t] = \sum_i \phi_i^N(m)\mathscr{G}_N^1(\phi_i^N)e^{-L_i^N t}. \tag{4.2}$$

From [15] we have the following.

PROPOSITION 4.1 (Estimates of eigenvalues). *For almost all realizations of* h,

$$\begin{aligned} &\frac{1}{N}\log L_1^N \to -\beta\Delta\mathscr{F}, \\ &\frac{1}{N}\log \Lambda_1^N \to -\beta\Delta\mathscr{F}. \end{aligned} \tag{4.3}$$

Estimating the eigenvalues Λ_2^N and L_2^N, one checks that only the first terms really contribute in (4.1) and (4.2). The next result is a consequence of the computation of [15], part 3.

PROPOSITION 4.2 (Spectral approximation). *There exists a deterministic constant K such that, for $t > 0$, if we define $T = t\exp(-K\sqrt{N}\log N) - K\log N$, then, for any realization of h and any $N \geq N_0$ s.t. $|S_N| \leq 2\sqrt{N\log N}$, for any $m \in T_1^N$, we have*

$$\left|P_m[\tau_N > t] - \phi_1^N(m)\mathscr{G}_N^1(\phi_1^N)e^{-L_1^N t}\right| \leq e^{-T}. \tag{4.4}$$

Moreover, for any continuous function ϕ bounded by 1, *and for $m \in \mathscr{M}_N$, we have*

$$\left|E_m[\phi(m_N(t))] - \left(\mathscr{G}_N(\phi) + \psi_1^N(m)\mathscr{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t}\right)\right| \leq e^{-T}. \tag{4.5}$$

Our next result is a precise estimate, Q-almost surely and in law of the fluctuations of the spectral gap of $\mathscr{L}_N$ and the first eigenvalue of $\mathscr{L}_N^K$.

THEOREM 4.3. *Let a be the constant defined in* (3.4). *For any $N \geq N_0$, for almost all realizations of h, on the set $|S_N| \leq 2\sqrt{N \log N}$, we have*

$$\left|\log L_1^N + \beta N \Delta\mathscr{F} + \beta a S_N\right| \leq C \log N \tag{4.6}$$

and

$$\left|\log \Lambda_1^N + \beta N \Delta\mathscr{F} - \beta a |S_N|\right| \leq C \log N. \tag{4.7}$$

As a consequence, if $\mathscr{N}$ is a normalized Gaussian random variable, then the following convergences hold in law w.r.t. Q:

$$N^{-1/2}\left(\log L_1^N + \beta N \Delta\mathscr{F}\right) \to -\beta a \mathscr{N}, \tag{4.8}$$

$$N^{-1/2}\left(\log \Lambda_1^N + \beta N \Delta\mathscr{F}\right) \to \beta a |\mathscr{N}|. \tag{4.9}$$

Let us recall some of the estimates proved in [15] that we shall need in the sequel. These are rough bounds on the exponential scale.

Let us choose $N \geq N_0$ and a realization of h s.t. $|S_N| \leq 2\sqrt{N \log N}$. Using Stirling's formula as in [15], part 4, it is not difficult to see that, for any $m \in [-1, +1]^2$,

$$|\mathscr{F}_N(m) - \mathscr{F}(m)| \leq C \frac{\log N}{\sqrt{N}}. \tag{4.10}$$

It is proved in [15] that, for any i, the following convergences hold almost surely:

$$\frac{1}{N} \log L_i^N \to -c_i,$$
$$\frac{1}{N} \log \Lambda_i^N \to -c_i,$$

where $c_1 = \beta \Delta\mathscr{F}$ and $c_i = 0$ for $i \geq 2$. Taking into account (4.10), it is immediate to prove that in fact

$$\begin{aligned} \left|\log L_i^N + N c_i\right| &\leq C\sqrt{N} \log N, \\ \left|\log \Lambda_i^N + N c_i\right| &\leq C\sqrt{N} \log N. \end{aligned} \tag{4.11}$$

At last we have some estimates of ϕ_1^N:

$$1 - \mathscr{G}_N^1\left(\phi_1^N\right) \leq e^{-CN} \tag{4.12}$$

and, for any given compact set A s.t. $\sup_{x \in A} \mathscr{F}(x) < \mathscr{F}(m_0)$, we have

$$\sup_{m \in A \cap T_1^N} \left|1 - \phi_1^N(m)\right| \leq e^{-C'N}, \tag{4.13}$$

where C' is a deterministic constant that depends on A. Then (4.12) and (4.13) can be proved as in [15], part 3.3, with the help of (4.10).

PROOF OF PROPOSITION 4.2. We first prove (4.5). For any continuous function ϕ bounded by 1, we have

$$\begin{aligned}(4.14)\qquad &\sqrt{\mathscr{G}_N\big(|E_\cdot[\phi(m_N(t))] - \{\mathscr{G}_N(\phi) + \psi_1^N(\cdot)\mathscr{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t}\}|^2\big)}\\ &\quad \le e^{-\Lambda_2^N t}\sqrt{\mathscr{G}_N(\phi^2)} \le e^{-t\Lambda_2^N}.\end{aligned}$$

Since, for any $m \in \mathscr{M}_N$, $|\mathscr{F}_N(m)| \le C$, we have, for any function ψ, $\psi(m)^2 \le e^{CN}\mathscr{G}_N(\psi^2)$. From (4.11), we deduce that

$$(4.15)\qquad \Lambda_2^N \ge e^{-C\sqrt{N}\log N}.$$

Therefore (4.14) implies that

$$\begin{aligned}(4.16)\qquad &\Big|E_m[\phi(m_N(t))] - \{\mathscr{G}_N(\phi) + \psi_1^N(m)\mathscr{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t}\}\Big|\\ &\quad \le N^C e^{-te^{-C\sqrt{N}\log N}} \equiv e^{-T},\end{aligned}$$

taking the constant $K = C$ in Proposition 4.2. The proof of (4.4) is identical: one has to consider the spectral decomposition of the process m^N killed at time τ_N [see (2.18)]. □

PROOF OF THEOREM 4.3. Equations (4.8) and (4.9) clearly follow from (4.6) and (4.7).

