TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 299, Number 1, January 1987

ON THE BEHAVIOR NEAR THE CREST OF WAVES OF EXTREME FORM

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ABSTRACT. The angle ϕ which the free boundary of an extreme wave makes with the horizontal is the solution of a singular, nonlinear integral equation that does not fit (as far as we know) into the theory of compact operators on Banach spaces. It has been proved only recently that solutions exist and that (as Stokes suggested in 1880) these solutions represent waves with sharp crests of included angle $2\pi/3$. In this paper we use the integral equation, known properties of solutions and the technique of the Mellin transform to obtain the asymptotic expansion

$$(*) \qquad \phi(s) = \frac{\pi}{6} + \sum_{n=1}^{k} a_n s^{\mu_n} + o(s^{\mu_k}) \quad \text{as } s \downarrow 0,$$

to arbitrary order; the coordinate s is related to distance from the crest as measured by the velocity potential rather than by length. The first few (and probably all) of the exponents μ_n are transcendental numbers. We are unable to evaluate the coefficients a_n explicitly, but define some in terms of global properties of ϕ, and the others in terms of earlier coefficients. It is proved in **[8]** that $a_1 < 0$, and follows here that $a_2 > 0$. The derivation of $(*)$ includes an assumption about a question in number theory; if that assumption should be false, logarithmic terms would enter the series at very large values of n.

1. Introduction. This paper concerns gravity waves, of permanent and extreme form, on the free surface of an ideal liquid, the flow being two-dimensional, irrotational, and in a vertical plane. By a wave of extreme form we mean one that is the "largest" member of a one-parameter family of such waves, and is characterized by a sharp crest of included angle $2\pi/3$, as shown in Figure 1(a). The existence of such waves was conjectured by Stokes in 1880 and has been proved recently. (For a fuller account, see the introduction to **[1]**.)

Letting ϕ denote the local wave angle (that is, the angle which the free boundary makes with the horizontal; $\phi(s) = \tan^{-1} Y'(x)$ in the notation of Figure 1), we shall seek the behavior as $s \to 0$ of a solution ϕ of the equation

$$(1.1) \qquad \phi(s) = \frac{1}{3}\int_0^\pi K(s,t)\frac{v(t)\sin\phi(t)}{\int_0^t v\sin\phi}\,dt, \qquad 0 < s \leqslant \pi,$$

Received by the editors April 24, 1985 and, in revised form, September 23, 1985.
1980 *Mathematics Subject Classification*. Primary 76B15; Secondary 45G05.
Key words and phrases. Water waves, nonlinear integral equations, asymptotic analysis.
[1]Part of this work was done while both authors were visiting the Mathematics Research Center, University of Wisconsin, Madison, and part while L. E. Fraenkel was a visitor to the Mathematics Department and to the Mathematical Sciences Research Institute, University of California, Berkeley.

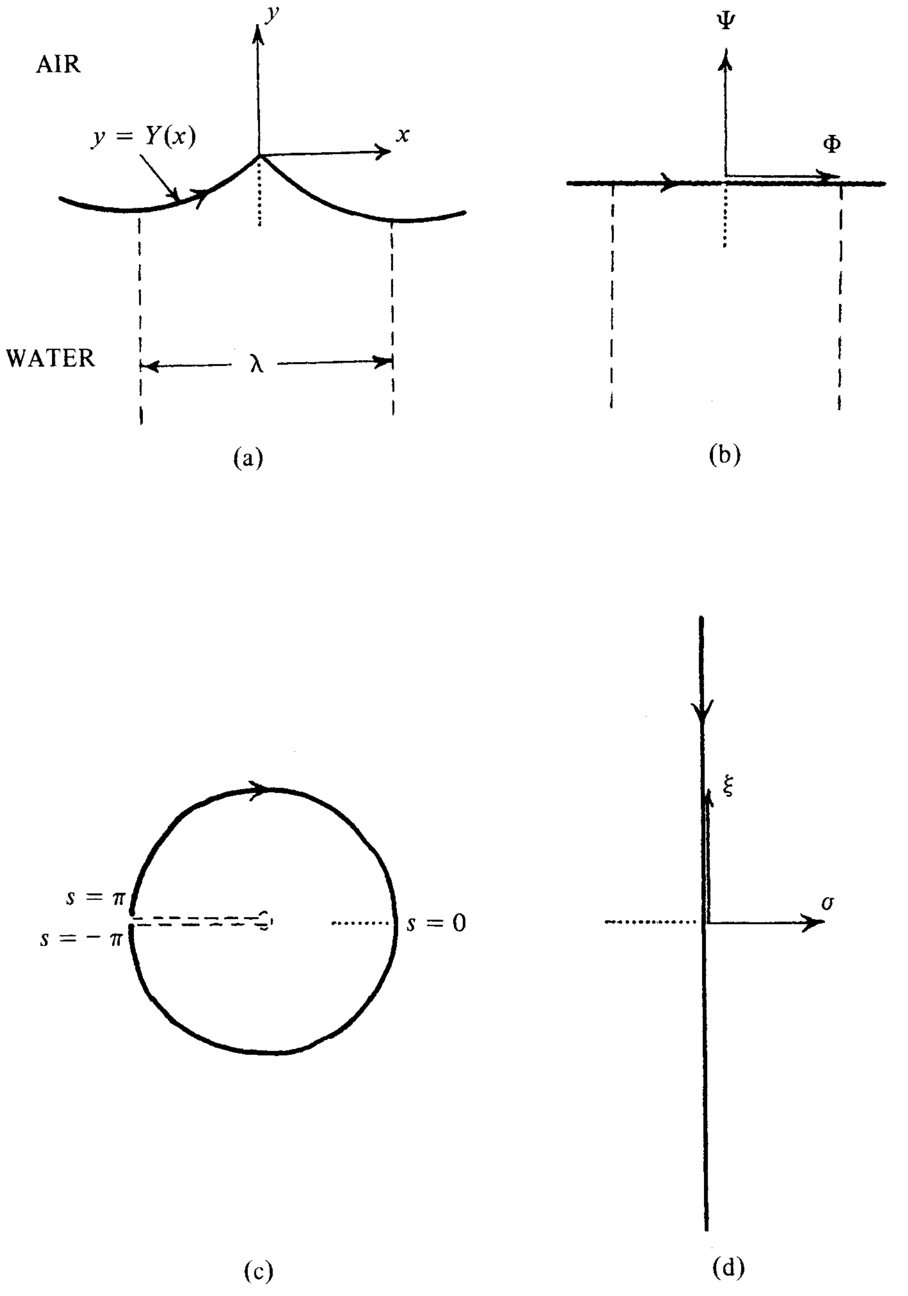

FIGURE 1. Images by various conformal maps of one period of the flow domain, say $-\frac{1}{2}\lambda \leqslant x < \frac{1}{2}\lambda$, $y < Y(x)$, for the case $b = 1$ of periodic waves in liquid of infinite depth. (a) The physical plane, that of $z = x + iy$; (b) plane of the complex potential $\chi = \Phi + i\Psi$; (c) plane of $\zeta = \rho e^{is}$, showing the variable s in (1.1); (d) plane of $\omega = \sigma + i\xi$, showing the variable ξ in (2.1).

where

$$K(s,t)=\frac{1}{\pi}\log\frac{\tan\frac{1}{2}s+\tan\frac{1}{2}t}{|\tan\frac{1}{2}s-\tan\frac{1}{2}t|}$$

and

$$v(t)=\tfrac{1}{2}\left(\cos^2\tfrac{1}{2}t+b\sin^2\tfrac{1}{2}t\right)^{-1/2}$$

for some $b\in[0,1]$ which we regard as fixed henceforth. Here $\int_0^t v\sin\phi$ stands for $\int_0^t v(u)\sin\phi(u)\,du$; such abbreviations will be used throughout the paper. Equation (1.1) refers only to waves of extreme form; the end-value of a parameter has already been chosen. The significance of the weighting function v is that $v(s)=-C\partial\Phi/\partial s$, where C is a positive constant and Φ denotes the velocity potential; the constant b distinguishes the families mentioned earlier: $b=0$ corresponds to solitary waves (of infinite wavelength and in liquid of finite depth), $b\in(0,1)$ to periodic waves in liquid of finite depth, and $b=1$ to periodic waves in liquid of infinite depth.

By a solution of (1.1) we mean a function ϕ satisfying the equation pointwise and such that

(1.2) $\quad 0<\phi(s)<\pi/2$ on $(0,\pi)$, $\phi(\pi)=0$, ϕ is real-analytic on $(0,\ \pi]$, and $\phi(s)\to\pi/6$ as $s\to0$.

Such solutions are now known to exist [**1, 2, 3, 7, 13**].

There have been many proposals for calculating the shape of the free boundary of an extreme wave by a combination of analytical and numerical methods (long lists of references are given in [**5** and **14**]). Such calculations were given a new direction by Grant [**4**], who sought the second term of an asymptotic expansion of $z(\chi)$ for $\chi\to0$ (here $z=x+iy$ denotes the complex coordinate in the physical plane, and $\chi=\Phi+i\Psi$ the complex potential), the first term being given by Stokes's corner flow. Grant pointed out that the second term must have an exponent that is irrational and "probably transcendental"; he concluded that "the structure near the corner is considerably more complicated than has been assumed in the past". Norman [**9**] contemplated terms beyond the two considered by Grant; inferred the nature of all the exponents; introduced the assumption that the numbers β_j, defined after (1.3) below, are linearly independent over the rationals; and established certain relationships between the coefficients of the series. (However, it seems that no coefficient after the first can be calculated by a merely local analysis.) Longuet-Higgins and Fox [**5**, **6**] used the exponents and functions arising in the work of Grant and Norman as one part of their theory of waves near an extreme one (the crest being smooth but of very large curvature).

