# ON THE AXIOMATIC THEORY OF SPECTRUM II.

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**Abstract.** We give a survey of results concerning various classes of bounded linear operators in a Banach space defined by means of kernels and ranges. We show that many of these classes define a spectrum that satisfies the spectral mapping property.

## Introduction

Denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators in a complex Banach space $X$. The identity operator in $X$ will be denoted by $I_X$, or simply by $I$ when no confusion can arise.

By [15], a non-empty subset $R \subset \mathcal{L}(X)$ is called a regularity if it satisfies the following two conditions:

(1) if $A \in \mathcal{L}(X)$ and $n \geq 1$ then $A \in R \Leftrightarrow A^n \in R$,
(2) if $A, B, C, D \in \mathcal{L}(X)$ are mutually commuting operators satisfying $AC + BD = I$ then $AB \in R \Leftrightarrow A, B \in R$.

A regularity defines in a natural way the spectrum $\sigma_R$ by $\sigma_R(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin R\}$ for every $A \in \mathcal{L}(X)$.

The axioms of regularity are usually easy to verify and there are many naturally defined classes of operators satisfying them, see [15]. Since the corresponding spectrum always satisfies the spectral mapping property, the notion of regularity enables to produce spectral mapping theorems in an easy way.

The aim of this paper is to give a survey of results for various classes of operators defined by means of kernels and ranges. For the sake of completeness we include also some well-known classes and results. On the other hand we obtain a great number of new results (especially spectral mapping theorems) for various classes of operators and introduce also new classes of operators which, in our opinion, deserve further attention.

The axioms of regularity (and consequently the spectral mapping property) provide a criterion for a decision which classes of operators are reasonable.

## I. Preliminaries

We start with basic properties of regularities, see [15].

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1) If $R \subset \mathcal{L}(X)$ is a regularity then $R$ contains all invertible operators, so that the corresponding spectrum is contained in the ordinary spectrum, $\sigma_R(A) \subset \sigma(A)$ for every $A \in \mathcal{L}(X)$.

2) In general $\sigma_R(A)$ is neither closed nor non-empty. In particular $R = \mathcal{L}(X)$ is also a regularity; the corresponding spectrum $\sigma_R(A)$ is empty for every $A \in \mathcal{L}(X)$.

3) If $(R_\alpha)_\alpha$ is any family of regularities then $R = \bigcap_\alpha R_\alpha$ is also a regularity. The corresponding spectra satisfy

$$\sigma_R(A) = \bigcup_\alpha \sigma_{R_\alpha}(A) \qquad (A \in \mathcal{L}(X)).$$

4) Let $R \subset \mathcal{L}(X)$ be a regularity and let $\sigma_R$ be the corresponding spectrum. Then

$$\sigma_R(f(A)) = f(\sigma_R(A))$$

for every $A \in \mathcal{L}(X)$ and every function $f$ analytic on a neighbourhood of $\sigma(A)$ which is non-constant on each component of its domain of definition.

5) Let $R \subset \mathcal{L}(X)$ be a regularity and let $X_1, X_2$ be a pair of complementary closed subspaces, $X = X_1 \oplus X_2$. Then there exist uniquely determined regularities

$$\begin{aligned} R_1 &= \{T_1 \in \mathcal{L}(X_1) : \quad T_1 \oplus I_{X_2} \in R\} \subset \mathcal{L}(X_1) \text{ and} \\ R_2 &= \{T_2 \in \mathcal{L}(X_2) : \quad I_{X_1} \oplus T_2 \in R\} \subset \mathcal{L}(X_2) \end{aligned}$$

such that

$$A_1 \oplus A_2 \in R \Leftrightarrow A_1 \in R_1 \text{ and } A_2 \in R_2 \qquad (A_1 \in \mathcal{L}(X_1),\ (A_2 \in \mathcal{L}(X_2)).$$

The corresponding spectra satisfy $\sigma_R(A_1 \oplus A_2) = \sigma_{R_1}(A_1) \cup \sigma_{R_2}(A_2)$. Suppose a regularity $R \subset \mathcal{L}(X)$ satisfies the following condition: if $X_1, X_2$ are closed complementary subspaces in $X$, $X = X_1 \oplus X_2$, such that $R_1 = \{T_1 \in \mathcal{L}(X_1) : T_1 \oplus I_{X_2} \in R\} \neq \mathcal{L}(X_1)$, then the corresponding spectrum $\sigma_{R_1}(A_1)$ is non-empty for every $A_1 \in \mathcal{L}(X_1)$. Then

$$\sigma_R(f(A)) = f(\sigma_R(A))$$

for every $A \in \mathcal{L}(X)$ and every function $f$ analytic on a neighbourhood of$\sigma(A)$.

**Remark.** In all reasonable situations (in particular in all situations considered in this paper) a regularity decomposes in the canonical way. For example, if $R = \{T \in \mathcal{L}(X) : T \text{ is onto}\}$ and $X = X_1 \oplus X_2$then $R_i = \{T_i \in \mathcal{L}(X_i) : \ T_i \text{ is onto}\}$ (i=1,2) and $T_1 \oplus T_2$ is onto $\Leftrightarrow$ $T_1, T_2$ are onto. Thus the condition above reduces to the question on the non-emptiness of the spectrum.

For an operator $T \in \mathcal{L}(X)$ denote by $N(T)$ and $R(T)$ its kernel $N(T) = \{x \in X : Tx = 0\}$ and range $R(T) = \{Tx : x \in X\}$, respectively. Clearly $N(T) \subset N(T^2) \subset \cdots$ and $R(T) \supset R(T^2) \supset \cdots$.

Denote further $N^\infty(T) = \bigcup_{n=0}^\infty N(T^n)$ and $R^\infty(T) = \bigcap_{n=0}^\infty R(T^n)$.

The following lemma enables an easy verification of axiom (2) of regularities for various classes of operators.

**Lemma 1.** *Let $A, B, C, D$ be mutually commuting operators in a Banach space $X$ satisfying $AC + BD = I$ and let $n \geq 0$. Then*

(1) $N(A^n B^n) = N(A^n) + N(B^n)$, $R(A^n B^n) = R(A^n) \cap R(B^n)$,
(2) $N^\infty(AB) = N^\infty(A) + N^\infty(B)$, $R^\infty(AB) = R^\infty(A) \cap R^\infty(B)$,
(3) $N^\infty(A) \subset R^\infty(B)$, $N^\infty(B) \subset R^\infty(A)$,
(4) $R(A^n B^n)$ *is closed* $\Leftrightarrow$ $R(A^n)$, $R(B^n)$ *are closed.*

**Proof.** The first 3 properties were proved in [15].

If $R(A^n)$, $R(B^n)$ are closed then clearly $R(A^n B^n) = R(A^n) \cap R(B^n)$ is closed.

Suppose $R(A^n B^n)$ is closed and $z \in \overline{R(A^n)}$, i.e there are $u_k \in X$, $k = 1, 2, \cdots$ such that $A^n u_k \to z$. Then $A^n B^n u_k \to B^n z = A^n B^n u$ for some $u \in X$. Thus $z - A^n u \in N(B^n) \subset R(A^n)$, so that $z \in R(A^n)$. Hence $R(A^n)$ is closed.