Following [15], let us introduce the following Dirichlet forms: for any function ϕ defined on $\mathscr{M}_N$, we denote by $\mathscr{E}_N$ the Dirichlet form of the operator $\mathscr{L}_N$ with respect to $\mathscr{G}_N$; that is,

$$\mathscr{E}_N(\phi) \equiv -\mathscr{G}_N(\phi[\mathscr{L}_N\phi]).$$

According to (2.25) in [15], $\mathscr{E}_N$ can also be written

$$\begin{aligned}(4.17)\qquad \mathscr{E}_N(\phi) = \frac{1}{2Z_N}\sum_{\substack{m,\tilde m\in\mathscr{M}_N\\ \tilde m\sim m}} &(\phi(\tilde m) - \phi(m))^2\\ &\times \Big(\widetilde{\mathscr{N}_N}(\tilde m, m)\Big)^{1/2} e^{(-\beta N/2)[\mathscr{F}_N(\tilde m)+\mathscr{F}_N(m)]},\end{aligned}$$

where $\widetilde{\mathscr{N}_N}$ is a correction factor bounded from below by $2/N$ and bounded from above by 1. Similarly let $\mathscr{E}_N^1$ be the Dirichlet form of the process m^N killed when reaching ∂T_1^N: the domain of $\mathscr{E}_N^1$ is the set of functions ϕ defined on $\overline{T}_1^N$ vanishing on ∂T_1^N, and we have

$$\begin{aligned}(4.18)\qquad \mathscr{E}_N^1(\phi) &\equiv -\mathscr{G}_N^1\big(\phi[\mathscr{L}_N(\phi)]\big)\\ &= \frac{1}{2Z_N^1}\sum_{\substack{m,\tilde m\in\overline{T}_1^N\\ \tilde m\sim m}}(\phi(\tilde m) - \phi(m))^2\big(\widetilde{\mathscr{N}_N}(\tilde m, m)\big)^{1/2}e^{-(\beta N/2)[\mathscr{F}_N(\tilde m)+\mathscr{F}_N(m)]}.\end{aligned}$$

The upper bound for L_1^N: L_1^N. This is given by the variational principle

$$L_1^N = \inf \frac{\mathscr{E}_N^1(\phi)}{\mathscr{G}_N^1(\phi^2)}, \tag{4.19}$$

where the inf is taken on the domain of $\mathscr{E}_N^1$. Choosing as a trial function $\phi(m) = \mathbb{1}(m \in T_1^N)$ in (4.19), the only nonzero terms in (4.18) come from neighboring points $(m, \tilde{m})$ such that $m \in T_1^N$ and $\tilde{m} \in \partial T_1^N$. For such points we have $3/N \geq \bar{m} \geq -3/N$. Therefore,

$$\mathscr{E}_N^1(\phi) \leq \frac{N^2}{Z_N^1} e^{-\beta N \inf \mathscr{F}_N(m)},$$

where the inf $\mathscr{F}_N(m)$ is computed for points $m \in \mathscr{M}_N$ s.t. $3/N \geq \bar{m} \geq -3/N$. From (3.22), we know that

$$\inf_{m \in \partial T_1^N} \mathscr{F}_N(m) \geq \mathscr{F}(m_0) - C \frac{\log N}{N}.$$

(The same holds true for ∂T_2^N.) Therefore,

$$\mathscr{E}_N^1(\phi) \leq \frac{N^2}{Z_N^1} e^{-\beta N \mathscr{F}(m_0)} N^C.$$

Using (3.6), we get that

$$\begin{aligned} \mathscr{E}_N^1(\phi) &\leq N^C e^{\beta N \mathscr{F}(m_1) - \beta a S_N} e^{-\beta N \mathscr{F}(m_0)} \\ &= N^C e^{-\beta N \Delta \mathscr{F} - \beta a S_N}. \end{aligned} \tag{4.20}$$

We also have $\mathscr{G}_N^1(\phi^2) = 1 - z_N^1 / Z_N^1 \geq 1/2$ provided that N_0 is chosen big enough (See Proposition 3.1.)

Therefore,

$$L_1^N \leq N^C e^{-\beta N \Delta \mathscr{F} - \beta a S_N}.$$

The lower bound for L_1^N. By definition of the eigenfunction ϕ_1^N, we have

$$L_1^N = \mathscr{E}_N^1(\phi_1^N).$$

Let $\gamma_1^N = (x_0, \dots, x_k)$ be the path defined before Lemma 3.3. We have

$$
\begin{aligned}
\left|\phi_1^N(m_1^N)\right|^2 &= \left|\sum_i \phi_1^N(x_i) - \phi_1^N(x_{i+1})\right|^2 \\
&\le \sum_i \left|\phi_1^N(x_i) - \phi_1^N(x_{i+1})\right|^2 \big(\widetilde{\mathscr{N}}_N(x_i, x_{i+1})\big)^{1/2} \\
&\qquad \times e^{-(\beta N/2)[\mathscr{F}_N(x_i)+\mathscr{F}_N(x_{i+1})]} \\
&\qquad \times \sum_i \big(\widetilde{\mathscr{N}}_N(x_i, x_{i+1})\big)^{-1/2} e^{(\beta N/2)[\mathscr{F}_N(x_i)+\mathscr{F}_N(x_{i+1})]} \\
&\le 2Z_N^1 \mathscr{E}_N^1(\phi_1^N)\sqrt{\frac{N}{2}}\, N^2 e^{\beta N \sup_{x\in\gamma_1^N} \mathscr{F}_N(x)} \\
&\le N^C L_1^N e^{\beta N \mathscr{F}(m_0) - \beta N \mathscr{F}(m_1) + \beta a S_N},
\end{aligned}
\tag{4.21}
$$

where we have used the results of Lemma 3.3 and Proposition 3.1. It only remains to note that, choosing N_0 large enough, we have $\phi_1^N(m_1^N) \ge 1/2$. This follows from (4.13). *The upper bound for* Λ_1^N*:* Λ_1^N. This is given by the variational principle

$$
\Lambda_1^N = \inf \frac{\mathscr{E}_N(\phi)}{\mathscr{G}_N[(\phi - \mathscr{G}_N(\phi))^2]}.
\tag{4.22}
$$

Choosing as a trial function $\phi(m) = \sqrt{Z_N^2/Z_N^1}\,\mathbb{1}(m \in \overline{T}_1^N) - \sqrt{Z_N^1/Z_N^2} \times \mathbb{1}(m \in \overline{T}_2^N)$ then we get $\mathscr{G}_N(\phi) = 0$ and

$$
\begin{aligned}
\mathscr{G}_N(\phi^2) &= \frac{Z_N^1 + Z_N^2}{Z_N} - 2\mathscr{G}_N(\overline{T}_N^1 \cap \overline{T}_N^2) \\
&\ge \frac{Z_N^1 - z_N^1 + Z_N^2 - z_N^2}{Z^N} \ge 1 - e^{-CN},
\end{aligned}
$$

according to Proposition 3.1.

