On the Base Problem for a Compact Integral Operator

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1. Introduction. The method of intermediate problems gives lower bounds for the eigenvalues of differential operators. This method was introduced by Weinstein in the theory of plates and other classical problems in mechanics, and was profoundly investigated by Aronszajn. Its range of applicability has been gradually expanded and Bazley, who was later joined by Fox, showed that it can easily be extended to problems in quantum mechanics. For recent expositions of the theory see the books of S. H. Gould [1] and G. Fichera [2] which contain detailed bibliographies. In classical mechanics the inverse operator is a compact integral operator (a Green's operator) and therefore of theoretical interest for questions such as existence, convergence, *etc.* However, as this Green's function is not in general explicitly known in two or more dimensions, the reciprocal eigenvalue problem for the corresponding integral equation cannot be used for the computation of upper bounds for eigenvalues. This fact, already mentioned by Trefftz in his paper on bounds for the eigenvalues of integral operators [5], and also noted in [3], [4], did not prevent in any way the successful application of the method of intermediate problems directly to the differential operator in one or many dimensions. As to the very important branch of intermediate problems concerning Schroedinger-type operators in quantum theory, there is no compact inverse operator at all, so the connection with integral equations does not occur.

It should be noted that in a recent publication [2], Fichera in a brilliant way overcame the difficulty just mentioned about the Green's functions in classical problems by giving a new representation of Green's operator which is of great interest by itself, but moreover, has proved to be a strong weapon in Fichera's work on the eigenvalues of compact operators having finite orthogonal invariants. In fact, starting with the fundamental solution, he uses projections to construct explicitly known intermediate operators. This part of his procedure is reminis-

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cent of the use of projections in the construction of the first type of intermediate problems introduced by Weinstein in 1935–37. Let us note that once explicit integral operators are obtained, Fichera uses a procedure which is no longer connected with the theory of intermediate problems.

In the publications [9], [2], [4] Fichera states that the method of intermediate problems has a serious limitation in its applicability to integral operators because we do not know how to construct a solvable integral base operator. The existence of a base problem which will be precisely defined in the next section, is a prerequisite to the applicability of the intermediate problems. Fichera's statements, which seemingly exclude explicit integral operators from the range of applicability of intermediate problems, stimulated the following investigation which should be of interest in its own right even though the alleged serious limitation has not made itself noticeable. The results given here, which obviously would apply to any compact operator will be for the most part formulated in the more explicit and important case of Hilbert–Schmidt integral operators. These results can also be interpreted as a modification of E. Schmidt's classical approximations by operators of finite rank by the use of the recent concept of truncated operators. In this way we shall show that while there may be no known compact base operator, there are known non-compact base operators which make it possible to apply the method.

2. Some properties of the classical E. Schmidt approximation. Let the symmetric kernel $K(s, t)$ belong to $\mathfrak{L}^2[(0, 1) \times (0, 1)]$ and consider the following eigenvalue problem on the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(0, 1)$,

$$\int_0^1 K(s, t)u(t)\, dt = \lambda u(s)$$

or using operator notation $Ku = \lambda u$.

For simplicity we assume that K is non-negative. Without loss of generality we denote by $||K||$ the maximum of (Ku, u) for $(u, u) = 1$. Then we have the eigenvalues $||K|| = \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq 0$ and the corresponding orthonormal set of eigenfunctions $u_1, u_2, \cdots$. The Rayleigh–Ritz method, see also [6], gives for non-negative operators lower bounds for the eigenvalues. In order to obtain upper bounds by the method of intermediate problems all we have to do, following the general theory, is to construct a base operator A° for which we know the eigenvalues and eigenfunctions and which is greater than K, so that $K = A^\circ + B^\circ$ where B° is strictly negative. It follows from the classical results of Weyl that the eigenvalues of A° are greater than the eigenvalues of K. By the method of intermediate problems we can improve these rough upper bounds. Let us note in passing that the converse is not true. A solvable operator A° whose eigenvalues are greater than the eigenvalues of K is not necessarily a base operator for which there are intermediate problems. In order to construct A° we proceed as follows. Let $v_1(s), v_2(s), \cdots$ be any complete orthonormal system for $\mathfrak{L}^2(0, 1)$. Then $\{v_i(s)\, v_j(t)\}$ is a complete orthonormal system for $\mathfrak{L}^2[(0, 1) \times$