All these calculations were heuristic rather than rigorous; in particular, Grant [**4**] and Norman [**9**] were concerned only to find analytic functions $z(\chi)$ that satisfy the nonlinear boundary condition of the problem in an asymptotic sense as $\Phi\to0$. In this paper we proceed from (1.1) and (1.2) to prove that, subject to an assumption about a question in number theory to be explained presently,

$$\phi(s)\sim\frac{\pi}{6}+\sum_{n=1}^{\infty}a_n s^{\mu_n}\quad\text{as } s\to0. \tag{1.3}$$

The exponents μ_n depend on the positive zeros $\beta_1, \beta_2, \beta_3, \ldots,$ with $\beta_j < \beta_{j+1}$, of $\sqrt{3}(1+\beta) - \tan(\beta\pi/2)$; the β_j are essentially the numbers discovered by Grant [**4**], and $\beta_1 \simeq 0.8027$, $\beta_2 \simeq 2.9066$. As Grant thought probable, and as we prove in the Appendix, each β_j is a transcendental number. Each exponent μ_n is a finite linear combination, with positive integer coefficients, of numbers in the set $\{2, \beta_1, \beta_2, \ldots\}$, and contains at least one β_j. The μ_n are ordered by $\mu_n < \mu_{n+1}$ (as the symbol $\sim$ of asymptoticity in (1.3) implies); the first few are β_1, $2\beta_1$, $3\beta_1$, $\beta_1 + 2$, β_2, $4\beta_1$, $2\beta_1 + 2$, $\beta_1 + \beta_2$, $5\beta_1$, $3\beta_1 + 2$, $2\beta_1 + \beta_2, \ldots$.

The assumption made in the derivation of (1.3) is stated precisely in §4; here we remark that it certainly holds if the set $\{1, \beta_1, \beta_2, \ldots\}$ is linearly independent over the rationals. Moreover, numerical calculation indicates that the assumption is true for the first hundred of the slightly larger set of exponents arising in the derivation of (1.3). If the assumption should be false, then our method would still be applicable, but logarithmic terms $a_{n,j}s^{\mu_n}(\log s)^{m_{n,j}}$, with $m_{n,j}$ a positive integer, would enter the series at large values of n.

We cannot evaluate the coefficients a_n in (1.3) when $\mu_n \in \{\beta_1, \beta_2, \beta_3, \ldots\}$; we define such coefficients by integrals involving the global behavior of ϕ. When $\mu_n \notin \{\beta_1, \beta_2, \beta_3, \ldots\}$, the corresponding coefficient a_n is determined by the previous coefficients $a_1, \ldots, a_{n-1}$; this agrees with Norman's results in [**9**]. The unpleasant possibility that all the $a_n = 0$ (not disproved in the present paper, although contrary to numerical evidence in [**6** and **14**]) is ruled out by McLeod's important result [**8**] that $a_1 < 0$, which is discussed further in Remark 2 before Theorem 3.3. It follows from this result and Theorem 3.3 that $a_2 > 0$.

The expansion (1.3) can be transformed and integrated to yield

$$z(\chi) \sim -ig^{-1/3}\left(\frac{3}{2}i\chi\right)^{2/3}\left\{1 + \sum_{n=1}^{\infty} b_n(i\chi)^{\mu_n}\right\} \quad \text{as } \chi \to 0, \tag{1.4}$$

where the constant g is the gravitational acceleration, $\arg(i\chi) \in [-\pi/2, \pi/2]$, and the coefficients b_n are real. If the set $\{1, \beta_1, \beta_2, \ldots\}$ is linearly independent over the rationals, then $b_n = 0$ whenever the linear combination defining μ_n contains a multiple of 2. The expansion (1.4) is then of the form proposed by Norman [**9**], if we interpret liberally certain tentative remarks in that paper (for example, that it is "possible to consider solutions...corresponding to combinations of terms from several roots" of $\sqrt{3}(1+\beta) = \tan(\beta\pi/2)$).

The plan of the paper is as follows. We begin §2 by making the transformation $\xi = \tan\frac{1}{2}s$, $\psi(\xi) = \phi(2\tan^{-1}\xi)$ in order to obtain a kernel $k(\xi, \eta)$ that is simpler than $K(s, t)$. (Note that, in effect, we map the unit disk in the plane of $\zeta = \rho e^{is}$ onto the half-plane $\{\omega = \sigma + i\xi\colon\ \sigma < 0\}$ by the conformal transformation $\omega = (\zeta - 1)/(\zeta + 1)$.) Since $\phi(s) = \psi(\tan\frac{1}{2}s)$, an asymptotic expansion of $\psi(\xi)$ for $\xi \to 0$ yields one of $\phi(s)$ for $s \to 0$. The next step is less obvious; we cast the integral equation (2.1) for $\psi(\xi)$ into a form, (2.10), that contains an elaborate nonlinearity but has the virtue of allowing us to construct the expansion of $\psi(\xi)$ by an inductive process. To begin this process, we show that $\psi(\xi) - \pi/6 = O(\xi^\alpha)$ for some $\alpha \in (\frac{1}{2}, \frac{3}{4})$.

In §3 we combine this preliminary estimate with the use of the Mellin transform to show first that

$$\psi(\xi) = \pi/6 + A_1\xi^{\beta_1} + O(\xi^l) \quad \text{as } \xi \to 0 \tag{1.5}$$

for any $l \in (\beta_1, 1)$, and then that this result implies the improved approximation

$$\psi(\xi) = \pi/6 + A_1\xi^{\beta_1} + A_2\xi^{2\beta_1} + O(\xi^m) \quad \text{as } \xi \to 0 \tag{1.6}$$

for any $m \in (2\beta_1, \beta_1 + l)$; here A_2 is a known function of A_1.

The step from (1.5) to (1.6) points the way to the long inductive proof, in §4, of the main results of the paper, which appear in Theorem 4.5 and Corollaries 4.6 and 4.7. The Appendix concerns some properties of the numbers β_j.

2. Preliminary transformations and estimates. Under the transformation $\xi = \tan\frac{1}{2}s$, $\eta = \tan\frac{1}{2}t$ and $\psi(\xi) = \phi(2\tan^{-1}\xi) = \phi(s)$, equation (1.1) becomes

$$\psi(\xi) = \frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{w(\eta)\sin\psi(\eta)}{\int_0^\eta w\sin\psi}\,d\eta, \qquad 0 < \xi < \infty, \tag{2.1}$$

where

$$k(\xi,\eta) = \frac{1}{\pi}\log\frac{\xi+\eta}{|\xi-\eta|} \quad \text{and} \quad w(\eta) = (1+\eta^2)^{-1/2}(1+b\eta^2)^{-1/2}.$$

By a solution of (2.1) we mean a function ψ satisfying (2.1) pointwise and such that

$$\begin{aligned} &0 < \psi(\xi) < \pi/2 \quad \text{on} \quad (0,\infty), \quad \psi(\xi) = O(\xi^{-1}) \quad \text{as} \quad \xi \to \infty, \\ &\psi \text{ is real-analytic on } (0,\infty), \text{ and } \psi(\xi) \to \pi/6 \text{ as } \xi \to 0. \end{aligned} \tag{2.2}$$

It is to be understood henceforth that $\xi \in (0,\infty)$. Occasionally we set $\xi = 0$, with the implication that $\psi(0) = \pi/6$ and $\psi \in C[0,\infty)$ (even though in the original problem ψ is an odd function on $\mathbf{R}\setminus\{0\}$).

Combining (2.1) and the formula [**1**, p. 197]

$$\frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{1}{\eta}\,d\eta = \frac{\pi}{6},$$

we obtain

$$\psi(\xi) - \frac{\pi}{6} = \frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{d}{d\eta}\log\left\{\frac{C}{\eta}\int_0^\eta w\sin\psi\right\}d\eta,$$

for any constant $C > 0$. Define

$$\gamma(\xi) = \psi(\xi) - \pi/6, \quad (E\gamma)(\eta) = 2w(\eta)\sin\{\pi/6 + \gamma(\eta)\}; \tag{2.3}$$

then

$$\gamma(\xi) = \frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{d}{d\eta}\log\left\{\frac{1}{\eta}\int_0^\eta E\gamma\right\}d\eta. \tag{2.4}$$

Our next transformation of the equation is more elaborate; it involves the integrated kernel (used extensively in [**1**])

$$q(\xi,\eta) = \int_0^\xi k(t,\eta)\,dt = \frac{1}{\pi}\left\{\xi\log\frac{\xi+\eta}{|\xi-\eta|} + \eta\log\frac{|\xi^2-\eta^2|}{\eta^2}\right\},$$

with

$$q_\eta(\xi,\eta) = \frac{1}{\pi}\log\frac{|\xi^2-\eta^2|}{\eta^2},$$

and the nonlinear operator F defined by

$$(2.5a)\qquad (F\gamma)(\eta) = \log\left\{\frac{1}{\eta}\int_0^\eta E\gamma\right\} + \frac{\eta(E\gamma)(\eta)}{\int_0^\eta E\gamma} - 1$$

$$(2.5b)\qquad = \frac{d}{d\eta}\left[\eta\log\left\{\frac{1}{\eta}\int_0^\eta E\gamma\right\}\right].$$