Using (4.17), we get that

$$
\begin{aligned}
\mathscr{E}_N(\phi) &\le \frac{1}{Z_N}\left(\sqrt{\frac{Z_N^1}{Z_N^2}} + \sqrt{\frac{Z_N^2}{Z_N^1}}\right)^2 N^C e^{-(\beta N'/2)\inf_{m\in\overline{T}_N^1, \tilde{m}\in\overline{T}_N^2, m\sim\tilde{m}}[\mathscr{F}_N(m)+\mathscr{F}_N(\tilde{m})]} \\
&\le N^C \frac{Z_N}{Z_N^1 Z_N^2} e^{-(\beta N/2)[\inf_{m\in\partial T_1^N} \mathscr{F}_N(m) + \inf_{m\in\partial T_2^N} \mathscr{F}_N(m)]}.
\end{aligned}
$$

Proposition 3.1 and Lemma 3.2 entail

$$
\Lambda_1^N \le N^C e^{-\beta N \Delta\mathscr{F} + \beta a |S_N|}.
$$

The lower bound for Λ_1^N. To be able to use the same strategy to bound Λ_1^N as we did for L_1^N, we need some estimates on the eigenfunction ψ_1^N. This is the content of the next lemma: we choose for ψ_1^N the normalized eigenfunction corresponding to Λ_1^N such that $\mathscr{G}_N(\mathbb{1}_{\overline{T}_1^N}\psi_1^N) > 0$. This last condition uniquely determines ψ_1^N. (When N is big enough, Λ_1^N has multiplicity 1 as follows from our estimates of Λ_1^N and Λ_2^N.)

LEMMA 4.4. *On the set $|S_N| \le 2\sqrt{N\log N}$, we have*

$$\frac{1}{N}\log\left|\psi_1^N(m_1^N) - \sqrt{\frac{Z_N^2}{Z_N^1}}\right| \le -C \tag{4.23}$$

and

$$\frac{1}{N}\log\left|\psi_1^N(m_2^N) + \sqrt{\frac{Z_N^1}{Z_N^2}}\right| \le -C. \tag{4.24}$$

This lemma will be proved later. We continue the proof of Theorem 4.3. Let γ_1^N be the path defined in Lemma 3.3. Define similarly a path γ_2^N in $\overline{T}_2^N$ from m_2^N to m_0^N such that $\sup_{x\in\gamma_2^N} \mathscr{F}_N(x) \le \mathscr{F}(m_0) + C\log N/N$. Let γ^N be the path from m_1^N to m_2^N obtained by gluing together γ_1^N and γ_2^N, say, $\gamma^N = (x_0, \dots, x_k)$. Therefore we have

$$\sup_{x\in\gamma^N} \mathscr{F}_N(x) \le \mathscr{F}(m_0) + C\frac{\log N}{N}. \tag{4.25}$$

As in (4.21), we have

$$\begin{aligned}
&\left|\psi_1^N(m_1^N) - \psi_1^N(m_2^N)\right|^2 \\
&\quad = \left| \sum_i \psi_1^N(x_i) - \psi_1^N(x_{i+1})\right|^2 \\
&\quad \le \sum_i \left|\psi_1^N(x_i) - \psi_1^N(x_{i+1})\right|^2 \left(\widetilde{\mathscr{N}}_N(x_i, x_{i+1})\right)^{1/2} e^{-(\beta N/2)[\mathscr{F}_N(x_i)+\mathscr{F}_N(x_{i+1})]} \\
&\qquad\quad \times \sum_i \left(\widetilde{\mathscr{N}}_N(x_i, x_{i+1})\right)^{-1/2} e^{(\beta N/2)[\mathscr{F}_N(x_i)+\mathscr{F}_N(x_{i+1})]} \\
&\quad \le 2Z_N \mathscr{E}_N(\psi_1^N)\sqrt{\frac{N}{2}}N^2 e^{\beta N \sup_{x\in\gamma^N}\mathscr{F}_N(x)} \\
&\quad \le N^C \Lambda_1^N e^{\beta N\mathscr{F}(m_0) - \beta N\mathscr{F}(m_1) + \beta a|S_N|} \\
&\quad = N^C \Lambda_1^N e^{\beta N\Delta\mathscr{F} + \beta a|S_N|}.
\end{aligned} \tag{4.26}$$

From Lemma 4.4 and Proposition 3.1, we have

$$\begin{aligned}
\left|\psi_1^N(m_1^N) - \psi_1^N(m_2^N)\right|^2 &\ge \frac{(Z_N)^2}{Z_N^1 Z_N^2}\left(1 - \frac{\sqrt{Z_N^1 Z_N^2}}{Z_N} e^{-2CN}\right)^2 \\
&\ge N^{-C} e^{2\beta a|S_N|}.
\end{aligned}$$

This entails

$$\Lambda_1^N \geq N^{-C} e^{-\beta N \Delta \mathscr{F} + \beta \alpha |S_N|}.$$

PROOF OF LEMMA 4.4. The proof makes use of Proposition 4.2 and the fact that ϕ_1^N converges to 1. Let $0 < \alpha < \alpha' < \Delta\mathscr{F}$ and $t = \exp(\alpha' \beta N)$. Define T as in Proposition 4.2. Clearly $T \geq e^{\alpha\beta N}$ for N big enough. Using (4.5) with the function $\phi(m) \equiv \mathbb{1}(m \in T_1^N)$, we get that

$$\left| P_m[m^N(t) \in T_1^N] - \mathscr{G}_N(T_1^N) - \psi_1^N(m) \mathscr{G}_N(\mathbb{1}_{T_1^N} \psi_1^N) e^{-\Lambda_1^N t} \right| \leq e^{-T}. \tag{4.27}$$

In particular,

$$\begin{aligned}
\psi_1^N(m)\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N) &\leq e^{\Lambda_1^N t}\Big(e^{-T} + P_m[m^N(t) \in T_1^N] - \mathscr{G}_N(T_1^N)\Big) \\
&\leq e^{\Lambda_1^N t}\Big(e^{-T} + 1 - \mathscr{G}_N(T_1^N)\Big) \\
&\leq e^{\Lambda_1^N t}\Big(e^{-T} + \mathscr{G}_N(\overline{T}_2^N)\Big) \\
&\leq e^{\Lambda_1^N t}\Big(e^{-T} + \frac{Z_N^2}{Z_N}\Big).
\end{aligned}$$

Taking into account that $T \geq e^{\alpha\beta N}$, $\alpha' < \Delta\mathscr{F}$ and the estimates (4.11) for Λ_1^N and Proposition 3.1, we obtain

$$\psi_1^N(m)\mathscr{G}_N\big(\mathbb{1}_{T_1^N}\psi_1^N\big) \leq \frac{Z_N^2}{Z_N}(1 + e^{-CN}).$$