Following Grabiner [7], consider for $T \in \mathcal{L}(X)$ and $n \geq 0$ the linear mapping $R(T^n)/R(T^{n+1}) \longrightarrow R(T^{n+1})/R(T^{n+2})$ induced by $T$. Denote by $k_n(T)$ the dimension of its kernel.

**Lemma 2.** ([7], Lemma 2.3) *Let $T \in \mathcal{L}(X)$ and $n \geq 0$. Then $k_n(T)$ is equal to any of the following quantities:*

(1) *the dimension of the kernel of the linear mapping*

$$R(T^n)/R(T^{n+1}) \longrightarrow R(T^{n+1})/R(T^{n+2})$$

*induced by $T$; this mapping is onto,*
(2) $\dim\big[\big(R(T^n) \cap N(T)\big)/\big(R(T^{n+1}) \cap N(T)\big)\big]$,
(3) *the codimension of the image of the linear mapping*

$$N(T^{n+2})/N(T^{n+1}) \longrightarrow N(T^{n+1})/N(T^n)$$

*induced by $T$; this mapping is injective,*
(4) $\dim\big[\big(R(T) + N(T^{n+1})\big)/\big(R(T) + N(T^n)\big)\big]$.

If $M_1$ and $M_2$ are (not necessarily closed) subspaces of a Banach space $X$ then we write for short $M_1{\subset}M_2$ ($M_1$ is essentially contained in $M_2$) if there is a finite dimensional subspace $F \subset X$ such that $M_1 \subset M_2 + F$. In this case we may assume that $F \subset M_1$. Clearly $M_1{\subset}M_2$ if and only if $\dim(M_1/(M_1 \cap M_2)) < \infty$. If $M_1{\subset}M_2$ and $M_2{\subset}M_1$, then we write $M_1{=}M_2$.

## II. Descent

For $T \in \mathcal{L}(X)$ and $n = 0, 1, \cdots$ denote

$$c_n(T) = \dim(R(T^n)/R(T^{n+1})).$$

By Lemma 2, we have $c_{n+1}(T) \leq c_n(T) \quad (n = 0, 1, \cdots)$.

The descent of an operator $T \in \mathcal{L}(X)$ is defined by

$$d(T) = \inf\{n : c_n(T) = 0\} = \inf\{n : R(T^n) = R(T^{n+1})\}$$

(the infimum of an empty set is defined to be $\infty$). If $d(T) < \infty$ then $R(T^{d(T)}) = R(T^{d(T)+1}) = \cdots = R^\infty(T)$.

Similarly we can define the essential descent of $T$ by

$$d_e(T) = \inf\{n : c_n(T) < \infty\} = \inf\{n : R(T^n){=}R(T^{n+1})\}.$$

If $d = d_e(T) < \infty$ then $R(T^d){=}R(T^n)$ for every $n \ge d$ (of course $R(T^d){=}R^\infty(T)$ is not true in general).

Denote by $\phi_-(X)$ the set of all lower semi-Fredholm operators in $X$, i.e., $T \in \phi_-(X)$ if and only if $c_0(T) < \infty$.

The following two lemmas enable the verification of axioms of regularity:

**Lemma 3.** *Let $T \in \mathcal{L}(X)$, $m \ge 1$, $n \ge 0$. Then $c_n(T^m) = \sum_{i=0}^{m-1} c_{mn+i}(T)$. In particular*

$$c_{mn}(T) \le c_n(T^m) \le m c_{mn}(T).$$

**Proof.** We have

$$\begin{aligned} c_n(T^m) &= \dim\bigl(R(T^{mn})/R(T^{mn+m})\bigr) \\ &= \sum_{i=0}^{m-1} \dim\bigl(R(T^{mn+i})/R(T^{mn+i+1})\bigr) = \sum_{i=0}^{m-1} c_{mn+i}(T)\ . \end{aligned}$$

**Lemma 4.** *Let $A, B, C, D$ be mutually commuting operators in a Banach space $X$ satisfying $AC + BD = I$ and let $n \ge 0$. Then*

$$\max\{c_n(A), c_n(B)\} \le c_n(AB) \le c_n(A) + c_n(B)\ .$$

**Proof.** We prove first $c_n(A) \le c_n(AB)$. This is clear if $c_n(AB) = \infty$. Suppose $c_n(AB) < \infty$. Set $m = c_n(AB) + 1$ and let $x_1, \cdots, x_m$ be arbitrary elements of $R(A^n)$. Then $B^n x_i \in R(A^n B^n)$ $(i = 1, \cdots, m)$ so that there exist a non-trivial linear combination

$$\sum_{i=1}^m \alpha_i B^n x_i = B^n(\sum_{i=1}^m \alpha_i x_i) \in R(A^{n+1}B^{n+1}).$$

Thus

$$\sum_{i=1}^m \alpha_i x_i \in R(A^{n+1}B) + N(B^n) \subset R(A^{n+1}).$$

Since the vectors $x_1, \cdots, x_m$ were arbitrary, we have $c_n(A) = \dim\bigl(R(A^n)/R(A^{n+1})\bigr) \le c_n(AB)$.

The second inequality is clear if $c_n(A) + c_n(B) = \infty$. Let $c_n(A) + c_n(B)$ be finite and consider the linear mapping $R(A^n B^n) \to R(A^n) \oplus R(B^n)$ defined by $x \mapsto x \oplus x$. If $m > c_n(A) + c_n(B)$ and $x_1, \cdots, x_m$ are arbitrary vectors in $R(A^n B^n)$ then there exists a non-trivial linear combination such that $\sum_{i=1}^m \alpha_i x_i \in R(A^{n+1})$ and $\sum_{i=1}^m \alpha_i x_i \in R(B^{n+1})$. By Lemma 1, $\sum_{i=1}^m \alpha_i x_i \in R(A^{n+1}B^{n+1})$ so that $c_n(AB) \le c_n(A) + c_n(B)$.

Let us consider the following classes of operators:
(1) $R_1^a = \{T \in \mathcal{L}(X) : d(T) = 0\}$. Other equivalent formulations: $c_0(T) = 0 \quad \Leftrightarrow c_n(T) = 0$ for every $n \quad \Leftrightarrow T$ is onto.
(2) $R_2^a = \{T \in \mathcal{L}(X) : d(T) < \infty$ and $d_e(T) = 0\}$. Equivalently: $\sum_{i=0}^\infty c_i(T) < \infty \Leftrightarrow c_0(T) < \infty$ and there exists $d \in \mathbb{N}$ such that $c_d(T) = 0 \Leftrightarrow T$ is lower semi-Fredholm and $T$ has a finite descent.
(3) $R_3^a = \{T \in \mathcal{L}(X) : d_e(T) = 0\}$. Equivalently: $c_0(T) < \infty \Leftrightarrow c_n(T) < \infty$ for every $n \Leftrightarrow T \in \phi_-(X)$.
(4) $R_4^a = \{T \in \mathcal{L}(X) : d(T) < \infty\}$. Equivalently, there exists $d \in \mathbb{N}$ such that $c_n(T) = 0 \ \ (n \ge d) \Leftrightarrow T$ has a finite descent.
(5) $R_5^a = \{T \in \mathcal{L}(X) : d_e(T) < \infty\}$. Equivalently, there exists $d \in \mathbb{N}$ such that $c_n(T) < \infty \ \ (n \ge d) \Leftrightarrow T$ has a finite essential descent.