The function $F\gamma$ is important throughout the paper; we note its behavior for $\eta \to 0$ and for $\eta \to \infty$. By definition (2.3),

$$(2.6)\qquad (E\gamma)(\eta) = 1 + \sqrt{3}\,\gamma(\eta) + O\big(\gamma(\eta)^2 + \eta^2\big) \quad \text{as } \eta \to 0,$$

and it follows from either form of (2.5) that

$$(2.7)\qquad (F\gamma)(\eta) = \sqrt{3}\,\gamma(\eta) + O\big(\gamma(\eta)^2 + \eta^2\big) \quad \text{as } \eta \to 0.$$

We claim that

$$(2.8)\qquad 0 < \int_0^1 E\gamma \leqslant \int_0^\eta E\gamma \leqslant \text{const.}, \qquad 1 \leqslant \eta < \infty,$$

where the constant depends on γ. The lower bound follows from (2.6), in which $\gamma(\eta) \to 0$ as $\eta \to 0$, and because $(E\gamma)(\xi) > 0$ on $(0,\infty)$ by (2.2). For the upper bound we have

$$\int_0^\eta E\gamma \leqslant 2\int_0^\infty \psi(\eta)(1+\eta^2)^{-1/2}\,d\eta = \int_0^\pi \phi(t)\sec\frac{t}{2}\,dt < \infty$$

by [**3**, p. 657], if $b \in (0,1]$, and by [**2**, Theorem 4.7(a)], for the case $b = 0$ of a solitary wave. Also, $\eta(E\gamma)(\eta)$ is bounded because $w(\eta) = O(\eta^{-1})$ as $\eta \to \infty$; hence (2.5a) and (2.8) show that

$$(2.9)\qquad |(F\gamma)(\eta)| \leqslant \text{const.}\log\eta, \qquad 2 \leqslant \eta < \infty.$$

LEMMA 2.1. *If $\pi/6 + \gamma$ is a solution of* (2.1), *then*

$$(2.10)\qquad \gamma(\xi) = \frac{1}{3}\int_0^\infty r(\xi,\eta)(F\gamma)(\eta)\,d\eta, \qquad 0 < \xi < \infty,$$

where

$$(2.11)\qquad r(\xi,\eta) = -\frac{1}{\xi}q_\eta(\xi,\eta) = -\frac{1}{\pi\xi}\log\frac{|\xi^2-\eta^2|}{\eta^2}.$$

PROOF. With the notation (2.3), equation (2.1) becomes

$$\frac{\pi}{6} + \gamma(\xi) = \frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{(E\gamma)(\eta)}{\int_0^\eta E\gamma}\,d\eta.$$

Differentiation of this with respect to ξ is legitimate if the resulting integral is written as a Cauchy principal value; noting that $\xi k_\xi(\xi,\eta) = -\eta k_\eta(\xi,\eta)$, and integrating by parts, we obtain

$$(2.12)\qquad \xi\gamma'(\xi) = \frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{d}{d\eta}\left\{\frac{\eta(E\gamma)(\eta)}{\int_0^\eta E\gamma}\right\}d\eta,$$

where the integral exists by the properties (2.2) of $\psi \equiv \pi/6 + \gamma$ and because $\psi'(\eta) = O(\eta^{-1})$ as $\eta \to 0$ (see [7]). We add (2.4) and (2.12), refer to the definition (2.5a) of $F\gamma$, and integrate with respect to ξ; there results

$$\xi\gamma(\xi) = \frac{1}{3}\int_0^\infty q(\xi,\eta)\frac{d}{d\eta}(F\gamma)(\eta)\,d\eta.$$

We may integrate by parts because (for fixed $\xi \in (0,\infty)$, as elsewhere) $q(\xi,\eta)$ is $O(\eta \log 1/\eta)$ as $\eta \to 0$, and is $O(\eta^{-1})$ as $\eta \to \infty$, and because we have the estimates (2.7) and (2.9) for $F\gamma$. Accordingly,

$$\xi\gamma(\xi) = -\frac{1}{3}\int_0^\infty q_\eta(\xi,\eta)(F\gamma)(\eta)\,d\eta,$$

and this is (2.10).

LEMMA 2.2. *If $\pi/6 + \gamma$ is a solution of* (2.1), *then there exists an exponent $\alpha \in (\frac{1}{2}, \frac{3}{4})$ such that $\gamma(\xi) = O(\xi^\alpha)$ as $\xi \to 0$.*

PROOF. In this proof we abbreviate $(F\gamma)(\eta)$ to $F_\gamma(\eta)$, and similarly for other functions of the same kind.

(i) For any $c \in (0,1]$, let $\xi \in (0,c]$ and rewrite (2.10) as

$$\gamma(\xi) = \frac{1}{3}\int_0^c r(\xi,\eta)F_\gamma(\eta)\,d\eta + R_\gamma(\xi,c), \qquad 0 < \xi \leqslant c, \tag{2.13a}$$

where, in view of (2.7) and (2.9),

$$\begin{aligned}
|R_\gamma(\xi,c)| &= \left|\frac{1}{3}\int_c^\infty r(\xi,\eta)F_\gamma(\eta)\,d\eta\right| \\
&\leqslant \text{const.}\left\{\int_c^2 \frac{1}{\xi}\left|\log\frac{|\xi^2-\eta^2|}{\eta^2}\right|d\eta + \int_2^\infty \frac{\xi}{\eta^2}\log\eta\,d\eta\right\} \\
&\leqslant \text{const.}\left\{\int_{c/\xi}^\infty \log\frac{u^2}{u^2-1}\,du + \xi\int_2^\infty \frac{\log\eta}{\eta^2}\,d\eta\right\} \\
&\leqslant \text{const.}\frac{\xi}{c},
\end{aligned} \tag{2.13b}$$

where the constant depends on γ but is independent of c. It is natural to define

$$G_\gamma(\eta) = 2\sin\{\pi/6 + \gamma(\eta)\} - 1;$$

then $G_\gamma(\eta) \sim \sqrt{3}\,\gamma(\eta)$ as $\eta \to 0$ and $\gamma(\eta) \to 0$, and, by definition (2.3),

$$E_\gamma(\eta) = w(\eta)\{1 + G_\gamma(\eta)\} = 1 + G_\gamma(\eta) + O(\eta^2) \quad \text{as } \eta \to 0.$$

Moreover, we can so define U_γ, V_γ, and W_γ that

$$\frac{\eta}{\int_0^\eta E_\gamma} = \left\{1 + \frac{1}{\eta}\int_0^\eta G_\gamma + O(\eta^2)\right\}^{-1} = 1 - \frac{U_\gamma(\eta)}{\eta}\int_0^\eta G_\gamma + O(\eta^2),$$

$$\log\left(\frac{1}{\eta}\int_0^\eta E_\gamma\right) = \log\left\{1 + \frac{1}{\eta}\int_0^\eta G_\gamma + O(\eta^2)\right\} = \frac{V_\gamma(\eta)}{\eta}\int_0^\eta G_\gamma + O(\eta^2),$$

$$G_\gamma(\eta) = \cos\gamma(\eta) - 1 + \sqrt{3}\sin\gamma(\eta) = \sqrt{3}\,W_\gamma(\eta)\gamma(\eta);$$

then

$$(2.14)\quad G_\gamma(\eta) \to 0, \quad U_\gamma(\eta) \to 1, \quad V_\gamma(\eta) \to 1 \quad \text{and} \quad W_\gamma(\eta) \to 1 \qquad \text{as } \eta \to 0,$$

and

$$\begin{aligned} F_\gamma(\eta) &= \{V_\gamma(\eta) - U_\gamma(\eta) - G_\gamma(\eta)U_\gamma(\eta)\}\frac{1}{\eta}\int_0^\eta G_\gamma + G_\gamma(\eta) + \delta_\gamma(\eta) \\ &= \sqrt{3}\,\{V_\gamma(\eta) - U_\gamma(\eta) - G_\gamma(\eta)U_\gamma(\eta)\}\frac{1}{\eta}\int_0^\eta W_\gamma\gamma + \sqrt{3}\,W_\gamma(\eta)\gamma(\eta) + \delta_\gamma(\eta), \end{aligned}$$

where $\delta_\gamma(\eta) = O(\eta^2)$ as $\eta \to 0$.