(0, 1)]. Let

$$c_{ij} = \int_0^1 \int_0^1 K(s, t)v_i(s)v_j(t)\, ds\, dt$$

be the Fourier coefficients of $K(s, t)$. Let us put

$$A_n(s, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}v_i(s)v_j(t).$$

Then $A_n(s, t)$ is a degenerate symmetric kernel in the sense of E. Schmidt and therefore its eigenvalues and eigenfunctions are explicitly known. We first show that the operator A_n is non-negative as follows. For any $u \in \mathfrak{H}$,

$$u = \sum_{i=1}^{n} x_i v_i + \sum_{i=n+1}^{\infty} x_i v_i = u^* + u^{**}.$$

Obviously we have

$$A_n u^* = \sum_{i,j=1}^{n} c_{ij}x_i v_j$$

and $A_n u^{**} = 0$.

Therefore $(A_n u^*, u^{**}) = 0$, so that $(A_n u, u) = (A_n u^*, u^*)$. Furthermore

$$(Ku^*, u^*) = \int_0^1 \int_0^1 K(s, t) \sum_{i,j=1}^{n} x_i x_j v_i(s)v_j(t)\, ds\, dt = \sum_{i,j=1}^{n} c_{ij}x_i x_j = (A_n u^*, u^*),$$

so we have

$$(A_n u, u) = (A_n u^*, u^*) = (Ku^*, u^*) \geqq 0.$$

Therefore the spectrum of A_n consists of n or less positive eigenvalues $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n \geqq 0, 0, \cdots$, with the eigenvalue 0 having infinite multiplicity. We denote a corresponding orthonormal set of eigenfunctions by $u_1^\circ, u_2^\circ, \cdots, u_n^\circ, u_{n+1}^\circ, u_{n+2}^\circ, \cdots$. All of these quantities of course depend upon the choice of n which will be fixed later. (We shall omit the index n whenever the notations become too cumbersome.) They can be determined explicitly, following E. Schmidt, in an elementary way. In fact, the non-trivial eigenvalues are the roots of the equation

$$\det\{(Kv_i, v_j) - \mu\delta_{ij}\} = 0, \qquad i, j = 1, 2, \cdots, n. \tag{1}$$

Remark. D. W. Fox, to whom we communicated the preceding results, kindly pointed out to us the seemingly new property of the special E. Schmidt approximation used here, namely that the eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ are Rayleigh–Ritz lower bounds for the first n eigenvalues of K. In fact, the determinant (1) is the same one used in the Rayleigh–Ritz method.

3. Construction of a family of non-compact base operators. Let us now put $K(s, t) = A_n(s, t) + B_n(s, t)$ and correspondingly $K = A_n + B_n$. Let us note

that for any function f, $(f, f) = 1$, we have (Rayleigh's Principle) $||K|| \geqq (Kf, f)$. We choose f such that (Kf, f), which we denote by R, is positive. Since $A_n(s, t)$ is the Fourier approximation of $K(s, t)$, the following inequalities are satisfied for a proper choice of $n \geqq 1$:

$$||B_n||^2 \leqq \int_0^1 \int_0^1 [B_n(s, t)]^2 \, ds \, dt < R^2 \leqq ||K||^2.$$

Therefore we can choose a number $c_n > 0$ such that

$$\int_0^1 \int_0^1 [B_n(s, t)]^2 \, ds \, dt < c_n^2 < R^2$$

which implies that

$$||B_n|| < c_n < ||K||. \tag{2}$$

Now let us write

$$K = (A_n + c_n I) + (B_n - c_n I).$$

Since $B_n^o = B_n - c_n I$ is strictly negative, the operator $A_n^o = A_n + c_n I$ is strictly greater than K. The operator A_n^o is a non-compact operator with known eigenvalues $\lambda_1^o \geqq \lambda_2^o \geqq \cdots$ where $\lambda_i^o = \mu_i + c_n$ for $i = 1, 2, \cdots, n$ and $\lambda_i^o = c_n$ for $i = n + 1, n + 2, \cdots$. The spectral formula for A_n^o is

$$A_n^o u = \sum_{i=1}^{n} \lambda_i^o (u, u_i^o) u_i^o + c_n \left[u - \sum_{i=1}^{n} (u, u_i^o) u_i^o \right]. \tag{3}$$