In case of ambiguity we shall write $R_i^a(X)$ instead of $R_i^a \ \ (i = 1, ..., 5)$.

It is easy to see, by Lemmas 3 and 4, that the sets $R_1^a, \cdots, R_5^a$ are regularities, so that the corresponding spectra satisfy the spectral mapping theorem (for locally non-constant analytic functions).

The conditions defining the sets $R_1^a, \cdots, R_5^a$ are purely algebraic (therefore we use the upper index a). We could define these classes for linear mappings in an arbitrary vector space. The spectral mapping theorem would remain true(of course only for non-constant polynomials).

An operator $T \in \mathcal{L}(X)$ with codim $R(T) < \infty$ has automatically closed range (and in this case also $R(T^n)$ is closed for every $n$). This is not the case for operators with a finite descent as the following example shows.

**Example 5.** There exists a bounded linear operator $T$ in a Hilbert space such that $R(T^2) = R(T)$ and $R(T)$ is not closed.

**Construction.** Consider the Hilbert space $H$ with an orthonormal basis $\{e_{ij}\}_{i,j=1}^\infty$ and the operator $T$ defined by

$$Te_{ij} = \begin{cases} 0 & \text{if } \ j = 1, \\ \dfrac{1}{i} e_{i,1} & \text{if } \ j = 2, \\ e_{i,j-1} & \text{otherwise .} \end{cases}$$

It is easy to check that $R(T^2) = R(T) = M_1 + TM_2$, where $M_1 = \bigvee\{e_{ij} : j \ge 2, i \ge 1\}$ and $M_2 = \bigvee\{e_{i,2} : i \ge 1\}$.

Further $R(T)$ is not closed since $R(T) \cap \left(\bigvee\{e_{i,1} : i \ge 1\}\right)$ is not closed.

It is more interesting from the point of view of the operator theory to combine the algebraic conditions defining regularities $R_4^a$ and $R_5^a$ with a topological condition — closeness of $R(T^d)$. It is easy to see that if $c_d(T) = \dim(R(T^d)/R(T^{d+1})) < \infty$ then $R(T^d)$ is closed if and only if $R(T^{d+1})$ is closed. Thus, by induction, if $c_d(T) < \infty$ and $R(T^n)$ is closed for some $n \ge d$ then $R(T^i)$ is closed for every $i \ge d$.

The classes of operators which we are really interested in are the following ones (the first 3 sets remain unchanged since a topological condition is already implicitly contained in the definition; we repeat them only in order to preserve symmetry with subsequent situations).

$$\begin{aligned}
R_1 &= \{T \in \mathcal{L}(X) : T \text{ is onto}\},\\
R_2 &= \{T \in \mathcal{L}(X) : T \in \phi_-(X) \text{ and } d(T) < \infty\},\\
R_3 &= \phi_-(X),\\
R_4 &= \{T \in \mathcal{L}(X) : d(T) < \infty \text{ and } R(T^{d(T)}) \text{ is closed}\},\\
R_5 &= \{T \in \mathcal{L}(X) : d_e(T) < \infty \text{ and } R(T^{d_e(T)}) \text{ is closed}\} .
\end{aligned}$$

Obviously $R_1 \subset R_2 = R_3 \cap R_4 \subset R_3 \cup R_4 \subset R_5$.

It is easy to see that the sets $R_1, \cdots, R_5$ are regularities. Denote by $\sigma_i$ $(i = 1, \cdots, 5)$ the corresponding spectra.

If $X = X_1 \oplus X_2$, $T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$ then we have

$$T_1 \oplus T_2 \in R_i(X) \Leftrightarrow T_1 \in R_i(X_1) \text{ and } T_2 \in R_i(X_2) \qquad (i = 1, \cdots, 5) .$$

Further

$$\sigma_1(T_1) \neq \emptyset \Leftrightarrow X_1 \neq \{0\} \Leftrightarrow R_1(X_1) \neq \mathcal{L}(X_1).$$

Similarly, for $i = 2, 3$,

$$\sigma_i(T_1) \neq \emptyset \Leftrightarrow \dim X_1 = \infty \Leftrightarrow R_j(X_1) \neq \mathcal{L}(X_1)$$

(see below). Thus we have

**Theorem 6.** *Let $T \in \mathcal{L}(X)$ and let $f$ be a function analytic in a neighbourhood of $\sigma(T)$. Then*

(1) $\sigma_i(f(T)) = f(\sigma_i(T)) \qquad (i = 1, 2, 3)$,

(2) *if $f$ is non-constant on each component of its domain of definition then*

$$\sigma_i(f(T)) = f(\sigma_i(T)) \qquad (i = 4, 5).$$

The spectra $\sigma_1$ and $\sigma_3$ are well-known — $\sigma_1$ is the defect spectrum (sometimes called also the surjective spectrum) and $\sigma_3$ is the lower semi-Fredholm spectrum. In the remaining cases the spectral mapping theorems seem to be new. Of particular interest is the case of $i = 2$, cf. [8].

We are now going to study further properties of regularities $R_i$ and the corresponding spectra $\sigma_i$. We will consider the following properties (to avoid trivialities we consider only infinite dimensional Banach spaces $X$):

(A) $\sigma_i(T) \neq \emptyset$ for every $T \in \mathcal{L}(X)$.

(B) $\sigma_i(T)$ is closed for every $T \in \mathcal{L}(X)$.

(C) If $T \in R_i$ then there exists $\varepsilon > 0$ such that $T + U \in R_i$ whenever $TU = UT$ and $\|U\| < \varepsilon$ (this means the upper semi-continuity of $\sigma_i$ on commuting elements, see [15], property (P3)).

(D) If $T \in R_i$ and $F \in \mathcal{L}(X)$ is a finite dimensional operator then $T + F \in R_i$.
(E) if $T \in R_i$ and $K$ is a compact operator commuting with $T$ then$T + K \in R_i$.
(F) If $T \in R_i$ and $Q \in \mathcal{L}(X)$ is a quasinilpotent operator commuting with $T$then $T + Q \in R_i$.