In particular, integrating over T_1^N,

$$\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N) \leq \frac{\sqrt{Z_N^1 Z_N^2}}{Z_N}(1 + e^{-CN}) \tag{4.28}$$

and

$$\psi_1^N(m_1^N)\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N) \leq \frac{Z_N^2}{Z_N}(1 + e^{-CN}). \tag{4.29}$$

From (4.27), we also get that, for $m \in T_1^N$,

$$\begin{aligned}
\psi_1^N(m)\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N) &\geq e^{\Lambda_1^N t}\Big(-e^{-T} + P_m[m^N(t) \in T_1^N] - \mathscr{G}_N(T_1^N)\Big) \\
&\geq -e^{-T} + P_m[\tau_N > t] - \mathscr{G}_N(T_1^N) \\
&\geq -2e^{-T} + \phi_1^N(m)\mathscr{G}_N^1(\phi_1^N)e^{-L_1^N t} - \mathscr{G}_N(T_1^N) \\
&= -2e^{-T} + \phi_1^N(m)\mathscr{G}_N^1(\phi_1^N)e^{-L_1^N t} - 1 + \mathscr{G}_N(\overline{T}_2^N),
\end{aligned}$$

where we used (2.20). In particular,

$$\psi_1^N(m_1^N)\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N) \geq -2e^{-T} + \phi_1^N(m_1^N)\mathscr{G}_N^1(\phi_1^N)e^{-L_1^N t} - 1 + \mathscr{G}_N(\overline{T}_2^N) \tag{4.30}$$

and, integrating over T_1^N,

$$\begin{aligned}
&\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N)^2\\
&\quad\geq \mathscr{G}_N(T_1^N)\biggl(-2e^{-T} + \frac{\mathscr{G}_N(\mathbb{1}_{T_1^N}\phi_1^N)}{\mathscr{G}_N(T_1^N)}\mathscr{G}_N^1(\phi_1^N)e^{-L_1^N t} - 1 + \mathscr{G}_N(\overline{T}_2^N)\biggr)\\
&\quad= \mathscr{G}_N(T_1^N)\biggl(-2e^{-T} + \frac{Z_N^1}{Z_N\mathscr{G}(T_1^N)}(\mathscr{G}_N^1(\phi_1^N))^2 e^{-L_1^N t} - 1 + \mathscr{G}_N(\overline{T}_2^N)\biggr).
\end{aligned} \tag{4.31}$$

Using (4.11) and (4.13), we deduce from (4.30) that

$$\psi_1^N(m_1^N)\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N) \geq \frac{Z_N^2}{Z_N}(1 - e^{-CN}). \tag{4.32}$$

Using (4.11) and (4.12), we deduce from (4.31) that

$$\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N)^2 \geq \mathscr{G}_N(T_1^N)\frac{Z_N^2}{Z_N}(1 - e^{-CN}).$$

Since it follows from Proposition 3.1 that

$$\mathscr{G}_N(T_1^N) = \frac{Z_1^N - z_1^N}{Z_N} \geq \frac{Z_N^1}{Z_N}(1 - e^{-CN}),$$

we get

$$\mathscr{G}_N(\mathbb{1}_{T_1^N}\psi_1^N) \geq \frac{\sqrt{Z_N^1 Z_N^2}}{Z_N}(1 - e^{-CN}). \tag{4.33}$$

One can now solve equations (4.28), (4.29), (4.32) and (4.33) to conclude the proof of (4.23). The proof for (4.24) is identical. □

We consider now the spectral decomposition of L_N in $L_2(S_N, \mu_N)$. Let us call (λ_i^N), $i = 0, 1\dots$ the eigenvalues of $-L_N$ in increasing order, with $\lambda_0^N = 0$. Our result is the lemma.

LEMMA 4.5. *There exists an integer N_0, s.t. Q a.s., on the set $|S_N| \leq 2\sqrt{N\log N}$; for $N \geq N_0$, we have*

$$\lambda_1^N = \Lambda_1^N. \tag{4.34}$$

Therefore,

$$|\log\lambda_1^N + \beta N\Delta\mathscr{F} - \beta a|S_N|| \leq C\log N \tag{4.35}$$

for some deterministic constant C. Besides

$$\lambda_2^N \geq e^{-C\sqrt{N}\log N}. \tag{4.36}$$

PROOF. Due to Theorem 4.3, (4.35) is indeed a consequence of (4.34). Let e_N denote the Dirichlet form of the operator L_N in $L_2(S_N, \mu_N)$, that is,

$$\begin{aligned} e_N(\phi) &\equiv -\mu_N(\phi[L_N\phi]) \\ &= \frac{1}{2NK_N} \sum_{\substack{\sigma,\tilde{\sigma}\in\mathcal{S}_N \\ \tilde{\sigma}\sim\sigma}} (\phi(\sigma)-\phi(\tilde{\sigma}))^2 e^{-(\beta/2)(H_N(\sigma)+H_N(\tilde{\sigma}))}, \end{aligned} \tag{4.37}$$

where $\tilde{\sigma} \sim \sigma$ means that for some i, $\tilde{\sigma} = T^i(\sigma)$. We then have

$$\lambda_1^N = \inf_{\phi} \frac{e_N(\phi)}{\mu_N[(\phi-\mu_N(\phi))^2]}. \tag{4.38}$$

Let $\mathcal{H}_N$ denote the set of functions of $\sigma \in \mathcal{S}_N$ that depend on the magnetization m_N only. Let $\mathcal{K}_N$ denote the orthogonal complement of $\mathcal{H}_N$ in $L_2(\mathcal{S}_N, \mu_N)$. Therefore $v \in \mathcal{K}_N$ iff, for any value $m \in \mathcal{M}_N$, we have $\mu_N(v\mathbb{1}(m_N = m)) = 0$. Because the Hamiltonian H_N only depends on the magnetization m_N, we also have that $v \in \mathcal{K}_N$ iff

$$\sum_{\sigma;m_N(\sigma)=m} \phi(\sigma) = 0 \tag{4.39}$$

for all $m \in \mathcal{M}_N$.