Finally, define a linear operator L on the space $C[0,c]$ by

$$(2.15)\quad (Lf)(\xi) = \frac{1}{\sqrt{3}}\int_0^c r(\xi,\eta)\Big[\{V_\gamma(\eta) - U_\gamma(\eta) - G_\gamma(\eta)U_\gamma(\eta)\} \times \frac{1}{\eta}\int_0^\eta W_\gamma f + W_\gamma(\eta)f(\eta)\Big]\,d\eta,$$

and the integral equation (2.13) becomes

$$(2.16\text{a})\qquad \gamma(\xi) = (L\gamma)(\xi) + S_\gamma(\xi,c), \qquad 0 < \xi \leqslant c,$$

where

$$(2.16\text{b})\qquad S_\gamma(\xi,c) = R_\gamma(\xi,c) + \frac{1}{3}\int_0^c r(\xi,\eta)\delta_\gamma(\eta)\,d\eta = O\Big(\frac{\xi}{c}\Big),$$

because

$$(2.17)\quad \left|\int_0^c r(\xi,\eta)\delta_\gamma(\eta)\,d\eta\right| \leqslant \text{const.}\,\xi^2\int_0^{c/\xi}\left|\log\frac{|1-u^2|}{u^2}\right|u^2\,du \leqslant \text{const.}\,c\xi.$$

(ii) For every $\varepsilon \in (0, \frac{1}{4}]$, choose $c = c(\varepsilon)$ so small that

$$|V_\gamma(\eta) - U_\gamma(\eta) - G_\gamma(\eta)U_\gamma(\eta)| + |1 - W_\gamma(\eta)| < \varepsilon \quad \text{for all } \eta \in [0, c(\varepsilon)];$$

this is possible by (2.14). For any $\alpha \in [0,1)$ and any $\varepsilon \in (0, \frac{1}{4}]$, define the Banach space

$$X_\alpha = X_\alpha(c(\varepsilon)) = \{f \in C[0,c(\varepsilon)] : \|f\|_\alpha < \infty\},$$

where

$$\|f\|_\alpha = \sup_{0<\xi<c(\varepsilon)} \xi^{-\alpha}|f(\xi)|.$$

Consider the linear operator L defined by (2.15); we wish to show that L maps X_α into itself, for ε sufficiently small and for some $\alpha > \frac{1}{2}$, and that

$$\|L\|_{(\alpha)} \equiv \sup_{f\in X_\alpha\setminus\{0\}} \frac{\|Lf\|_\alpha}{\|f\|_\alpha} < 1.$$

It is easy to show that $Lf \in C[0, c(\varepsilon)]$ when $f \in X_\alpha$, and we estimate $\|Lf\|_\alpha$ as follows.

$$\begin{aligned}|(Lf)(\xi)| &\leqslant \frac{1}{\sqrt{3}}\int_0^{c(\varepsilon)}|r(\xi,\eta)|\left[\frac{\varepsilon(1+\varepsilon)}{\eta}\int_0^\eta|f| + (1+\varepsilon)|f|\right]d\eta \\ &\leqslant \frac{1}{\sqrt{3}}(1+\varepsilon)\|f\|_\alpha\int_0^{c(\varepsilon)}|r(\xi,\eta)|\left(\frac{\varepsilon}{1+\alpha}\eta^\alpha + \eta^\alpha\right)d\eta \\ &\leqslant \frac{1}{\sqrt{3}\,\pi}(1+\varepsilon)\left(1+\frac{\varepsilon}{1+\alpha}\right)\|f\|_\alpha\int_0^\infty \frac{1}{\xi}\left|\log\frac{|\xi^2-\eta^2|}{\eta^2}\right|\eta^\alpha\,d\eta \\ &= \frac{1}{\sqrt{3}\,\pi}(1+\varepsilon)\left(1+\frac{\varepsilon}{1+\alpha}\right)\|f\|_\alpha\xi^\alpha\int_0^\infty\left|\log\frac{|1-u^2|}{u^2}\right|u^\alpha\,du,\end{aligned}$$

where $0 < \xi \leqslant c(\varepsilon)$. It follows that

$$\|L\|_{(\alpha)} \leqslant (1+\varepsilon)\left(1+\frac{\varepsilon}{1+\alpha}\right)\mu(\alpha),$$

where

$$\begin{aligned}\mu(\alpha) &= \frac{1}{\sqrt{3}\,\pi}\int_0^\infty\left|\log\frac{|1-u^2|}{u^2}\right|u^\alpha\,du \\ &= \frac{1}{\sqrt{3}\,\pi}\left\{\int_0^\infty \log\frac{u^2}{|1-u^2|}\,u^\alpha\,du + 2\int_0^{1/\sqrt{2}}\log\frac{1-u^2}{u^2}\,u^\alpha\,du\right\}.\end{aligned}$$

The first of these two integrals is evaluated in §3, see (3.11a); in the second, we set $u = 1/x$ and integrate by parts; then

$$\mu(\alpha) = \frac{1}{\sqrt{3}\,\pi}\left\{\frac{\pi}{1+\alpha}\tan\frac{\alpha\pi}{2} + \frac{4}{1+\alpha}\int_{\sqrt{2}}^\infty \frac{dx}{x^\alpha(x^2-1)}\right\}. \tag{2.18}$$

To obtain a simple majorant, replace x^α by 1 in the last integral, which is then easily evaluated; thus

$$\mu(\alpha) \leqslant \frac{1}{\sqrt{3}(1+\alpha)}\left\{\tan\frac{\alpha\pi}{2} + \frac{4}{\pi}\log(1+\sqrt{2})\right\} = \tau(\alpha), \quad \text{say.}$$

Now $\tau(\frac{1}{2}) < 0.82$, $\tau(\frac{3}{4}) > 1.16$ and $\tau'(\alpha) > 0$ on $(0,1)$; hence there exists a number $\tilde{\alpha} \in (\frac{1}{2}, \frac{3}{4})$ such that $\mu(\alpha) \leqslant \tau(\alpha) \leqslant \tau(\tilde{\alpha}) < 1$ for all $\alpha \in [0, \tilde{\alpha}]$. If we choose ε sufficiently small, then $\|L\|_{(\alpha)} < 1$ for all $\alpha \in [0, \tilde{\alpha}]$.

(iii) With ε and $c = c(\varepsilon)$ now fixed, the estimate (2.16b) of S_γ may be written $S_\gamma(\xi) = O(\xi)$ as $\xi \to 0$. Hence $S_\gamma \in X_\alpha$ for all $\alpha \in [0,1)$, and so the equation $f(\xi) = (Lf)(\xi) + S_\gamma(\xi)$ has a unique solution $f_\alpha \in X_\alpha$ for each $\alpha \in [0, \tilde{\alpha}]$. Since $X_\alpha \subset X_0$, these solutions are identical; in other words, $f_\alpha = f_0$ for all $\alpha \in [0, \tilde{\alpha}]$ because each $f_\alpha \in X_0$. We know that $\gamma \in X_0$ and satisfies (2.16a), whence $\gamma = f_0 \in X_\alpha$ for all $\alpha \in [0, \tilde{\alpha}]$. In particular, $\gamma \in X_{\tilde{\alpha}}$, so that $\tilde{\alpha}$ may be chosen as the exponent in the statement of the lemma.

REMARK. A more careful treatment of the integral in (2.18) is not worthwhile, because we shall see from Lemmas 3.1 and 3.2 that the best possible exponent is the number β_1 introduced after (1.3), so that

$$\frac{1}{\sqrt{3}(1+\beta_1)}\tan\frac{\beta_1\pi}{2} = 1,$$

and the left-hand member of this equation corresponds to only the first term of $\mu(\alpha)$ in (2.18).

3. The asymptotic expansion of $\psi(\xi)$ to three terms. Our observation in (2.7) that $(F\gamma)(\eta) \sim \sqrt{3}\,\gamma(\eta)$ as $\eta \to 0$ suggests that the integral equation (2.10) be written

$$\gamma(\xi) = \frac{1}{\sqrt{3}}\int_0^1 r(\xi,\eta)\gamma(\eta)\,d\eta + \rho_\gamma(\xi), \qquad 0 < \xi \leqslant 1, \tag{3.1}$$

where

$$\rho_\gamma(\xi) = \frac{1}{3}\int_0^1 r(\xi,\eta)\{(F\gamma)(\eta) - \sqrt{3}\,\gamma(\eta)\}\,d\eta + \frac{1}{3}\int_1^\infty r(\xi,\eta)(F\gamma)(\eta)\,d\eta. \tag{3.2}$$

From (2.7) and Lemma 2.2 we have

$$(F\gamma)(\eta) - \sqrt{3}\,\gamma(\eta) = O(\eta^{2\alpha}) \quad \text{as } \eta \to 0, \text{ with } \alpha \in \left(\tfrac{1}{2}, \tfrac{3}{4}\right),$$

and it follows (by an estimate like (2.17) for the first integral in (3.2), and an estimate like (2.13b) for the second) that $\rho_\gamma(\xi) = O(\xi)$ as $\xi \to 0$.

All the results to come will be consequences of (3.1) and (3.2). Our plan is to bootstrap from some expansion of $\psi(\xi)$ for $\xi \to 0$ (at present, $\pi/6 + O(\xi^\alpha)$) to an estimate of $\rho_\gamma(\xi)$ (at present, $O(\xi)$), and then to derive a more complete expansion of $\psi(\xi)$ from (3.1), regarded as a linear equation for γ in which ρ_γ is "known". The proof by induction in §4 will follow this program to exhaustion; without the more explicit first steps in this section, it would probably be incomprehensible.

The *Mellin transform* [**12**, p. 7] will be our main tool for analysis of (3.1). Let f be piecewise continuous on $(0,\infty)$, define $\phi(x,s) = x^{s-1}f(x)$ for $x > 0$, where $s = \sigma + it \in \mathbf{C}$, and assume that $\phi(\cdot,s) \in L_1(0,\infty)$ for $a < \sigma < b$. Then, for s in this strip, we define the Mellin transform $\hat{f}$ of f by

$$\hat{f}(s) = \int_0^\infty x^{s-1}f(x)\,dx; \tag{3.3}$$

we also write this as $\hat{f}(s) \risingdotseq f(x)$. Note that $\hat{f}$ is analytic in the strip $\operatorname{Re} s \in (a,b)$. The inversion formula [**12**, p. 46] is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\hat{f}(s)\,ds, \qquad a < c < b, \tag{3.4}$$

at points of continuity of f. The product formula [**12**, p. 54] is

$$\hat{f}(-s+A)\hat{g}(s+B) \risingdotseq x^B\int_0^\infty u^{A+B-1}f(u)g(xu)\,du, \tag{3.5}$$

provided that $\hat{f}(-s + A)$ and $\hat{g}(s + B)$ have a common strip of convergence. Finally, we record the following property [**12**, p. 118] of the Mellin transform.