The properties of $R_i \quad (i = 1, \cdots 5)$ are summarized in the following table:

|  | (A)<br>$\sigma_i \neq \emptyset$ | (B)<br>$\sigma_i$closed | (C)<br>small commut. perturbations | (D)<br>finite dim. perturb. | (E)<br>commut.comp. perturbations | (F)<br>commut. quasinilp. pert. |
|---|---|---|---|---|---|---|
| $R_1$<br>onto | yes | yes | yes | no | no | yes |
| $R_2$<br>$\phi_-(X)$ and $d(T) < \infty$ | yes | yes | yes | no | yes | yes |
| $R_3$<br>$\phi_-(X)$ | yes | yes | yes | yes | yes | yes |
| $R_4$<br>$d(T) < \infty$ | no | yes | no | no | no | no |
| $R_5$<br>$d_e(T) < \infty$ | no | yes | no | yes | no | no |

Tab. 1

All these properties are known and some of them are trivial. Nevertheless we indicate briefly by the following observations how the table can be filled in.

1) the zero operator $0 \in R_i$ and $\sigma_i(0) = \emptyset \quad (i = 4,5)$. Since every operator commutes with 0, $R_4$ and $R_5$ cannot have properties (A), (C), (E), (F).

2) Consider the identity operator in a Hilbert space and let $P$ be a 1-dimensional orthogonal projection. Then $I - P$ is not onto and $R_1$ does not have properties (D), (E).

3) Consider the bilateral shift $T$ in a Hilbert space $H$ with an orthonormal basis $\{e_i\}_{i=-\infty}^{\infty}$ defined by $Te_i = e_{i+1}$ and let $Fx =< x, e_0 > e_1$. Then $d(T - F) = \infty$ so that $R_2$ and $R_4$ do not have property (D).

The remaining properties are true.

4) It is well-known that $\sigma_1(T)$ and $\sigma_3(T)$ are non-empty and closed. Further $\sigma_2(T) \neq \emptyset$, since $\sigma_2(T) \supset \sigma_3(T)$.

5) It is well-known that $R_3$ is stable under (not necessary commuting)compact perturbations. Also both $R_1$ and $R_3$ are stable under small (not necessary commuting) perturbations.

6) The stability of $R_2$ under commuting compact perturbations was proved in [6].

7) Let $T \in \mathcal{L}(X)$ be onto and let $Q$ be a quasinilpotent commuting with $T$. By the spectral mapping property for the joint defect spectrum (see e.g. [9], [25]) we have

$$\begin{aligned}\sigma_1(T+Q) &= \{\lambda+\mu : (\lambda,\mu) \in \sigma_1(T,Q)\} \subset \{\lambda+\mu : \lambda \in \sigma_1(T), \mu \in \sigma_1(Q)\} \\ &= \sigma_1(T) \not\ni 0 \ .\end{aligned}$$

Thus $T+Q \in R_1$.

Analogous considerations can be done also for $R_3$ (for the spectral mapping property see [3]).

8) If $T, F \in \mathcal{L}(X)$ and $F$ is a finite dimensional operator, then

$$\begin{aligned}(T+F)^n - T^n &= \sum_{i=0}^{n-1} \left[T^i(T+F)^{n-i} - T^{i+1}(T+F)^{n-i-1}\right] \\ &= \sum_{i=0}^{n-1} T^i F (T+F)^{n-i-1}\end{aligned}$$

so that $(T+F)^n - T^n$ is a finite dimensional operator. Consequently

$$R((T+F)^n) {=} R(T^n)$$

for every $n$ and $c_n(T) < \infty \Leftrightarrow c_n(T+F) < \infty$,i.e. $T \in R_5 \Leftrightarrow T+F \in R_5$.

9) Clearly $T \in R_2$ if and only if $\operatorname{codim} R^\infty(T) < \infty$.

Let $T \in R_2$ and $UT = TU$. Then $UR^\infty(T) \subset R^\infty(T)$ and $T|R^\infty(T)$ is onto. If $\|U\|$ is a small enough or $U$ is a quasinilpotent then $(T+U)|R^\infty(T)$ is still onto (see observation 7) so that $R^\infty(T+U) \supset R^\infty(T)$ and $T+U \in R_2$. Thus $R_2$ has properties (C) and (F). Consequently $\sigma_2(T)$ is closed.

10) Let $T \in R_5(X)$. Denote $M = R(T^{d_e(T)})$. Then $T|M \in \phi_-(M)$ and the operator $\hat{T} : X/M \to X/M$ induced by $T$ is nilpotent. Let $\lambda$ be a non-zero complex number small enough. Then $\widehat{T+\lambda} : X/M \to X/M$ is invertible and $(T+\lambda)|M \in \phi_-(M)$. Let $F$ be a finite dimensional subspace of $M$ such that $R((T+\lambda)|M)+F = M$.

If $x \in X$ then $x+M \in R(\widehat{T+\lambda})$, so that $x \in R(T+\lambda)+M \subset R(T+\lambda)+F$. Thus $\operatorname{codim} R(T+\lambda) < \infty$ and $T+\lambda \in \phi_-(X) \subset R_5(X)$. Hence $\sigma_5(T)$ is closed (moreover $\sigma_3(T) - \sigma_5(T)$ consists of at most countably many isolated points).

Similar considerations can be done for $T \in R_4$ (with $F = \{0\}$). Thus $\sigma_4(T)$ is closed and $\sigma_1(T) - \sigma_4(T)$ consists of at most countably many isolated points.

## III. Ascent

Similar considerations can be done for the dual situation. For $T \in \mathcal{L}(X)$ and $n = 0, 1, \cdots$, define $c_n'(T) = \dim(N(T^{n+1})/N(T^n))$. By Lemma 2 we have $c_0'(T) \geq c_1'(T) \geq \cdots$ . Moreover, if $c_n'(T) < \infty$ then $k_n(T) = c_n'(T) - c_{n+1}'(T)$.

The ascent of $T$ is defined by

$$a(T) = \inf\{n : c_n'(T) = 0\} = \inf\{n : N(T^{n+1}) = N(T^n)\}$$

and the essential ascent by

$$a_e(T) = \inf\{n : c'_n(T) < \infty\} = \inf\{n : N(T^{n+1}) = N(T^n)\}.$$

As in Lemmas 3 and 4 it is possible to show that

$$c'_n(T^m) = \sum_{i=0}^{m-1} c'_{nm+i}(T) \qquad (m \geq 1, n \geq 0)$$

and, for commuting $A, B, C, D$ satisfying $AC + BD = I$,

$$\max\{c'_n(A), c'_n(B)\} \leq c'_n(AB) \leq c'_n(A) + c'_n(B)\ .$$

Denote by $\phi_+(X)$ the set of all upper semi-Fredholm operators, in a Banach space $X$, i.e.$\phi_+(X) = \{T \in \mathcal{L}(X) : \dim N(T) < \infty$ and $R(T)$ is closed$\}$.

The dual versions of the regularities $R_1^a, \cdots, R_5^a$ are the following:

$$\begin{aligned}
R_6^a &= \{T \in \mathcal{L}(X) : T \text{ is injective}\},\\
R_7^a &= \{T \in \mathcal{L}(X) : \dim N(T) < \infty \text{ and } a(T) < \infty\},\\
R_8^a &= \{T \in \mathcal{L}(X), \dim N(T) < \infty\},\\
R_9^a &= \{T \in \mathcal{L}(X), a(T) < \infty\},\\
R_{10}^a &= \{T \in \mathcal{L}(X), a_e(T) < \infty\}\ .
\end{aligned}$$

It is easy to see that the sets $R_6^a, \cdots, R_{10}^a$ are regularities, so that the corresponding spectra satisfy the spectral mapping theorem (for locally non-constant analytic functions).