Note that $\mathcal{H}_N$ is left invariant by L_N. By symmetry $\mathcal{K}_N$ is also left invariant by L_N. Therefore if $\phi = u + v$, with $u \in \mathcal{H}_N$ and $v \in \mathcal{K}_N$, we have $e_N(\phi) = e_N(u) + e_N(v)$. Therefore,

$$\begin{aligned} \lambda_1^N &= \inf \frac{e_N(\phi)}{\mu_N[(\phi-\mu_N(\phi))^2]} \\ &= \inf_{u\in\mathcal{H}_N; v\in\mathcal{K}_N} \frac{e_N(u)+e_N(v)}{\mu_N[(u-\mu_N(u))^2]+\mu_N(v^2)} \\ &\geq \inf\left(\inf_{u\in\mathcal{H}_N} \frac{e_N(u)}{\mu_N[(u-\mu_N(u))^2]}; \inf_{u\in\mathcal{K}_N} \frac{e_N(v)}{\mu_N(v^2)}\right). \end{aligned} \tag{4.40}$$

The reverse inequality is trivially true. Besides,

$$\Lambda_1^N = \inf_{u\in\mathcal{H}_N} \frac{e_N(u)}{\mu_N[(u-\mu_N(u))^2]}.$$

Therefore we have

$$\lambda_1^N = \inf\left(\Lambda_1^N; \inf_{v\in\mathcal{K}_N} \frac{e_N(v)}{\mu_N(v^2)}\right). \tag{4.41}$$

Let us now compute the last term in (4.41). For $i \neq j$, let $T^{i,j}(\sigma)$ be defined by $T^{i,j}(\sigma)_k = \sigma_k$ for $k \neq i, j$, $T^{i,j}(\sigma)_i = \sigma_j$ and $T^{i,j}(\sigma)_j = \sigma_i$. Let

$A^+ = \{i \text{ s.t. } h_i = 1\}$ and $A^- = \{i \text{ s.t. } h_i = -1\}$. Consider the operator

$$\mathscr{A}_N \phi(\sigma) = \frac{1}{|A^+|} \sum_{i,\, j \in A^+} \phi(T^{i,\, j}\sigma) - \phi(\sigma) + \frac{1}{|A^-|} \sum_{i,\, j \in A^-} \phi(T^{i,j}\sigma) - \phi(\sigma).$$

$\mathscr{A}_N$ is symmetric w.r.t. the uniform measure restricted to the set $\{m \text{ s.t. } m_N(\sigma) = m\}$. $\mathscr{A}_N$ is the sum of two operators acting on different coordinates. Under the dynamics induced by $\mathscr{A}_N$, the coordinates in A^+ (resp. A^-) perform a simple exclusion process. The spectral gap of each of the two terms defining $\mathscr{A}_N$ is known to be bounded from below by 1/2, whatever the value of m; see [12]. Therefore we have the Poincaré inequality [Remember that $\sum_{\sigma; m_N(\sigma)=m} v(\sigma) = 0$]:

$$\begin{aligned}\sum_{\sigma; m_N(\sigma)=m} v^2(\sigma) &\le \frac{1}{4} \sum_{\sigma; m_N(\sigma)=m} \Bigg(\frac{1}{|A^+|} \sum_{i,j\in A^+} (v(T^{i,j}\sigma) - v(\sigma))^2 \\ &\qquad\qquad + \frac{1}{|A^-|} \sum_{i,j\in A^-} (v(T^{i,j}\sigma) - v(\sigma))^2 \Bigg) \\ &\le \frac{1}{4} \sum_{\sigma; m_N(\sigma)=m} \Bigg(\frac{1}{|A^+|} \sum_{i,j\in A^+} (v(T^i T^j\sigma) - v(\sigma))^2 \\ &\qquad\qquad + \frac{1}{|A^-|} \sum_{i,j\in A^-} (v(T^i T^j\sigma) - v(\sigma))^2 \Bigg),\end{aligned} \tag{4.42}$$

where we used for the last equality the fact that if $T^{i,j}\sigma \ne \sigma$ then $T^{i,j}\sigma = T^i T^j \sigma$.

Writing $(v(T^iT^j\sigma) - v(\sigma))^2 \le 2(v(T^iT^j\sigma) - v(T^j\sigma))^2 + 2(v(T^j\sigma) - v(\sigma))^2$, we deduce from (4.42) that

$$\begin{aligned}\sum_{\sigma; m_N(\sigma)=m} v^2(\sigma) &\le \sum_{\sigma; m_N(\sigma)=m} \Bigg(\sum_{i\in A^+} (v(T^i\sigma) - v(\sigma))^2 + \sum_{i\in A^-} (v(T^i\sigma) - v(\sigma))^2 \Bigg) \\ &\le 2 \sum_{\substack{\sigma,\tilde\sigma \in \mathscr{S}_N \\ \tilde\sigma\sim\sigma}} \big(v(\tilde\sigma) - v(\sigma)\big)^2.\end{aligned} \tag{4.43}$$

Taking the sum over all possible values of m, we get that

$$\begin{aligned}\mu_N(v^2) &\le \frac{2}{K_N} \sum_{\substack{\sigma,\tilde\sigma \in \mathscr{S}_N \\ \tilde\sigma\sim\sigma}} (v(\tilde\sigma) - v(\sigma))^2 e^{-\beta H_N(\sigma)} \\ &\le \frac{2}{K_N} e^{\beta(\theta+1)} \sum_{\substack{\sigma,\tilde\sigma \in \mathscr{S}_N \\ \tilde\sigma\sim\sigma}} (v(\tilde\sigma) - v(\sigma))^2 e^{-(\beta/2)(H_N(\sigma)+H_N(\tilde\sigma))} \\ &= 4Ne^{\beta(\theta+1)} e_N(v),\end{aligned} \tag{4.44}$$

where we have used $|H_N(\sigma) - H_N(\tilde\sigma)| \le 2\theta + 2$.

Choose now N_0 big enough so that $\Lambda_1^N \le 4N^{-1}e^{-\beta(\theta+1)}$. For any $N \ge N_0$, we then have $\lambda_1^N = \Lambda_1^N$.

We have now completed the proof of the first part of the lemma. For the second part, we start with the min–max characterization of λ_2^N,

$$\lambda_2^N = \sup_{f,g} \inf_{\phi;\mu_N(\phi f)=\mu_N(\phi g)=0} \frac{e_N(\phi)}{\mu_N(\phi^2)}.$$

Let us choose $f = 1$ and $g = \psi_N^1 \, o \, m_N$, where ψ_1^N is the eigenfunction corresponding to the eigenvalue Λ_1^N for the magnetization process. Then

$$\begin{aligned}\lambda_2^N &\ge \inf_{v\in\mathscr{K}_N} \inf_{u\in\mathscr{H}_N;\mu_N(u)=\mu_N(u\psi_N^1 o m_N)=0} \frac{e_N(u+v)}{\mu_N((u+v)^2)}\\ &\ge \inf\left(\inf_{u\in\mathscr{H}_N;\mu_N(u)=\mu_N(u\psi_N^1 o m_N)=0} \frac{e_N(u)}{\mu_N(u^2)}; \inf_{v\in\mathscr{K}_N} \frac{e_N(v)}{\mu_N(v^2)}\right)\\ &= \inf\left(\Lambda_2^N; \inf_{v\in\mathscr{K}_N} \frac{e_N(v)}{\mu_N(v^2)}\right).\end{aligned} \tag{4.45}$$

Combining this inequality with (4.44) and (4.11) leads to the second assertion of the lemma. □

5. Proof of Theorems.

PROOF OF THEOREM 2.1. From (4.35) we get that, on the set $|\mathscr{S}_N| \le 2\sqrt{N\log N}$,

$$\lambda_1^N \ge N^{-C} e^{-\beta N\Delta\mathscr{F} + \beta a|\mathscr{S}_N|} \tag{5.1}$$

if N is large enough. Here C is some deterministic constant.