From this representation it is clear that the operator A_n^o is of the same type as a truncated operator in that it has a finite number of eigenvalues of finite multiplicity followed by an eigenvalue of infinite multiplicity. The theory of truncated operators was developed by Weinberger and Bazley and Fox. See the books [1] and [2], to which we shall refer the reader. There is, however, an essential difference. In their theory, given an operator T^o having infinitely many eigenvalues and eigenfunctions which for the moment we also denote by $\lambda_1^o, \lambda_2^o, \cdots$ and $u_1^o, u_2^o, \cdots$ the truncated operator T_n^o is defined by the same formula (3) as A_n^o with the only change being that c_n is replaced by λ_{n+1}^o. If the index n is increased, the first terms in the truncated operator remain the same, while in our case, for every index n, we have to recompute c_n as well as the eigenvalues $\lambda_1^o, \lambda_2^o, \cdots, \lambda_n^o$ and eigenfunctions $u_1^o, u_2^o, \cdots, u_n^o$ which all depend on n because they are, up to the addition of c_n, essentially the eigenvalues and eigenfunctions of the degenerate kernel A_n. In view of the remark by Fox, these computations for greater values of n would lead at the same time automatically to improved Rayleigh–Ritz lower bounds. It is now obvious that for every n for which (2) holds, the operator A_n^o can be used as a base operator and that the method of intermediate problems applies in exactly the same manner as in the cases treated by Bazley and Fox in which a truncated operator served as a base operator. Since our constructions are explicit we have reached the result that

the method of intermediate problems can be applied theoretically and practically to a given operator. Even if it turns out that there is no known way of constructing an integral operator which could serve in our case as a base operator, we have proved that there is always a family of non-compact operators which can be used for the same purpose. The use of non-compact operators was emphasized in our short abstract presented to the International Congress of Mathematicians, 1966.

4. Convergence. Let us note that by the results of Bazley and Fox, also mentioned in [1] and [2, p. 128], the eigenvalues of the intermediate problems converge to the eigenvalues of the given integral operator if $c_n \to 0$ for $n \to \infty$ and at the same time the index of the intermediate problem tends to infinity.

Remark. W. Stenger kindly communicated to me the observation that in the present case the proof of convergence can be simplified in the following way. In fact, we have

$$||K - A_n^\circ|| = ||K - A_n - c_n I|| \leqq ||K - A_n|| + c_n .$$

Since A_n is the Fourier approximation to K it follows that as $n \to \infty$ both $||K - A_n|| \to 0$ and $c_n \to 0$. Therefore $||K - A_n^\circ|| \to 0$, which is sufficient to establish the convergence of the eigenvalues λ_i° of A_n°, or to be more precise we write $\lambda_i^\circ(n) = \lambda_i^\circ$, to the eigenvalues of K, *i.e.* $\lambda_i^\circ(n) \to \lambda_i$ as $n \to \infty$ for all i. This in turn proves the convergence of the eigenvalues of the intermediate operators, which are no greater than the eigenvalues of A_n°, to the eigenvalues of K.

It may not be superfluous to point out that after the degenerate kernel $A_n(s, t)$ has been obtained our procedure of intermediate problems is completely different from the classical method of E. Schmidt for the solution of the given integral equation.

From the preceding discussion it is obvious that any decomposition of $K = A_n + B_n$ would be useful provided A_n is of finite rank and that $||B_n||$ is small. For instance if $K(s, t)$ (or an iterate of $K(s, t)$) is continuous we may use a Weierstrass-type approximation.

5. Numerical example. Let us now illustrate our procedure on the following integral operator. We shall use here the theory of intermediate problems for which we refer to the books of Fichera [2] and Gould [1]. Let

$$K(s, t) = e^{st} = \sum_{j=0}^{\infty} \frac{(st)^j}{j!} , \qquad 0 \leqq s, t \leqq 1.$$

We have

$$\int_0^1 \int_0^1 K(s, t)u(s)u(t)\, ds\, dt = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\int_0^1 s^j u(s)\, ds \right)^2$$

which is strictly positive for $u \neq 0$ in the $\mathfrak{L}^2$ sense. We have by Rayleigh–Ritz, using a constant as a test function,

$$||K|| \geqq \int_0^1 \int_0^1 e^{st}\, ds\, dt \geqq 1.$$

Using the obvious decomposition of $K(s, t)$ we put for $n = 1, 2, \cdots$

$$A_n(s, t) = \sum_{j=0}^{n-1} \frac{(st)^j}{j!}$$

and

$$B_n(s, t) = (st)^n \sum_{j=0}^{\infty} \frac{(st)^j}{(j+n)!}. \tag{4}$$

To get an upper bound on $||B_n||$ we note that

$$0 \leqq B_n(s, t) \leqq (st)^n e^{st} \leqq (st)^n e.$$

Therefore

$$||B_n||^2 \leqq \int_0^1 \int_0^1 [B_n(s, t)]^2\, ds\, dt \leqq e^2 \left(\int_0^1 s^{2n}\, ds \right)^2 = \frac{e^2}{(2n+1)^2} < ||K||.$$