If we consider the topological versions of these regularities, there is a small difference from the dual case — the ranges of operators in $R_6^a, R_7^a$ and $R_8^a$ need not be closed. The dual versions of $R_1, \cdots R_5$ are then:

$$\begin{aligned}
R_6 &= \{T \in \mathcal{L}(X) : T \text{ is bounded below}\},\\
R_7 &= \{T \in \mathcal{L}(X) : T \in \phi_+(X) \text{ and } a(T) < \infty\},\\
R_8 &= \phi_+(X),\\
R_9 &= \{T \in \mathcal{L}(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\},\\
R_{10} &= \{T \in \mathcal{L}(X) : a_e(T) < \infty \text{ and } R(T^{a_e(T)+1}) \text{ is closed}\}.
\end{aligned}$$

Obviously $R_6 \subset R_7 = R_8 \cap R_9 \subset R_8 \cup R_9 \subset R_{10}$.

The following lemma explains the exponents in the definitions of $R_9$ and $R_{10}$ (cf [7]).

**Lemma 7.** *Let $T$ be an operator in a Banach space $X$ with $a_e(T) < \infty$. Then the following two statements are equivalent:*

(1) *there exists $n \geq a_e(T) + 1$ such that $R(T^n)$ is closed,*

(2) *$R(T^n)$ is closed for every $n \geq a_e(T)$.*

**Proof.** (2) ⇒ (1) is trivial.

(1) $\Rightarrow$ (2): Let $n \geq a_e(T)+1$ so that $\dim(N(T^n)/N(T^{n-1})) < \infty$ and let $R(T^n)$ be closed. We prove first that also $R(T^{n-1})$ is closed. To see this, note, that $R(T) + N(T^{n-1}) = T^{-(n-1)}(R(T^n))$ is closed. Further $R(T^n) \cap N(T)$ is closed and it is of finite codimension in $R(T^{n-1}) \cap N(T)$ by Lemma 2, so that $R(T^{n-1}) \cap N(T)$ is closed. By the lemma of Neubauer (see [16], Proposition 2.1.1) we conclude that $R(T^{n-1})$ is closed.

By repeating these considerations we get that $R(T^i)$ is closed for every $i$ with

$$a_e(T) \leq i \leq n.$$

Further, $T|R(T^{n-1})$ is upper semi-Fredholm operator, so that

$$R(T^i) = R\big((T|R(T^{n-1}))^{i-n+1}\big)$$

is closed for every $i \geq n$.

It is easy to see that the sets $R_i$ $(i = 6, \cdots 10)$ are regularities, so that the corresponding spectra $\sigma_i(T) = \{\lambda : T - \lambda \notin R_i\}$ satisfy the spectral mapping theorem (in case of $i = 6, 7, 8$ for all analytic functions; in case of $i = 9, 10$ for analytic functions which are locally non-constant).

Moreover, since intersection of two (or more) regularities is again a regularity, we can obtain the spectral mapping theorem for a large number of combinations of $R_1, \cdots, R_{10}$.

It is easy to see that $T \in \mathcal{L}(X)$ belongs to $R_i(X)$ $(i = 1, \cdots 5)$ if and only if $T^* \in R_{i+5}(X^*)$. Similarly $T \in R_i(X)$ $(i = 6, \cdots, 10)$ if and only if $T^* \in R_{i-5}(X^*)$.

Since the properties $(A), \cdots, (F)$ considered in the previous section are also preserved by taking adjoints, the regularities $R_6 \cdots R_{10}$ satisfy exactly those properties as $R_1, \cdots, R_5$. So the Table 1 remains valid for $R_1, \cdots R_5$ replaced by $R_6, \cdots, R_{10}$.

## IV. Semi-regular, essentially semi-regular and quasi-Fredholm operators

In this section we replace the numbers $c_n(T) = \dim(R(T^n)/R(T^{n+1}))$ and $c'_n(T) = \dim(N(T^{n+1})/N(T^n))$ by the numbers

$$\begin{aligned} k_n(T) &= \dim\big[(R(T) + N(T^{n+1}))/(R(T) + N(T^n))\big] \\ &= \dim\big[(N(T) \cap R(T^n))/(N(T) \cap R(T^{n+1}))\big]. \end{aligned}$$

By Lemma 2, $k_n(T) = c_n(T) - c_{n+1}(T)$ if $c_n(T) < \infty$ and $k_n(T) = c'_n(T) - c'_{n+1}(T)$ if $c'_n(T) < \infty$. On the other hand it is possible that $k_n(T) < \infty$ while both $c_n(T)$ and $c'_n(T)$ are infinite.

We start with an analogue of Lemmas 3 and 4.

**Lemma 8.** *Let $A, B, C, D$ be mutually commuting operators in a Banach space $X$ satisfying $AC + BD = I$ and let $n \geq 0$. Then*
(1) $R(A^nB^n) \cap N(AB) = [R(A^n) \cap N(A)] + [R(B^n) \cap N(B)]$,
(2) $\max\{k_n(A), k_n(B)\} \leq k_n(AB) \leq k_n(A) + k_n(B)$.

**Proof.** (1) We have

$$\begin{aligned} &R(A^nB^n) \cap N(AB) = R(A^n) \cap R(B^n) \cap [N(A) + N(B)] \\ \supset&[R(A^n) \cap R(B^n) \cap N(A)] + [R(A^n) \cap R(B^n) \cap N(B)] \\ =&[R(A^n) \cap N(A)] + [R(B^n) \cap N(B)]. \end{aligned} \tag{1}$$

On the other hand, if $x \in R(A^n) \cap R(B^n) \cap [N(A) + N(B)]$ then $x = y + z$ for some $y \in N(A) \subset R(B^n)$ and $z \in N(B) \subset R(A^n)$. Thus also $y = x - z \in R(A^n)$ and $z = x - y \in R(B^n)$, so that

$$x \in [R(A^n) \cap R(B^n) \cap N(A)] + [R(A^n) \cap R(B^n) \cap N(B)]$$

and we have equality in (1).

(2a) We prove $k_n(A) \leq k_n(AB)$. If $x_1, \cdots, x_m \in R(A^n) \cap N(A)$ where$m > k_n(AB)$ then $B^nx_i \in R(A^nB^n) \cap N(A) \subset R(A^nB^n) \cap N(AB)$ $(i = 1, \cdots, m)$. Thus there exists their non-trivial linear combination

$$\sum_{i=1}^{m} \alpha_i B^n x_i \in R(A^{n+1}B^{n+1}) \subset B^n R(A^{n+1}.$$

So

$$\sum_{i=1}^{m} \alpha_i x_i \in R(A^{n+1}) + N(B^n) \subset R(A^{n+1}).$$

(2b) To prove the second inequality, let

$$x_1, \ldots, x_m \in R(A^nB^n) \cap N(AB) = (R(A^n) \cap N(A)) + (R(B^n) \cap N(B)),$$

where $m > k_n(A) + k_n(B)$. Then there exist $y_i \in R(A^n) \cap N(A)$, $z_i \in R(B^n) \cap N(B)$ such that $x_i = y_i + z_i$ $(i = 1, \ldots, m)$. Thus there exists a non-trivial linear combination such that $\sum_{i=1}^{m} \alpha_i y_i \in R(A^{n+1}) \cap N(A)$ and $\sum_{i=1}^{m} \alpha_i y_i \in R(B^{n+1}) \cap N(AB)$.