(3.6) If $\int_0^\infty |x^\sigma f(x)|^p \frac{dx}{x} < \infty$ for some $\sigma \in (a, b)$ and some $p \in (1, 2)$, then $\hat{f}(\sigma + i\,\cdot) \in L'_p(-\infty, \infty)$, where $1/p + 1/p' = 1$.

To evaluate certain contour integrals arising from (3.4), and to state our theorems, we shall need the following lemma, a variant of which has already been given in [**5**] for a different purpose. The significance of the exponents β_j in the lemma is evident from the observation (a particular case of Lemma 4.2) that, if $\sqrt{3}(1 + \beta) = \tan(\beta\pi/2)$ and $\beta > -1$, then

$$\frac{1}{\sqrt{3}} \int_0^1 r(\xi, \eta)\eta^\beta \, d\eta = \xi^\beta + a(\xi), \qquad 0 < \xi \leqslant 1, \tag{3.7}$$

where $a(\xi)$ is $O(\xi)$ and real-analytic on $[0, 1)$, and contains only odd powers of ξ in its Taylor series about the origin. In other words, *the linear integral operator in* (3.1) *leaves the functions* ξ^{β_j} *almost invariant*, merely adding to them such functions a, and these latter turn out to be unimportant.

LEMMA 3.1. *The only zeros of* $\sqrt{3}(1 - s) + \tan(s\pi/2)$ *in the half-plane* $\operatorname{Re} s < 1$ *are simple zeros on the negative real axis. We denote such points by* $s = -\beta_j$, $j = 1, 2, 3, \ldots,$ *with* $\beta_j < \beta_{j+1}$. *Then* $\beta_j \in (2j - 2, 2j - 1)$ *for all* j, *and* $\beta_j = 2j - 1 + O(j^{-1})$ *as* $j \to \infty$ *(the O-term is negative). Also,*

$$\beta_1 \simeq 0.8027, \quad \beta_2 \simeq 2.9066, \quad \beta_3 \simeq 4.9383.$$

PROOF. Set $s = 1 - 2z/\pi$ and $z = x + iy$; we have to solve

$$(2\sqrt{3}/\pi)z + \cot z = 0, \qquad x > 0,$$

or equivalently, since no zero or pole of $\tan z$ is a solution,

$$z \tan z = -\pi/2\sqrt{3}, \qquad x > 0.$$

If this equation has a solution with $y \neq 0$, then the imaginary part of the equation gives

$$(\sinh 2y)/2y = -(\sin 2x)/2x,$$

which is impossible (because $(\sinh 2y)/2y > 1$, while $-(\sin 2x)/2x \leqslant 0$ for $0 < x \leqslant \pi/2$ and $-(\sin 2x)/2x < 1/\pi$ for $x > \pi/2$). The remaining assertions now follow from elementary analysis of the points where the graphs of $\tan x$ and of $-\pi/2\sqrt{3}\,x$ intersect, for $x > 0$.

LEMMA 3.2. *If* ψ *is a solution of* (2.1), *then*

$$\psi(\xi) = \pi/6 + A_1\xi^{\beta_1} + O(\xi^l) \quad \text{as } \xi \to 0,$$

for some constant A_1 *and any* $l \in (\beta_1, 1)$. *Here* β_1 *is as in Lemma* 3.1.

PROOF. (i) Define

$$g(\xi) = \begin{cases} \gamma(\xi), & 0 \leqslant \xi \leqslant 1, \\ 0, & 1 < \xi, \end{cases}$$

and

$$h(\xi) = \begin{cases} \rho_\gamma(\xi), & 0 \leqslant \xi \leqslant 1, \\ -\dfrac{1}{\sqrt{3}} \displaystyle\int_0^1 r(\xi,\eta)\gamma(\eta)\,d\eta, & 1 < \xi, \end{cases}$$

so that (3.1) becomes

$$g(\xi) = \frac{1}{\sqrt{3}} \int_0^\infty r(\xi,\eta) g(\eta)\,d\eta + h(\xi), \qquad 0 < \xi < \infty,$$

since the integrand is zero for $\eta > 1$, and both sides are zero for $\xi > 1$. Setting $\eta = \xi u$, we arrive at a form suitable for the Mellin transform:

$$\text{(3.8a)} \qquad g(\xi) = \frac{1}{\sqrt{3}} \int_0^\infty R(u) g(\xi u)\,du + h(\xi), \qquad 0 < \xi < \infty,$$

where

$$\text{(3.8b)} \qquad R(u) = r(1,u) = -\frac{1}{\pi} \log \frac{|1-u^2|}{u^2}.$$

Because of the plethora of notation, we emphasize that *g is a truncated form of $\gamma = \psi - \pi/6$, that $h(\xi)$ is presumably of smaller order than $g(\xi)$ as $\xi \to 0$, and that the dependence of h on γ is now implicit.*

(ii) The term $h(\xi)$ may be estimated as follows. Let $\alpha \in (\frac{1}{2}, \frac{3}{4})$ be the exponent in Lemma 2.2; then for $\xi \geqslant 2$ (and hence $u \leqslant \frac{1}{2}$ when we set $\eta = \xi u$)

$$\begin{aligned} |h(\xi)| &\leqslant \frac{1}{\sqrt{3}} \int_0^1 |r(\xi,\eta)|\,|\gamma(\eta)|\,d\eta \\ &\leqslant \text{const.}\, \xi^\alpha \int_0^{1/\xi} \left| \log \frac{|1-u^2|}{u^2} \right| u^\alpha\,du \\ &\leqslant \text{const.}\, \xi^\alpha \int_0^{1/\xi} \log \frac{1}{u}\, u^\alpha\,du \\ &\leqslant \text{const.}\, \xi^{-1} \log \xi, \end{aligned}$$

and for $1 \leqslant \xi \leqslant 2$ the integral is bounded by a constant. Since $\rho_\gamma(\xi) = O(\xi)$ as $\xi \to 0$, we have

$$\text{(3.9)} \qquad |h(\xi)| \leqslant \begin{cases} \text{const.}\, \xi, & 0 \leqslant \xi \leqslant 1, \\ \text{const.}\, \xi^{-1}(1 + \log \xi), & 1 < \xi. \end{cases}$$

It follows that $\hat{h}(s)$ exists and is analytic for $-1 < \sigma < 1$, where $s = \sigma + it$, and from (3.6) that

$$\text{(3.10)} \qquad \hat{h}(\sigma + i\,\cdot) \in L_{p'}(-\infty, \infty) \quad \text{for all } \sigma \in (-1,1) \text{ and all } p' > 2.$$

(iii) To find the Mellin transform of the function R in (3.8), we recall that our original kernel in (2.1) was

$$k(\xi,\eta) = \frac{1}{\pi} \log \frac{\xi + \eta}{|\xi - \eta|},$$

and note from [12, p. 192], that (with $s = \sigma + it$)

$$\int_0^\infty u^{s-1} k(1,u)\,du = \frac{1}{s}\tan\frac{s\pi}{2}, \qquad -1 < \sigma < 1.$$

Setting $u = \eta/\xi$ for fixed $\xi \in (0,\infty)$, multipying both sides by ξ^s, and then integrating with respect to ξ, we obtain

$$\int_0^\infty \eta^{s-1} q(\xi,\eta)\,d\eta = \frac{\xi^{s+1}}{s(s+1)}\tan\frac{s\pi}{2}, \qquad -1 < \sigma < 1.$$

We may integrate by parts because $\eta^s q(\xi,\eta)$ is $O(\eta^{\sigma+1}\log 1/\eta)$ as $\eta \to 0$, and is $O(\eta^{\sigma-1})$ as $\eta \to \infty$, and there results

$$-\int_0^\infty \eta^s q_\eta(\xi,\eta)\,d\eta = \frac{\xi^{s+1}}{s+1}\tan\frac{s\pi}{2}, \qquad -1 < \sigma < 1.$$

Set $\eta = \xi u$ and $s = z - 1$; then

$$\text{(3.11a)} \qquad -\frac{1}{\pi}\int_0^\infty u^{z-1}\log\frac{|1-u^2|}{u^2}\,du = \frac{1}{z}\tan\frac{(z-1)\pi}{2}, \qquad 0 < \operatorname{Re} z < 2,$$

that is,

$$\text{(3.11b)} \qquad \hat{R}(z) = \frac{1}{z}\tan\frac{(z-1)\pi}{2}, \qquad 0 < \operatorname{Re} z < 2.$$

(iv) We are now in a position to take the Mellin transform of the integral equation (3.8a). By Lemma 2.2, $g(\xi) = O(\xi^\alpha)$ as $\xi \to 0$, with $\alpha \in (\frac{1}{2}, \frac{3}{4})$; therefore $\hat{g}(s)$ exists and is analytic for $\sigma > -\alpha$. We have just shown that $\hat{h}(s)$ and $\hat{R}(-s+1)$ exist and are analytic for $-1 < \sigma < 1$. Hence, if $-\alpha < \sigma < 1$, we may apply the Mellin transform to (3.8a) and use the product formula (3.5) with $A = 1$ and $B = 0$:

$$\hat{g}(s) = (1/\sqrt{3})\hat{R}(1-s)\hat{g}(s) + \hat{h}(s), \qquad -\alpha < \sigma < 1,$$

or, equivalently,

$$\text{(3.12)} \qquad \hat{g}(s) - \hat{h}(s) = Q(s)\hat{h}(s), \qquad -\alpha < \sigma < 1,$$

where

$$\text{(3.13)} \qquad Q(s) = \frac{\hat{R}(1-s)}{\sqrt{3} - \hat{R}(1-s)} = -\frac{\tan(s\pi/2)}{\sqrt{3}(1-s) + \tan(s\pi/2)}.$$

The inversion formula (3.4) gives, for all $\xi \in (0,\infty)\setminus\{1\}$,

$$\text{(3.14)} \qquad g(\xi) - h(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-s} Q(s)\hat{h}(s)\,ds, \qquad -\alpha < c < 1.$$

(v) Let $\xi \in (0,1)$, so that $g(\xi) = \gamma(\xi)$. Since $h(\xi) = O(\xi)$ as $\xi \to 0$, it suffices to show that the right-hand side of (3.14) behaves like $A_1\xi^{\beta_1} + O(\xi^l)$ for some constant A_1 and any $l \in (\beta_1, 1)$. We prove this by moving the path of integration as far to the left as our knowledge of $\hat{h}$ allows, and taking the residue at any pole of the integrand between the path in (3.14) and the new one. The details are as follows.