For any function ϕ bounded by 1, we have

$$\sqrt{\mu_N\big(|E_\cdot[\phi(\sigma_N(t))] - \mu_N(\phi)|^2\big)} \le e^{-\lambda_1^N t}\sqrt{\mu_N(\phi^2)} \le e^{-\lambda_1^N t}.$$

Moreover, for any function ψ and any $\sigma \in \mathscr{S}_N$, since $(1/N)H_N \le C$, we have $|\psi(\sigma)| \le \exp(CN)\sqrt{\mu_N(\psi^2)}$. Therefore,

$$\big|E_\sigma[\phi(\sigma_N(t))] - \mu_N(\phi)\big| \le e^{CN}e^{-\lambda_1^N t}. \tag{5.2}$$

Replacing t by t_N as defined in Theorem 2.1, and using (5.1), we conclude the proof of (2.16).

The proof of (2.17) is similar: for any continuous function ϕ bounded by 1, we have

$$\sqrt{\mathscr{G}_N\big(|E_\cdot[\phi(m_N(t_N)] - \mathscr{G}_N(\phi)|^2\big)} \le e^{-\Lambda_1^N t_N}\mathscr{G}_N(\phi^2) \le e^{-t_N\Lambda_1^N}. \tag{5.3}$$

Since, for any $m \in \mathscr{M}_N$, $|\mathscr{F}_N(m)| \le C$, we have, for any function ψ, $\psi(m)^2 \le e^{CN}\mathscr{G}_N(\psi^2)$. From Theorem 4.3, we deduce that

$$\Lambda_1^N \ge e^{-\beta N\Delta\mathscr{F} + \beta a|\mathscr{S}_N|}N^{-C},$$

hence

$$\Lambda_1^N t_N \geq N^{K-C}.$$

Therefore (5.3) implies that

$$|E_m[\phi(m_N(t_N))] - \mathscr{G}_N(\phi)| \leq e^{-N^{K-C}}.$$

(2.17) follows by choosing $K > C$. □

PROOF OF THEOREM 2.2. The proof is quite similar to the proof of Theorem 2.6 in [15], so we only sketch it. First note that it follows from Lemma (4.5) and (4.11) that, for N large enough, $\lambda_1^N \neq \lambda_2^N$. Therefore, since $\lambda_1^N = \Lambda_1^N$ is of multiplicity one, the corresponding eigenvector is $\psi_1^N \, o \, m_N$. (Remember that ψ_1^N is the eigenvector corresponding to Λ_1^N in the spectral decomposition of the magnetization process.) Since $\lambda_2^N \geq \exp(-C\sqrt{N}\log N)$, using the same argument as in the proof of Theorem 2.1, we get that

$$E_{\sigma^N}[\phi(\sigma_{(N)}(t/\Lambda_1^N))] \sim \mu_N(\phi) + e^{-t}\psi_1^N(m_N(\sigma^N))\mu_N((\psi_1^N \, o \, m_N)\phi). \tag{5.4}$$

Writing $\mu_N(\phi|m_N)$ for the conditional expectation given m_N and denoting by $\mathscr{G}_N$ the law of m_N, the last term in (5.4) can be rewritten

$$\mu_N(\phi) + e^{-t}\psi_1^N(m_N(\sigma^N))\mathscr{G}_N(\psi_1^N \mu_N(\phi|m_N)). \tag{5.5}$$

To compute the expression (5.5), note that we only have to evaluate functions of the magnetization. The same arguments as in the proof of Theorem 2.6 in [15] then lead to

$$\begin{aligned} &\psi_1^N(m_N(\sigma^N))\mathscr{G}_N(\psi_1^N \mu_N(\phi|m_N)) \\ &\quad \sim \mathscr{G}_N^1(\mu_N(\phi|m_N)) - \mathscr{G}_N(\mu_N(\phi|m_N)) \\ &\quad = \mu_N^1(\phi) - \mu_N(\phi), \end{aligned} \tag{5.6}$$

where we use the notation μ_N^1 for the Gibbs measure μ_N conditioned to the set $\widetilde{T}_N^1 = \{\sigma \in \mathscr{S}_N \text{ s.t. } \bar{m}_N(\sigma) > 0\}$. Similarly μ_N^2 denotes the measure μ_N conditioned to the set $\widetilde{T}_N^2 = \{\sigma \in \mathscr{S}_N \text{ s.t. } \bar{m}_N(\sigma) < 0\}$.

Using the expression (5.6) in (5.4), we get that

$$E_{\sigma^N}[\phi(\sigma_{(N)}(t/\Lambda_1^N))] \sim \mu_N(\phi) + e^{-t}(\mu_N^1(\phi) - \mu_N(\phi)).$$

To conclude the proof, we use the fact that, on a set of full Q probability, μ_N^1 weakly converges to μ_∞^1, μ_N^2 weakly converges to μ_∞^2 and $\mu_N - (\alpha_N \mu_\infty^1 + (1 - \alpha_N)\mu_\infty^2)$ weakly converges to 0. (See [14].) □

We now turn to the proof of Theorem (2.3).

PROOF OF (2.20). Let $\alpha \in \mathbb{R}$ and $t_N = \exp(\beta N \Delta\mathscr{F} + \alpha\sqrt{N})$. We write

$$P_{m^N}[N^{-1/2}(\log \tau_N - \beta N \Delta\mathscr{F}) \geq \alpha] = P_{m^N}[\tau_N \geq t_N]$$

$$= P_{m^N}[\tau_N \ge t_N] - \phi_1^N(m^N)\mathscr{G}_N^1(\phi_1^N)e^{-L_1^N t_N} + \Big(\phi_1^N(m^N)\mathscr{G}_N^1(\phi_1^N) - 1\Big)e^{-L_1^N t_N} + e^{-L_1^N t_N}.$$