**Lemma 9.** *Let $T \in \mathcal{L}(X)$, $n \geq 0$ and $m \geq 1$. Then*

$$\begin{aligned} k_n(T^m) = {}& k_{mn}(T) + 2k_{mn+1}(T) + 3k_{mn+2}(T) + \cdots + mk_{mn+m-1}(T) \\ &+ (m-1)k_{mn+m}(T) + (m-2)k_{mn+m+1}(T) + \cdots + k_{mn+2m-1}(T). \end{aligned}$$

*In particular,*

$$\max_{0 \leq i \leq 2m-1} k_{mn+i}(T) \leq k_n(T^m) \leq m^2 \max_{0 \leq i \leq 2m-1} k_{mn+i}(T).$$

**Proof.** Consider the mapping

$$\hat{T}_j : R(T^j)/R(T^{j+m}) \to R(T^{j+1})/R(T^{j+m+1})$$

induced by $T$. Its kernel is $[(N(T)\cap R(T^j))+R(T^{j+m})]/R(T^{j+m})$ which is naturally isomorphic to $(N(T)\cap R(T^j))/(N(T)\cap R(T^{j+m}))$, see [7], Lemma 2.1 (b). Thus

$$\begin{aligned}&\dim N(\hat{T}_j)=\dim\bigl[(N(T)\cap R(T^j))/(N(T)\cap R(T^{j+m})\bigr]\\ =&\sum_{i=0}^{m-1}\dim\bigl[(N(T)\cap R(T^{j+i}))/(N(T)\cap R(T^{j+i+1}))\bigr]=\sum_{i=0}^{m-1}k_{j+i}(T).\end{aligned}$$

Since the mapping $R(T^{mn})/R(T^{mn+m})\to R(T^{mn+m})/R(T^{mn+2m})$ induced by $T^m$ is the composition of mappings $\hat{T}_{mn+m-1}\hat{T}_{mn+m-2}\cdots\hat{T}_{mn}$ and all these mappings are onto, we have

$$k_n(T^m)=\sum_{j=mn}^{mn+m-1}\dim N(\hat{T}_j)=\sum_{j=mn}^{mn+m-1}\sum_{i=0}^{m-1}k_{j+i}(T)$$

which gives the statement of the lemma.

We now define the classes of operators analogous to $R_1^a,\cdots,R_5^a$:

**Notation.** Let $X$ be a Banach space. Denote

$$\begin{aligned}&R_{11}^a=\{T\in\mathcal{L}(X):k_n(T)=0\ \text{ for every }\ n\in\mathbb{N}\},\\ &R_{12}^a=\{T\in\mathcal{L}(X):\sum_{i=0}^{\infty}k_i(T)<\infty\},\\ &R_{13}^a=\{T\in\mathcal{L}(X):k_n(T)<\infty\ \text{ for every }\ n\in\mathbb{N}\},\\ &R_{14}^a=\{T\in\mathcal{L}(X):\text{ there exists }\ d\in\mathbb{N}\ \text{ such that }k_n(T)=0\ \ (n\ge d)\}\\ &R_{15}^a=\{T\in\mathcal{L}(X):\text{ there exists }\ d\in\mathbb{N}\ \text{ such that }\ k_n(T)<\infty\ \ (n\ge d)\}.\end{aligned}$$

The condition in $R_{11}^a$ means that

$$N(T)=N(T)\cap R(T)=N(T)\cap R(T^2)=\cdots=N(T)\cap R^\infty(T)$$

so that $R_{11}^a=\{T:N(T)\subset R^\infty(T)\}$.

Similarly $\sum_{i=0}^\infty k_i(T)<\infty$ means that there is $d\in\mathbb{N}$ such that

$$N(T){=}N(T)\cap R(T){=}N(T)\cap R(T^2){=}\cdots{=}N(T)\cap R(T^d)=N(T)\cap R^\infty(T),$$

so that $R_{12}^a=\{T:N(T){\subset}R^\infty(T)\}$ . These or similar conditions were studied by many authors, see e.g. [5], [7], [10], [11], [17], [20], [21], [23].

Operators $T\in R_{14}^a$ were called in [7] “Operators with eventually uniform descent”. The condition defining $R_{14}^a$ can be rewritten as $N(T)\cap R(T^d)=N(T)\cap R^\infty(T)$ and it was studied also in connection with quasi-Fredholm operators, see [16].

The condition in $R_{13}^a$ can be rewritten as $N(T^m){\subset}R(T^n)$ for all $m,n\in\mathbb{N}$. This condition appeared implicitly already in [20]. The conditions in $R_{15}^a$ probably has not been considered yet.

It follows from Lemmas 8 and 9 that the sets $R^a_{11} \cdots R^a_{15}$ are regularities, so that the corresponding spectra satisfy the spectral mapping theorem (for locally non-constant analytic functions).

Before we introduce the topological version of $R^a_{11}, \cdots, R^a_{15}$ we state several simple lemmas.

**Lemma 10.** *Let $T \in \mathcal{L}(X)$ and let $m \ge 0$, $n \ge i \ge 1$. If $R(T^n) + N(T^m)$ is closed then $R(T^{n-i}) + N(T^{m+i})$ is closed.*

**Proof.** It is sufficient to show that

$$R(T^{n-i}) + N(T^{m+i}) = T^{-i}[R(T^n) + N(T^m)]. \tag{2}$$

The inclusion $\subset$ is clear. Conversely, suppose that $T^i z \in R(T^n) + N(T^m)$, so that $T^i z = T^n x + u$ for some $x \in X$ and $u \in N(T^m)$. Then $u \in R(T^i)$, so that $u = T^i v$ for some $v \in N(T^{m+i})$. Then $z - T^{n-i}x - v \in N(T^i)$, so that $z \in R(T^{n-i}) + N(T^{m+i}) + N(T^i) = R(T^{n-i}) + N(T^{m+i})$ and we have equality in (2).

**Lemma 11.** *Let $T \in \mathcal{L}(X)$ and let $n \ge 0$.*

*If $R(T^n)$ is closed and $R(T) + N(T^n)$ is closed then $R(T^{n+1})$ is closed.*

**Proof.** Let $u_j \in X$ $(j = 1, 2, \cdots)$ and let $T^{n+1}u_j \to z$ as$j \to \infty$. Then $z \in R(T^n)$, $z = T^n u$ for some $u \in X$ and $T^n(u - Tu_j) \to 0$.