Given $c \in (-\alpha, 1)$ and $l \in (\beta_1, 1)$, define Γ_M to be the (positively directed) rectangular contour with corners at $c \pm iM$ and $-l \pm iM$ for some large $M > 0$, and consider the contour integral

$$\frac{1}{2\pi i}\int_{\Gamma_M} \xi^{-s} Q(s)\hat{h}(s)\,ds.$$

The contribution of the horizontal parts ($-l \leqslant \sigma \leqslant c$, $t = \pm M$) of Γ_M tends to zero as $M \to \infty$, because (a) $|\xi^{-s}| = \xi^{-\sigma}$ with $0 < \xi < 1$; (b) the fact that $\tan(s\pi/2) \to \pm i$ as $t \to \pm\infty$ with σ fixed implies, in view of (3.13), that $Q(s) = O(t^{-1})$; (c) the Riemann-Lebesgue lemma, applied to the definition integral (3.3) of $\hat{h}(s)$, shows that $\hat{h}(s) \to 0$ as $t \to \pm\infty$ with σ fixed.

Since $\hat{h}(s)$ is analytic for $-1 < \sigma < 1$, we conclude from (3.13) and Lemma 3.1 that the only singularity of $\xi^{-s}Q(s)\hat{h}(s)$, with $-1 < \sigma < 1$, is a simple pole at $s = -\beta_1$. Accordingly,

$$\begin{aligned} &\frac{1}{2\pi i}\left\{\int_{c-i\infty}^{c+i\infty} - \int_{-l-i\infty}^{-l+i\infty} \xi^{-s}Q(s)\hat{h}(s)\,ds\right\} \\ &\quad = \lim_{M\to\infty}\frac{1}{2\pi i}\int_{\Gamma_M} \xi^{-s}Q(s)\hat{h}(s)\,ds \\ &\quad = \xi^{\beta_1}\operatorname{res} Q(s)\,|_{s=-\beta_1}\hat{h}(-\beta_1) = A_1\xi^{\beta_1}, \quad \text{say,} \end{aligned} \tag{3.15}$$

and so

$$g(\xi) - h(\xi) = A_1\xi^{\beta_1} + \frac{\xi^l}{2\pi}\int_{-\infty}^{\infty} \xi^{-it}Q(-l+it)\hat{h}(-l+it)\,dt. \tag{3.16}$$

We bound this last integral (without the factor $\xi^l/2\pi$) by means by Hölder's inequality, noting that $Q(-l + i\,\cdot\,) \in L_{3/2}(-\infty, \infty)$ because $Q(s) = O(t^{-1})$ as $t \to \pm\infty$, and that $\hat{h}(-l + i\,\cdot\,) \in L_3(-\infty, \infty)$ by (3.10). As we remarked earlier, the desired result now follows, because $g(\xi) = \gamma(\xi)$ for $\xi \leqslant 1$ and $h(\xi) = O(\xi)$ as $\xi \to 0$.

REMARKS. 1. As was to be expected from (3.1) and (3.7), we cannot evaluate the coefficient A_1 in Lemma 3.2; in (3.15) we have defined it in terms of the global behavior of γ.

2. McLeod [**8**] has observed that the logarithm in the integral equation (2.4) is always *less* than its leading term for $\eta \to 0$: by (2.3),

$$\log\left\{\frac{1}{\eta}\int_0^\eta E\gamma\right\} < \frac{\sqrt{3}}{\eta}\int_0^\eta \gamma \quad \text{for all } \eta > 0,$$

and has deduced thereby that $A_1 < 0$. This result has two important consequences for the present theory, as follows. (a) As was mentioned in the introduction, the fact that $A_1 \neq 0$ ensures that our asymptotic approximations to γ are not identically zero. (b) Lemma 3.2 and analysis of the equation that results from differentiation of (2.1) imply that $\xi\psi'(\xi) = A_1\beta_1\xi^{\beta_1} + O(\xi^l)$; therefore, the result $A_1 < 0$ is consistent with Stokes's conjecture (see [**1**, pp. 194 and 199]) that the profile of an extreme wave is strictly convex between the crest and the trough, in other words, that $\psi(\xi)$ is decreasing on $(0, \infty)$.

The earlier numerical calculations in [**6**] (equation (2.10), with the value $B = 0.131$ on p. 776) and in [**14**, Table 1] are in accord with McLeod's result; they give $b_1 > 0$ in (1.4), which corresponds to $A_1 < 0$.

3. The term $A_2\xi^{2\beta_1}$ in our next result has a character quite different from that of $A_1\xi^{\beta_1}$; instead of arising from (3.7), it arises from terms that are essentially squares of the first perturbation, and so A_2 is determined uniquely by A_1.

THEOREM 3.3. *If* ψ *is a solution of* (2.1), *then*

$$\psi(\xi) = \pi/6 + A_1\xi^{\beta_1} + A_2\xi^{2\beta_1} + O(\xi^m) \quad \text{as } \xi \to 0,$$

for any $m \in (2\beta_1, \beta_1 + l)$. *Here* A_1, β_1 *and* l *are as in Lemma* 3.2, *so that* $l \in (\beta_1, 1)$, *and*

$$A_2 = -\frac{(4 + 8\beta_1 + \beta_1^2)(\tan\beta_1\pi)A_1^2}{2\sqrt{3}(1 + \beta_1)^2\{\sqrt{3}(1 + 2\beta_1) - \tan\beta_1\pi\}} \simeq 0.1341A_1^2.$$

PROOF. We shall use the notation in the proof of Lemma 3.2, and shall begin by showing that the previous estimate (3.9) of h can be sharpened, by means of the result of Lemma 3.2, to

$$h(\xi) = C_1\xi + C_2\xi^{2\beta_1} + O(\xi^{\beta_1 + l}), \qquad 0 \leqslant \xi \leqslant 1, \tag{3.17a}$$

$$|h(\xi)| \leqslant \text{const.}\,\xi^{-1}(1 + \log\xi), \qquad 1 < \xi, \tag{3.17b}$$

where C_1 is a constant of no interest, because ultimately it will drop out of the integral equation, and C_2 is a constant that is known in terms of A_1. With this estimate in hand, we can move the path of integration in (3.14) further to the left than was possible before. Taking due account of the poles of the integrand between the path in (3.14) and the new one, we shall obtain the result of the theorem.

(i) Let $\xi \in (0, 1]$. A straightforward calculation gives the following.

$$\begin{aligned}(E\gamma)(\eta) &= 2w(\eta)\sin\{\pi/6 + \gamma(\eta)\} \\ &= 1 + \sqrt{3}\gamma(\eta) - \tfrac{1}{2}A_1^2\eta^{2\beta_1} + O(\eta^{\beta_1 + l}),\end{aligned}$$

$$\begin{aligned}(F\gamma)(\eta) - \sqrt{3}\gamma(\eta) &= \log\left\{\frac{1}{\eta}\int_0^\eta E\gamma\right\} + \frac{\eta(E\gamma)(\eta)}{\int_0^\eta E\gamma} - 1 - \sqrt{3}\gamma(\eta) \\ &= -\frac{4 + 8\beta_1 + \beta_1^2}{2(1 + \beta_1)^2}A_1^2\eta^{2\beta_1} + O(\eta^{\beta_1 + l}).\end{aligned}$$

Since $2\beta_1$ and $\beta_1 + l$ are not odd positive integers, Lemmas 4.2 and 4.3 yield

$$\frac{1}{3}\int_0^1 r(\xi, \eta)\{(F\gamma)(\eta) - \sqrt{3}\gamma(\eta)\}\,d\eta = C_0\xi + C_2\xi^{2\beta_1} + O(\xi^{\beta_1 + l}), \tag{3.18}$$

where the constant C_0 will be a part of C_1 (and is therefore of no interest), and

$$C_2 = -\frac{(4 + 8\beta_1 + \beta_1^2)(\tan\beta_1\pi)A_1^2}{6(1 + \beta_1)^2(1 + 2\beta_1)}. \tag{3.19}$$

The other term in the definition (3.2) of $\rho_\gamma(\eta)$ is

$$(3.20)\quad \frac{1}{3}\int_1^\infty r(\xi,\eta)(F\gamma)(\eta)\,d\eta = -\frac{1}{3\pi\xi}\int_1^\infty \log\left(1-\frac{\xi^2}{\eta^2}\right)(F\gamma)(\eta)\,d\eta$$
$$= \frac{1}{3\pi}\sum_{k=1}^\infty \frac{\xi^{2k-1}}{k}\int_1^\infty \frac{(F\gamma)(\eta)}{\eta^{2k}}\,d\eta,$$

where the integral is $O(k^{-1})$ as $k \to \infty$, so that the series converges for $\xi \leqslant 1$. Equations (3.18) and (3.20) imply that

$$\rho_\gamma(\xi) = C_1\xi + C_2\xi^{2\beta_1} + O(\xi^{\beta_1+l}),$$

and thus prove (3.17a), while (3.17b) is a previous estimate.