Therefore

$$\begin{aligned}\mathbb{P}_{m^N}\big[\tau_N \ge t_N\big] = {}& \mathbb{P}_{m^N}[\tau_N \ge t_N; |S_N| \ge 2\sqrt{N\log N}] \\ & + Q\Big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}\big(P_{m^N}[\tau_N \ge t_N] \\ & \qquad - \phi_1^N(m^N)\mathscr{G}_N^1(\phi_1^N)e^{-L_1^N t_N}\big)\Big] \\ & + Q\Big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}\big(\phi_1^N(m^N)\mathscr{G}_N^1(\phi_1^N) - 1\big)e^{-L_1^N t_N}\Big] \\ & + Q\Big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}e^{-L_1^N t_N}\Big].\end{aligned} \tag{5.7}$$

In (5.7), the first term converges to 0 since $Q[|S_N| \ge 2\sqrt{N\log N}] \le 2/N^2$. By Proposition 4.2, the second term is bounded by $\exp(-T)$ with $T = t_N \times \exp(-K\sqrt{N}\log N) - K\log N \to +\infty$. Therefore the second term also tends to 0. From (4.12) and (4.13), it follows that the third term is bounded by $\exp(-CN)$ and therefore goes to 0. Thus

$$\mathbb{P}_{m^N}\big[\tau_N \ge t_N\big] - Q\big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}e^{-L_1^N t_N}\big] \to 0.$$

For any $\varepsilon > 0$, write

$$\begin{aligned} & Q\big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}e^{-L_1^N t_N}\big] \\ & \quad = Q\big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}e^{-L_1^N t_N}\,\mathbb{1}_{\alpha\sqrt{N}-\beta a S_N \le -\varepsilon\sqrt{N}}\big] \\ & \qquad + Q\big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}e^{-L_1^N t_N}\,\mathbb{1}_{\alpha\sqrt{N}-\beta a S_N \ge \varepsilon\sqrt{N}}\big] \\ & \qquad + Q\big[\mathbb{1}_{|S_N|\le 2\sqrt{N\log N}}e^{-L_1^N t_N}\,\mathbb{1}_{-\varepsilon\sqrt{N}\le\alpha\sqrt{N}-\beta a S_N \le +\varepsilon\sqrt{N}}\big].\end{aligned} \tag{5.8}$$

Note that, from Theorem 4.3, on the set where $|S_N| \le 2\sqrt{N\log N}$ and $\alpha\sqrt{N} - \beta a S_N \le -\varepsilon\sqrt{N}$ hold, we have $|\exp(-L_1^N t_N) - 1| \le N^C\exp(-\varepsilon\sqrt{N}) \to 0$. Therefore the first term in (5.8) is close to $Q[|S_N| \le 2\sqrt{N\log N};\ \alpha\sqrt{N} - \beta a S_N \le -\varepsilon\sqrt{N}]$. On the set where $|S_N| \le 2\sqrt{N\log N}$ and $\alpha\sqrt{N} - \beta a S_N \ge \varepsilon\sqrt{N}$ hold, we have $L_1^N t_N \ge N^{-C}\exp(\varepsilon\sqrt{N}) \to +\infty$. Therefore the second term in (5.8) converges to 0. The third term in (5.8) is bounded by $Q[-\varepsilon\sqrt{N} \le \alpha\sqrt{N} - \beta a S_N \le +\varepsilon\sqrt{N}]$. So far we have proved that

$$\begin{aligned}\limsup & \Big|\mathbb{P}_{m^N}[\tau_N \ge t_N] - Q[|S_N| \le 2\sqrt{N\log N}; \alpha\sqrt{N} - \beta a S_N \le -\varepsilon\sqrt{N}]\Big| \\ & \le \limsup Q[-\varepsilon\sqrt{N} \le \alpha\sqrt{N} - \beta a S_N \le +\varepsilon\sqrt{N}].\end{aligned}$$

The central limit theorem for S_N entails that

$$\limsup \Big|\mathbb{P}_{m^N}[\tau_N \ge t_N] - Q[\beta a\mathscr{N} \ge \alpha + \varepsilon]\Big| \le Q[\alpha - \varepsilon \le \beta a\mathscr{N} \le \alpha + \varepsilon]. \tag{5.9}$$

Since (5.9) holds for any $\varepsilon > 0$, we also have

$$\mathbb{P}_{m^N}[\tau_N \geq t_N] \to Q[\beta a \mathscr{N} \geq \alpha]. \qquad \square$$

Now we prove Theorem 2.4.

PROOF OF (2.21). Let $0 < \alpha' < \Delta\mathscr{F}$ and define $s = \exp(\beta\alpha' N)$ and

$$S = s \exp(-K\sqrt{N}\log N) - K \log N,$$

where K is the constant in Proposition 4.2. Then S tends to $+\infty$ and, from (4.11), on the set $|S_N| \leq 2\sqrt{N \log N}$, we have

$$\Lambda_1^N s \leq N^C e^{(\alpha' - \Delta\mathscr{F})\beta N + C\sqrt{N}\log N} \to 0.$$

ϕ being bounded by 1, we deduce from Proposition 4.2 that, on the set $|S_N| \leq 2\sqrt{N \log N}$, we have, for $N \geq N_0$,

$$\left|\psi_1^N(m^N)\mathscr{G}_N(\phi\psi_1^N)\right| \leq C. \tag{5.10}$$

Proceeding as in the proof of (2.20) and using (5.10), it is easy to see that

$$\begin{aligned} &\limsup \Big|\mathbb{E}_{m^N}[\phi(m_N(t_N))] \\ &\quad - Q\big[\mathbb{1}_{|S_N| \leq 2\sqrt{N\log N}}\big(\mathscr{G}_N(\phi) + \psi_1^N(m^N)\mathscr{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t_N}\big)\big]\Big| = 0. \end{aligned} \tag{5.11}$$

Therefore, using Theorem 4.3 and (5.10), we have

$$\begin{aligned} &\limsup \Big|\mathbb{E}_{m^N}[\phi(m_N(t_N))] \\ &\qquad - Q\big[\mathbb{1}_{|S_N| \leq 2\sqrt{N\log N}}\big(\mathscr{G}_N(\phi) + \psi_1^N(m^N)\mathscr{G}_N(\phi\psi_1^N) \\ &\qquad\qquad \times \mathbb{1}_{\beta a|S_N| + \alpha\sqrt{N} \leq -\varepsilon\sqrt{N}}\big)\big]\Big| \\ &\quad \leq C \limsup Q[-\varepsilon\sqrt{N} \leq \beta a|S_N| + \alpha\sqrt{N} \leq \varepsilon\sqrt{N}]. \end{aligned} \tag{5.12}$$

From [15], formula (5.25), we know that, for almost all realization of h,

$$\psi_1^N(m^N)\mathscr{G}_N(\phi\psi_1^N) + \mathscr{G}_N(\phi) - \phi(m_1) \to 0.$$