Consider the operator $\widehat{T^n} : X/N(T^n) \to X$ induced by $T^n$.

Clearly $\widehat{T^n}$ is injective and has closed range, therefore it is bounded below and $\widehat{T^n}(u - Tu_j + N(Tn)) \to 0$ $(j \to \infty)$ implies $u - Tu_j + N(T^n) \to 0$ in $X/N(T^n)$. Thus there are elements $v_j \in N(T^n)$ such that $Tu_j + v_j \to u \in R(T) + N(T^n)$. Hence $z \in R(T^{n+1})$.

**Lemma 12.** (cf. [7], Theorem 3.2). *Let $T \in \mathcal{L}(X)$, $d \in \mathbb{N}$ and let $k_i(T) < \infty$ for every $i \ge d$. Then the following statements are equivalent:*

(1) *there exists $n \ge d + 1$ such that $R(T^n)$ is closed,*
(2) *$R(T^n)$ is closed for every $n \ge d$,*
(3) *$R(T^n) + N(T^m)$ is closed for all $m, n$ with $m + n \ge d$.*

**Proof.** Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1). The implication (2) $\Rightarrow$ (3) follows from Lemma 10.

(1) $\Rightarrow$ (2) : If $R(T^n)$ is closed then, by Lemma 10, $R(T) + N(T^{n-1})$ is closed. Since $R(T) + N(T^{n-1}) \subset R(T) + N(T^n) \subset \cdots$ we get that $R(T) + N(T^i)$ is closed for every $i \ge n$. Thus by Lemma 11 we get inductively that $R(T^i)$ is closed for every $i \ge n$.

To show that $R(T^i)$ is closed for every $i$, $d \le i \le n$ we can proceed exactly as in the proof of Lemma 7.

**Notation.** We denote

$$R_{11} = \{T \in \mathcal{L}(X) : N(T) \subset R^\infty(T) \text{ and } R(T) \text{ is closed}\},$$

$$R_{12} = \{T \in \mathcal{L}(X) : N(T) \subset R^\infty(T) \text{ and } R(T) \text{ is closed}\},$$

$$R_{13} = \{T \in \mathcal{L}(X) : k_n(T) < \infty \text{ for every } n \in \mathbb{N} \text{ and } R(T) \text{ is closed}\},$$

$$R_{14} = \{T \in \mathcal{L}(X) : \text{ there exists } d \in \mathbb{N} \text{ such that } R(T) + N(T^d) \\ = R(T) + N^\infty(T) \text{ and } R(T^{d+1}) \text{ is closed}\},$$

$$R_{15} = \{T \in \mathcal{L}(X) : \text{ there exists } d \in \mathbb{N} \text{ such that } \\ k_n(T) < \infty \ (n \ge d) \text{ and } R(T^{d+1}) \text{ is closed}\}.$$

Clearly $R_{11} \subset R_{12} = R_{13} \cap R_{14} \subset R_{13} \cup R_{14} \subset R_{15}$,$R_1 \cup R_6 \subset R_{11}$, $R_2 \cup R_7 \subset R_3 \cup R_8 \subset R_{12}$, $R_4 \cup R_9 \subset R_{14}$ and $R_5 \cup R_{10} \subset R_{15}$.

It is easy to see that the sets $R_{11} \cdots R_{15}$ are regularities.

Let $\sigma_i$ $(i = 11, \cdots, 15)$ be the corresponding spectra defined by $\sigma_i(T) = \{\lambda : T - \lambda \notin R_i\}$. If $X = X_1 \oplus X_2$ is a decomposition of $X$ with closed $X_1, X_2$ and if $T_1 \in \mathcal{L}(X_1)$,$T_2 \in \mathcal{L}(X_2)$ then

$$\sigma_i(T_1 \oplus T_2) = \sigma_i(T_1) \cup \sigma_i(T_2) \qquad (i = 11, \cdots, 15)\ .$$

Since $\sigma_{11}(T_1) \neq \emptyset \Leftrightarrow X_1 \neq \{0\}$, and for $i = 12, 13$, $\sigma_i(T_1) \neq \emptyset \Leftrightarrow \dim X_1 = \infty$ (see below), we have the following spectral mapping theorems:

**Theorem 13.** *Let $T \in \mathcal{L}(X)$ and let $f$ be a function analytic on a neighbourhood of $\sigma(T)$. Then*

$$\sigma_i(f(T)) = f(\sigma_i(T) \qquad (i = 11, 12, 13).$$

*If $f$ is non-constant on each component of its domain of definition then*

$$\sigma_i(f(T)) = f(\sigma_i(T)) \qquad (i = 14, 15).$$

**Remark.** The operators of class $R_{11}$ and $R_{12}$ will be called semi-regular and essentially semi-regular, respectively. These classes are well-known and so is the spectral mapping theorem for the corresponding spectra, see [1], [17], [19], [24] and [20], [21].

The operators of class $R_{14}$ will be called quasi-Fredholm. In case of Hilbert space operators this definition coincides with the definition of Labrousse [16]. The spectral mapping theorem for $\sigma_{14}$ in Hilbert space case was proved in [2]. For Banach space operators the definition of quasi-Fredholm operators is new and so are, as far as we know, the classes $R_{13}$ and $R_{15}$.

**Example 14.** A typical example of an operator of class $R_{13}$ is the operator

$$S = \bigoplus_{n=1}^{\infty} S_n \in \mathcal{L}\Big(\bigoplus_{n=1}^{\infty} H_n\Big),$$

where $H_n$ is an n-dimensional Hilbert space and $S_n$ is a shift in $H_n$. In this case $k_n(S) = 1$ for every $n$.

The properties $(A)-(F)$ for regularities $R_{11},\dots,R_{15}$ are summarized in the following table:

| | (A)<br>$\sigma_i \neq \emptyset$ | (B)<br>$\sigma_i$closed | (C)<br>small commut. perturbations | (D)<br>finite dim. perturbations | (E)<br>commut.comp. perturbations | (F)<br>commut. quasinilp. pert. |
|---|---|---|---|---|---|---|
| $R_{11}$<br>semi-reg | yes | yes | yes | no | no | yes |
| $R_{12}$<br>ess.s-reg. | yes | yes | yes | no | yes | yes |
| $R_{13}$ | yes | ? | ? | yes | ? | ? |
| $R_{14}$<br>$q\phi$ | no | yes | no | yes | no | no |
| $R_{15}$ | no | ? | no | yes | no | no |

Tab.2

**Comments.** 1) It is well-known that $\sigma_{11}(T), \sigma_{12}(T)$ are closed for every $T \in \mathcal{L}(X)$, $\sigma_{11}(T) \supset \partial\sigma(T)$, $\sigma_{12}(T) \supset \partial\sigma_e(T)$, so that both spectra are non-empty (for infinite dimensional Banach spaces). Here $\sigma_e$ denotes the essential spectrum, $\sigma_e(T) = \{\lambda : T-\lambda \text{ is not Fredholm}\}$.

For property (C) for $R_{11}$ and $R_{12}$ see [15].