(ii) Define

$$h_1(\xi) = \begin{cases} C_1\xi + C_2\xi^{2\beta_1}, & 0 \leqslant \xi \leqslant 1, \\ 0, & 1 < \xi, \end{cases}$$

and $h_2 = h - h_1$. Then, for $\sigma > -1$,

$$\hat{h}_1(s) = \frac{C_1}{s+1} + \frac{C_2}{s+2\beta_1},$$

while $\hat{h}_2(s)$ exists and is analytic for $-\beta_1 - l < \sigma < 1$, and $\hat{h}_2(\sigma + i\,\cdot\,) \in L_{p'}(-\infty,\infty)$ for all $\sigma \in (-\beta_1 - l, 1)$ and all $p' > 2$. The obvious analytic (more precisely, meromorphic) continuations of $\hat{h}_1$ and $\hat{h}$ into the strip of convergence of $\hat{h}_2$ will also be denoted by $\hat{h}_1$ and $\hat{h}$, respectively.

We can now proceed from (3.14) as in the proof of Lemma 3.2, step (v), except that now we move the path of integration to $\sigma = -m$ for any $m \in (2\beta_1, \beta_1 + l)$, and collect the residues of the poles of $\xi^{-s}Q(s)\hat{h}(s)$ at $s = -\beta_1$, -1 and $-2\beta_1$. Since $Q(-1) = -1$, there results

$$g(\xi) - h(\xi) = A_1\xi^{\beta_1} - C_1\xi + C_2Q(-2\beta_1)\xi^{2\beta_1}$$
$$+ \frac{\xi^m}{2\pi}\int_{-\infty}^\infty \xi^{-it}Q(-m+it)\hat{h}(-m+it)\,dt;$$

the last integral is bounded as before since $\hat{h}_1(-m + i\,\cdot\,)$ and $\hat{h}_2(-m + i\,\cdot\,)$ are both in $L_3(-\infty,\infty)$. Substituting for $h(\xi)$ from (3.17a), we have

$$\gamma(\xi) = A_1\xi^{\beta_1} + C_2\{Q(-2\beta_1) + 1\}\xi^{2\beta_1} + O(\xi^m) \quad \text{as } \xi \to 0,$$

where C_2 is given by (3.19) and $Q(-2\beta_1)$ by (3.13).

4. The asymptotic expansion of $\psi(\xi)$ to any number of terms.

4.1. *Notation.* Let $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbf{N} = \{1, 2, 3, \ldots\}$, and $B = \{\beta_1, \beta_2, \beta_3, \ldots\}$, where the numbers β_j are as in Lemma 3.1. We shall prove in the Appendix that the β_j are transcendental numbers.

An elaborate notation is needed for the exponents in various series that we shall construct, because (a) the nonlinearity in (2.1), and the presence of the weighting function w there, cause the exponents in the expansion of ψ to form a fairly complicated sequence, (b) further exponents enter, and finally depart, during the course of the construction.

In what follows $(m_1, \dots, m_{r+1})$ is a multi-index: $m_i \in \mathbf{N}_0$ for $i = 1, \dots, r+1$ and r is any element of $\mathbf{N}$. Define

$$\Lambda = \{ m_1\beta_1 + \cdots + m_r\beta_r + m_{r+1}2 \colon m_1 + \cdots + m_r + m_{r+1} \geqslant 1 \};$$

since Λ is a countable set, we may write $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$ with $\lambda_j < \lambda_{j+1}$. The first few numbers λ_j are

$$\beta_1, 2\beta_1, 2, 3\beta_1, \beta_1 + 2,\ \beta_2, 4\beta_1, 2\beta_1 + 2,\ \beta_1 + \beta_2, 4, 5\beta_1, 3\beta_1 + 2, 2\beta_1 + \beta_2, \dots.$$

The following subsets of Λ will be needed:

$$\Lambda_\beta = \{ m_1\beta_1 + \cdots + m_r\beta_r + m_{r+1}2 \colon m_1 + \cdots + m_r \geqslant 1 \},$$

so that $\Lambda = \Lambda_\beta \cup \{2, 4, 6, \dots\}$, and

$$\Lambda_3 = \{2\} \cup \{ m_1\beta_1 + \cdots + m_r\beta_r + m_{r+1}2 \colon m_1 + \cdots + m_r + m_{r+1} \geqslant 2 \},$$

so that $\Lambda = \Lambda_3 \cup B$. Thus each element of Λ_β involves at least one of the β_j; we expect the exponents in the expansion of ψ to be in Λ_β. The exponents in Λ_3 arise if, for example, we combine three series as follows: form the product of two series with exponents in Λ and add a series with exponents in $\{2, 4, 6, \dots\}$.

Truncated subsets will be denoted by

$$\Lambda_\beta(n) = \{ \lambda_j \in \Lambda_\beta \colon \lambda_j \leqslant \lambda_n \} = \{ \lambda_j \in \Lambda_\beta \colon j \leqslant n \},$$
$$\Lambda_3(n) = \{ \lambda_j \in \Lambda_3 \colon j \leqslant n \}, \quad B(n) = \{ \beta_j \in B \colon \beta_j \leqslant \lambda_n \}.$$

4.2. *Bootstrapping from γ to ρ_γ.* We now prove that a given expansion of γ, with exponents in $\Lambda_\beta(n)$, implies an expansion of ρ_γ with exponents in $\Lambda_3(N) \cup \{1, 3, 5, \dots\}$, where $N \geqslant n+1$. This will allow us to improve the given approximation to γ by the method that we used to pass from Lemma 3.2 to Theorem 3.3. We shall prove in Theorem 4.5 that, if $\Lambda_3(N) \cap B(N) = \varnothing$, then γ has an expansion with exponents in the larger set $\Lambda_\beta(N)$. Hence, if $\Lambda_3 \cap B = \varnothing$ (which is a number-theoretic problem addressed in the Appendix), then γ has an infinite asymptotic expansion with exponents in Λ_β.

We begin by expanding the function $F\gamma - \sqrt{3}\,\gamma$ in the definition (3.2) of ρ_γ; then we establish two basic properties of the linear integral operator in (3.1) and in the first term of ρ_γ. The second term of ρ_γ has a very simple expansion. Combining these results, we obtain the expansion of ρ_γ, for a given expansion of γ.

LEMMA 4.1. *Suppose that, for certain constants A_i,*

$$\gamma(\eta) = \sum_{\lambda_i \in \Lambda_\beta(n)} A_i \eta^{\lambda_i} + O(\eta^\mu) \quad \textit{as } \eta \to 0, \tag{4.1}$$

where $\lambda_n < \mu < \lambda_{n+1}$ and $\mu + \beta_1 \notin \Lambda \cup \{1, 3, 5, \dots\}$. Define $N = N(n, \mu)$ by

$$\lambda_N = \max\{ \lambda_j \in \Lambda \colon \lambda_j < \mu + \beta_1 \}.$$

(Since $\lambda_{n+1} \leqslant \lambda_n + \beta_1 < \mu + \beta_1$, we have $N(n, \mu) \geqslant n+1$.) Then

$$(F\gamma)(\eta) - \sqrt{3}\,\gamma(\eta) = \sum_{\lambda_j \in \Lambda_3(N)} B_j \eta^{\lambda_j} + O(\eta^{\mu + \beta_1}) \quad \textit{as } \eta \to 0, \tag{4.2}$$

where the coefficients B_j depend only on the A_i in (4.1).

PROOF. Again we abbreviate $(F\gamma)(\eta)$ to $F_\gamma(\eta)$, and similarly for other functions of the same kind. We have

$$E_\gamma(\eta) = 2w(\eta)\sin\{\pi/6 + \gamma(\eta)\},$$

where

$$w(\eta) = (1+\eta^2)^{-1/2}(1+b\eta^2)^{-1/2} = 1 + \sum_{k=1}^{\infty} C_k\eta^{2k}, \quad \text{say, for } \eta < 1.$$

Define two new operators H and J by

$$\begin{aligned} H_\gamma(\eta) &= E_\gamma(\eta) - 1 - \sqrt{3}\,\gamma(\eta) \\ &= w(\eta)\{\cos\gamma(\eta) - 1 + \sqrt{3}\,[\sin\gamma(\eta) - \gamma(\eta)]\} \\ &\quad + \{w(\eta) - 1\}\{1 + \sqrt{3}\,\gamma(\eta)\}, \end{aligned} \tag{4.3}$$

and

$$J_\gamma(\eta) = \frac{1}{\eta}\int_0^\eta \{E_\gamma - 1\}.$$

Let us now restrict attention to those η, say $\eta \leqslant \eta_0$, for which $|J_\gamma(\eta)| \leqslant \frac{1}{2}$. Then

$$\begin{aligned} F_\gamma(\eta) &= \frac{d}{d\eta}\left[\eta\log\{1 + J_\gamma(\eta)\}\right] \\ &= \frac{d}{d\eta}\left[\eta\left\{J_\gamma - \frac{1}{2}J_\gamma^2 + \frac{1}{3}J_\gamma^3 - \cdots\right\}\right], \qquad J_\gamma \equiv J_\gamma(\eta), \\ &= E_\gamma(\eta) - 1 + \frac{d}{d\eta}\left\{-\frac{1}{2}\eta J_\gamma^2 + \frac{1}{3}\eta J_\gamma^3 - \cdots\right\}, \end{aligned}$$

whence

$$F_\gamma(\eta) - \sqrt{3}\,\gamma(\eta) = H_\gamma(\eta) + \frac{d}{d\eta}\left\{-\frac{1}{2}\eta J_\gamma^2 + \frac{1}{3}\eta J_\gamma^3 - \cdots\right\}. \tag{4.4}$$

Although this may seem a very involved formula, it provides a good way of expanding $F_\gamma(\eta) - \sqrt{3}\,\gamma(\eta)$, given an expansion of $\gamma(\eta)$.