Let us take the simplest case of $n = 1$. In this case $c = .9334$ satisfies the condition (2). Then using our previous notation we have $\mu_1^\circ = 1$ and $\lambda_1^\circ = 1.9334$ and $u_1^\circ(s) = 1$. We have in this way an acceptable rough upper bound for λ_1. Instead of taking better base problems by using $n = 2, 3, \cdots$, we improve the upper bound for λ_1 by using the first intermediate problem with the base problem corresponding to $n = 1$. To simplify the notation let $A = A_1^\circ$ and $B = B_1^\circ$. By the general theory we form the first intermediate problem by selecting a function $p_1(s)$ or for short $p(s)$ which satisfies the condition $(Bp, u_1^\circ) \neq 0$, which is a distinguished choice in the sense of Weinstein, and we write the resolvent of Bp as follows:

$$R_\lambda Bp = \frac{(Bp, u_1^\circ)}{\lambda_1 - \lambda} u_1^\circ + \frac{1}{c - \lambda} [Bp - (Bp, u_1^\circ) u_1^\circ].$$

Then we set the usual determinant, which in this case is one by one, equal to zero. In this way we obtain the equation

$$(Bp, p) + (R_\lambda Bp, Bp) = 0. \tag{5}$$

Since we have taken a distinguished choice for p, λ_1° is not a persistent eigenvalue. Therefore the intermediate eigenvalue $\lambda_1^1 (< \lambda_1^\circ)$ will be the unique root of the equation (5) in the interval $c < \lambda < \lambda_1^\circ$. Referring to equation (4) for $n = 1$ and selecting the distinguished choice $p(s) = 1$ we obtain by elementary calculations $\lambda_1^1 = 1.5285$ which gives an improved upper bound for λ_1. It is clear that this bound, as well as the Rayleigh–Ritz lower bound, could be indefinitely

improved if a need would arise as has been done in so many differential problems [7].

A short summary of some of the results of this paper appeared in the American Mathematical Society Notices, Volume 13, 1966, p. 861.

Bibliography

[1] Gould, S. H., *Variational Methods for Eigenvalue Problems. An Introduction to the Weinstein Method of Intermediate Problems*, second edition, University of Toronto Press, Toronto, 1966.

[2] Fichera, G., *Linear Elliptic Differential Systems and Eigenvalue Problems*, Springer-Verlag, New York, 1965.

[3] Fichera, G., *Sul Calcolo degli Autovalori, Atti del Simposio "Applicazioni dell' Analisi alla Fisica Matematica"*, Cagliari-Sassari, 1964, Ed. Cremonese, Roma, 1965.

[4] Fichera, G., *Approximation and Estimates for Eigenvalues, Symposium on the Numerical Solution of Partial Differential Equations*, Academic Press, New York, 1966.

[5] Trefftz, E., Über Fehlerschätzung bei Berechnung von Eigenwerten, *Math. Ann.*, **108** (1933).

[6] Stenger, W., On Poincaré's Bounds for Higher Eigenvalues, *Bull. Amer. Math. Soc.*, **72** (1966).

[7] Weinstein, A., *Some Numerical Results in Intermediate Problems for Eigenvalues, Symposium on the Numerical Solution of Partial Differential Equations*, Academic Press, New York, 1966.

[8] Diaz, J. B., *Upper and Lower Bounds for Eigenvalues*, Proceedings of the Eighth Symposium in Applied Mathematics of the American Mathematical Society, held at the University of Chicago, April 12–13, 1956; American Mathematical Society, Providence, Rhode Island, 1958, pp. 53–78.

[9] Fichera, G., On the Compactness of the Base Operator in the Theory of Intermediate Problems, *Rendiconti della Classe di Scienzefisiche, Mathematiche e Naturali*, Accademia Nazionale dei Lincei, Roma, **41** (1966).

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