Using (5.10) and the bounded convergence lemma, from (5.12) we deduce that

$$\begin{aligned} &\limsup \Big|\mathbb{E}_{m^N}[\phi(m_N(t_N))] \\ &\qquad - Q\big[\mathscr{G}_N(\phi) + (\phi(m_1) - \mathscr{G}_N(\phi))\mathbb{1}_{\beta a|S_N| + \alpha\sqrt{N} \leq -\varepsilon\sqrt{N}}\big]\Big| \\ &\quad \leq C \limsup Q[-\varepsilon\sqrt{N} \leq \beta a|S_N| + \alpha\sqrt{N} \leq \varepsilon\sqrt{N}]. \end{aligned} \tag{5.13}$$

Let A_N be the set $\beta a|S_N| + \alpha\sqrt{N} \leq -\varepsilon\sqrt{N}$. From Proposition 3.1, we have

$$Q[\mathscr{G}_N(\phi)\mathbb{1}_{A_N}] - \phi(m_1)Q[\alpha_N \mathbb{1}_{A_N}] - \phi(m_2)Q[(1 - \alpha_N)\mathbb{1}_{A_N}] \to 0.$$

By symmetry,

$$Q[\alpha_N \mathbb{1}_{A_N}] = Q[(1 - \alpha_N)\mathbb{1}_{A_N}] = \tfrac{1}{2}Q[A_N].$$

Therefore,

$$Q[\mathscr{G}_N(\phi)\,\mathbb{1}_{\beta a|S_N|+\alpha\sqrt{N}\le -\varepsilon\sqrt{N}}] - \tfrac{1}{2}(\phi(m_1) \\ +\phi(m_2))Q[\beta a|S_N| + \alpha\sqrt{N} \le -\varepsilon\sqrt{N}] \to 0.$$

The central limit theorem for S_N entails that

$$\begin{aligned} &\limsup \big|\mathbb{E}_{m^N}[\phi(m_N(t_N))] \\ &\quad - \big(\tfrac{1}{2}(\phi(m_1) + \phi(m_2)) + \tfrac{1}{2}(\phi(m_1) - \phi(m_2))Q[\beta a|\mathscr{N}| \le -\alpha - \varepsilon]\big)\big| \\ &\quad \le\ Q[-\varepsilon - \alpha \le \beta a|\mathscr{N}| \le -\alpha + \varepsilon]. \end{aligned} \tag{5.14}$$

Since (5.14) is true for all $\varepsilon > 0$, we have

$$\begin{aligned} &\mathbb{E}_{m^N}[\phi(m_N(t_N))] \\ &\quad \to \tfrac{1}{2}(\phi(m_1) + \phi(m_2)) + \tfrac{1}{2}(\phi(m_1) - \phi(m_2))Q[\beta a|\mathscr{N}| \le -\alpha]. \end{aligned}$$ □

PROOF OF (2.22). The proof is similar to (2.21). Using Lemma (4.5) and (5.6) one gets that

$$\begin{aligned} &\mathbb{E}_{\sigma^N}\big[\phi\big(\sigma_{(N)}(e^{\beta N\Delta\mathscr{F}+\alpha\sqrt{N}})\big)\big] \\ &\quad \sim Q\big[\mu_N(\phi) + \exp(-e^{\beta N\Delta\mathscr{F}+\alpha\sqrt{N}}\lambda_1^N)(\mu_N^1(\phi) - \mu_N(\phi))\big] \\ &\quad \sim Q\big[\mu_N(\phi) + \mathbb{1}_{\beta a|S_N|+\alpha\sqrt{N}\le 0}(\mu_N^1(\phi) - \mu_N(\phi))\big] \end{aligned} \tag{5.15}$$

In view of Proposition 3, part 4 of [14], one can replace μ_N in (5.15) by $\alpha_N\mu_N^1 + (1-\alpha_N)\mu_N^2$. Separating the expectation into two pieces according to $S_N \ge N^{1/4}$ (then $\alpha_N \sim 1$) or $S_N \le -N^{1/4}$ (then $\alpha_N \sim 0$), we get that

$$\begin{aligned} &\mathbb{E}_{\sigma^N}\big[\phi(\sigma_{(N)}(e^{\beta N\Delta\mathscr{F}+\alpha\sqrt{N}}))\big] \\ &\quad \sim\ Q\big[\mathbb{1}_{S_N\ge 0}\mu_N^1(\phi)\big] + Q\big[\mathbb{1}_{S_N\le 0}\mu_N^2(\phi)\big] \\ &\quad + Q\big[\mathbb{1}_{0\ge\beta a S_N\ge\alpha\sqrt{N}}(\mu_N^1(\phi) - \mu_N^2(\phi))\big]. \end{aligned} \tag{5.16}$$

Using the results of [14], we can replace μ_N^1 (resp. μ_N^2) by μ_∞^1 (resp. μ_∞^2) and we get

$$\begin{aligned} &\sim Q[\mathbb{1}_{S_N\ge 0}\mu_\infty^1(\phi)] + Q[\mathbb{1}_{S_N\le 0}\mu_\infty^2(\phi)] \\ &\quad + Q[\mathbb{1}_{0\ge\beta a S_N\ge\alpha\sqrt{N}}(\mu_\infty^1(\phi) - \mu_\infty^2(\phi))]. \end{aligned} \tag{5.17}$$

Now note that, without loss of generality, we can assume that ϕ depends only on a finite number of coordinates of σ, say, ϕ is a function of $\sigma_1, \dots, \sigma_k$. Then

$\mu^1_\infty(\phi)$ is a function of $h_1, \dots, h_k$ only. Therefore,

$$Q[\mathbb{1}_{S_N \geq 0}\mu^1_\infty(\phi)] \sim Q[S_N \geq 0]Q[\mu^1_N(\phi)].$$

One can use the same argument for the other terms in (5.17), and the central limit theorem for S_N, to get that

$$\begin{aligned} &\mathbb{E}_{\sigma^N}\big[\phi(\sigma_{(N)}(e^{\beta N \Delta \mathscr{F} + \alpha\sqrt{N}}))\big] \\ &\quad \sim \tfrac{1}{2}Q[\mu^1_\infty(\phi)] + \tfrac{1}{2}Q[\mu^2_\infty(\phi)] \\ &\qquad + Q[0 \geq \beta a \mathscr{N} \geq \alpha]Q[(\mu^1_\infty(\phi) - \mu^2_\infty(\phi))] \end{aligned} \tag{5.18}$$

It remains to note that $Q[\mu^1_\infty(\phi)] = \mathscr{B}^1(\phi)$ and $Q[\mu^2_\infty(\phi)] = \mathscr{B}^2(\phi)$ to conclude the proof of (2.22). □

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