2) Since $0 \in R_{14}, R_{15}$, one can see easily that (A), (C), (E) and (F) fail for $R_{14}$ and $R_{15}$.

3) Observation 2 after Table 1 shows that (D) and (E) fail for semi-regular operators.

4) As in observation 8, one can see easily that $R_{13}$ and $R_{15}$ are closed under finite dimensional perturbations. For essentially semi-regular operators this was proved in [13], for quasi-Fredholm operators this will be showed below. Also the non-emptiness of $\sigma_{13}$ will be proved below.

5) Semi-regular and essentially semi-regular operators are stable under commuting quasinilpotent perturbations by [14].

6) The stability of essentially semi-regular operators under commuting compact perturbations was shown in [7], Theorem 5.9. By Theorem 4.7 of the same paper $\sigma_{14}(T)$ is closed (moreover $R_{11}(T) \setminus R_{14}(T)$ consists of at most countable many isolated points).

The boxes marked by ? represent open problems. Especially interesting question is whether $R_{13}(T)$ is closed (our conjecture is yes).

Note also that Tables 1 and 2 (as far as is as filled in) are quite similar, with only two differences.

We finish with the two promised results:

**Theorem 15.** *Let $T \in \mathcal{L}(X)$ be a quasi-Fredholm operator (i.e. $T \in R_{14}$) and let $F \in \mathcal{L}(X)$ be a finite dimensional operator. Then $T+F$ is also quasi-Fredholm.*

**Proof.** Clearly if is sufficient to consider only the case of $\dim R(F) = 1$.

Since $R((T+F)^n){=}R(T^n)$ for every $n$ (see observation 8), $R((T+F)^n)$ is closed if and only if $R(T^n)$ is closed and it is sufficient to show only the algebraic condition of $R_{14}$ for $T+F$.

Since $T$ is quasi-Fredholm, there exists $d \in \mathbb{N}$ such that $N(T) \cap R(T^d) \subset R^\infty(T)$ and $R(T^d), R(T^{d+1})$ are closed. Denote $M = R(T^d)$ and $T_1 = T|M$. Then $N(T_1) = N(T) \cap R(T^d) \subset R^\infty(T) = R^\infty(T_1)$ so that $T_1$ is semi-regular.

It is sufficient to show that $N(T_1) \subset R^\infty(T+F)$. Indeed, since

$$N(T_1) = N(T) \cap R(T^d){=}N(T+F) \cap R((T+F)^d),$$

we have

$$N(T+F) \cap R((T+F)^d){\subset}R^\infty(T+F),$$

i.e., $N(T+F) \cap R((T+F)^d){=}N(T+F) \cap R^\infty(T+F)$.

This means that $N(T+F) \cap R((T+F)^n) = N(T+F) \cap R^\infty(T+F)$ for some $n \geq d$.

Let $x_0 \in N(T_1)$. We prove the following statement:

(a) For every $n$ there exist vectors $x_1, \dots, x_n \in R^\infty(T_1)$ such that $Tx_i = x_{i-1}$ and $Fx_i = 0 \quad (i = 1, \dots, n)$.

If (a) is proved then of course

$$(T+F)^n x_n = (T+F)^{n-1} x_{n-1} = \ldots = (T+F)x_1 = x_0,$$

so that $x_0 \in R(T+F)^n$ for every $n$. Thus $N(T_1) \subset R^\infty(T+F)$ and the theorem is proved.

We prove (a) by induction on $n$. For $n = 0$ the statement is trivial. Suppose (a) is true for $n$, i.e. there are vectors$x_1, \dots, x_n \in R^\infty(T_1)$ such that $Tx_i = x_{i-1}$ and $Fx_i = 0$ $(i = 1, \dots, n)$. Since $T_1$ is semi-regular, we can find $x_{n+1} \in R^\infty(T_1)$ such that $T_1 x_{n+1} = x_n$.

If $Fx_{n+1} = 0$ then we have statement (a) for $n+1$. Let $Fx_{n+1} \neq 0$.

Let $k$ be the smallest integer with the property $N(T_1^k) \not\subset N(F)$ (clearly $k \leq n+1$ since $x_{n+1} \in N(T_1^{n+1}) \setminus N(F)$). Since $F$ is one-dimensional, we can find $z \in N(T_1^k) \subset R^\infty(T_1)$ such that $F(x_{n+1} - z) = 0$. Set

$$\begin{aligned} &x'_{n+1} = x_{n+1} - z, \ x'_n = T_1 x'_{n+1}, \ x'_{n-1} = T_1^2 x'_{n+1}, \dots \\ \dots, &x'_{n+1-k} = T_1^k x'_{n+1} = T_1^k x_{n+1} = x_{n+1-k}, \ x'_{n-k} = x_{n-r}, \dots, x'_1 = x_1. \end{aligned}$$

Clearly $x'_i \in R^\infty(T_1)$, $T_1 x'_i = x'_{i-1} \quad (i = 1, \dots, n+1)$, $Fx'_{n+1} = 0$ and $Fx_i = 0$ for $1 \le i \le n+1-k$. If $n+2-k \le i \le n$ then $Fx'_i = F(x'_i - x_i) + Fx_i = F(x'_i - x_i) = 0$ since $x'_i - x_i \in N(T_1^{k-1})$, see the definition of $k$.

This finishes the proof of (a) and also of the theorem.

**Theorem 16.** *Let $T \in \mathcal{L}(X)$. Then $\partial\sigma_e(T) \subset \sigma_{13}(T)$.*

**Proof.** We use the construction of Sadoskii [22], see also [3]. Denote by $l^\infty(X)$ the Banach space of all bounded sequences of elements of $X$ with the sup-norm and let $J(X)$ be the closed subspace of $l^\infty(X)$ consisting of all sequences $\{x_n\}_{n=1}^\infty$ such that the set $\{x_n : n = 1, 2, \dots\}$ is precompact. Denote $P(X) = l^\infty(X)/J(X)$.

An operator $T$ defines pointwise an operator$T^\infty : l^\infty(X) \to l^\infty(X)$ such that $T^\infty J(X) \subset J(X)$, so that we can define naturally an operator $P(T) : P(X) \to P(X)$. For properties of the functor $P$ see [3], [4] or [22].

Let $T \in R_{13}$. Then $R(T^n)$ is closed for every $n$, so that $R((P(T))^n) = R(P(T^n))$ is closed.

It is easy to verify that $N(P(T)) = l^\infty(N(T)) + J(X)$ and

$$R(P(T^n)) = l^\infty(R(T^n)) + J(X).$$

Since $\dim\big[N(T)/(N(T) \cap R(T^n))\big] < \infty$ for every $n$, we have

$$N(P(T)) = l^\infty(N(T) \cap R(T^n)) + J(X) \subset N(P(T)) \cap R(P(T^n)),$$

so that $P(T)$ is semi-regular.

If $\lambda \in \partial\sigma_e(T) = \partial\sigma(P(T))$, then$P(T) - \lambda I_{P(X)} = P(T - \lambda I_X)$ is not semi-regular,so that $T - \lambda I_X \notin R_{13}(X)$.

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