Equation (4.3) yields

$$\begin{aligned} H_\gamma(\eta) &= \left\{1 + \sum_{k=1}^{\infty} C_k\eta^{2k}\right\}\left\{-\frac{\gamma^2}{2!} + \frac{\gamma^4}{4!} - \cdots + \sqrt{3}\left[-\frac{\gamma^3}{3!} + \frac{\gamma^5}{5!} - \cdots\right]\right\} \\ &\quad + \sum_{k=1}^{\infty} C_k\eta^{2k}\{1 + \sqrt{3}\,\gamma(\eta)\} \\ &= \sum_{\lambda_j \in \Lambda_3(N)} D_j\eta^{\lambda_j} + O(\eta^{\mu+\beta_1}), \end{aligned} \tag{4.5}$$

where the constants D_j depend only on the A_i in (4.1) (and on the C_k). Consider now the remaining terms on the right of (4.4). Since

$$J_\gamma(\eta) = \frac{1}{\eta}\int_0^\eta \{\sqrt{3}\,\gamma + H_\gamma\},$$

equation (4.5) ensures that the expansion of J_γ has exponents in Λ, and hence that the expansion of

$$\frac{d}{d\eta}\{\eta J_\gamma^m\} = (1-m)J_\gamma^m + mJ_\gamma^{m-1}\{\sqrt{3}\gamma + H_\gamma\}, \qquad m \geqslant 2,$$

has exponents in Λ_3. Since $\sqrt{3}\gamma + H_\gamma$, and hence J_γ, are known with error $O(\eta^\mu)$, it follows that J_γ^m and $J_\gamma^{m-1}\{\sqrt{3}\gamma + H_\gamma\}$ are known with error $O(\eta^{\mu+\beta_1})$ for $m \geqslant 2$. We apply these two conclusions to the terms in (4.4) that follow H_γ, and the lemma is proved.

LEMMA 4.2. *For $\xi \in (0,1)$ and $p > -1$,*

$$\int_0^1 r(\xi,\eta)\eta^p\,d\eta = \begin{cases} \dfrac{\tan(p\pi/2)}{1+p}\xi^p + \dfrac{1}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{\xi^{2k-1}}{k(1+p-2k)} & \text{if } p \notin \{1,3,5,\dots\}, \\ \dfrac{2}{\pi(1+p)}\xi^p \log\dfrac{1}{\xi} + \dfrac{2\xi^p}{\pi(1+p)^2} & \\ \qquad + \dfrac{1}{\pi}\displaystyle\sum_{k\in\mathbf{N}\setminus\{(1+p)/2\}}\frac{\xi^{2k-1}}{k(1+p-2k)} & \text{if } p \in \{1,3,5,\dots\}. \end{cases}$$

PROOF. Denote the integral by $I_p(\xi)$, and define

$$\phi_p(\xi) = \begin{cases} \xi^p, & 0 < \xi \leqslant 1, \\ 0, & 1 < \xi, \end{cases}$$

so that $\hat{\phi}_p(s) = 1/(s+p)$ for $\sigma > -p$. Then

$$\begin{aligned} I_p(\xi) &= \int_0^\infty r(\xi,\eta)\phi_p(\eta)\,d\eta \qquad (0 < \xi < 1) \\ &= \int_0^\infty r(1,u)\phi_p(\xi u)\,du \\ &\risingdotseq \hat{R}(1-s)\hat{\phi}_p(s) \quad \text{for } -\min\{p,1\} < \sigma < 1, \end{aligned}$$

where we have used the product rule, the notation $r(1,u) = R(u)$, and the statement in (3.11b) that $\hat{R}(1-s)$ exists and is analytic for $-1 < \sigma < 1$. By the inversion formula and (3.11b),

$$I_p(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-s}\frac{\tan(s\pi/2)}{(s-1)(s+p)}\,ds, \qquad -\min\{p,1\} < c < 1.$$

Consider the integral (of this integrand) around the (positively directed) rectangular contour with corners at $c \pm i2n$ and $-2n \pm i2n$; here $n \in \mathbf{N}$. On the parts $\sigma = -2n$, $|t| \leqslant 2n$, and $-2n \leqslant \sigma \leqslant c$, $t = \pm 2n$ of this path, $|\tan(s\pi/2)|$ is bounded; hence the integrand is $O(|s|^{-2})$ there, and the contribution of these three sides tends to zero as $n \to \infty$. Accordingly, $I_p(\xi)$ equals the sum of the residues at poles with $\sigma < c$. If $p \notin \{1,3,5,\dots\}$, these are simple poles at $s = -p$ and $s = -1,-3,-5,\dots$; if $p \in \{1,3,5,\dots\}$, there is a double pole at $s = -p$, and simple poles at $s = -(2k-1)$, $k \in \mathbf{N} \setminus \{(1+p)/2\}$. Evaluating the residues, we obtain the result of the lemma.

LEMMA 4.3. *If* $p \in (2m+1, 2m+3)$ *for some* $m \in \mathbf{N}_0$ *and* $|f(\eta)| \leqslant \text{const.}\, \eta^p$, *then, for* $\xi \in (0,1)$,

$$\int_0^1 r(\xi,\eta) f(\eta)\, d\eta = \frac{1}{\pi} \sum_{k=1}^{m+1} \frac{\xi^{2k-1}}{k} \int_0^1 f(\eta) \eta^{-2k}\, d\eta + O(\xi^p).$$

PROOF. We have

$$\begin{aligned}\int_0^1 r(\xi,\eta) f(\eta)\, d\eta &= \int_0^\xi r(\xi,\eta) f(\eta)\, d\eta - \frac{1}{\pi\xi} \int_\xi^1 \log\left(1 - \frac{\xi^2}{\eta^2}\right) f(\eta)\, d\eta \\ &= \int_0^1 r(1,u) f(\xi u)\, du + \frac{1}{\pi} \sum_{k=1}^{m+1} \frac{\xi^{2k-1}}{k} \left\{ \int_0^1 - \int_0^\xi f(\eta) \eta^{-2k}\, d\eta \right\} \\ &\quad + \frac{1}{\pi} \sum_{k=m+2}^{\infty} \frac{\xi^{2k-1}}{k} \int_\xi^1 f(\eta) \eta^{-2k}\, d\eta.\end{aligned}$$

The assumption that $|f(\eta)| \leqslant$ const. η^p, in which p is not an odd integer, ensures that the first, third, and fourth integrals in this expression are, respectively, $O(\xi^p)$, $O(\xi^{p-2k+1})$ with $p > 2k-1$, and $O(\xi^{p-2k+1})$ with $p < 2k-1$; the fourth is $O(\xi^{p-2k+1}/k)$ as $\xi \to 0$ and $k \to \infty$. This proves the lemma.

LEMMA 4.4. *Let the hypotheses in Lemma* 4.1 *hold, let* $\{\text{odd}\}$ *denote the set* $\{1, 3, 5, \dots\}$ *of odd positive integers, and let* $2P-1$ *be the largest odd integer less than* $\mu + \beta_1$. *Then, as* $\xi \to 0$,

$$\begin{aligned}\rho_\gamma(\xi) = \sum_{k=1}^{P} E_k \xi^{2k-1} &+ \sum_{\lambda_j \in \Lambda_3(N) \setminus \mathbf{N}} F_j \xi^{\lambda_j} \\ &+ \sum_{\lambda_j \in \Lambda_3(N) \cap \{\text{odd}\}} G_j \xi^{\lambda_j} \log \frac{1}{\xi} + O\left(\xi^{\mu+\beta_1}\right),\end{aligned} \tag{4.6}$$

where

$$F_j = \frac{\tan(\lambda_j \pi/2)}{3(1+\lambda_j)} B_j, \quad G_j = \frac{2}{3\pi(1+\lambda_j)} B_j, \tag{4.7}$$

but the coefficients E_k *are defined in terms of the global behavior of* γ (*and not merely in terms of the previous* A_i *and* B_j).

REMARK. In (4.6), the set $\Lambda_3(N) \cap \{\text{odd}\}$ is probably empty for all N, but, if it is not, the exponents in it cause no complication (in contrast to any exponents that there may be in the set $\Lambda_3(N) \cap B(N)$ to be considered presently), because the logarithmic terms in (4.6) will be cancelled when we compute the corresponding expansion of γ.

PROOF. Recall from (3.2) that

(4.8)

$$\rho_\gamma(\xi) = \frac{1}{3} \int_0^1 r(\xi,\eta)\left\{(F\gamma)(\eta) - \sqrt{3}\,\gamma(\eta)\right\} d\eta + \frac{1}{3} \int_1^\infty r(\xi,\eta)(F\gamma)(\eta)\, d\eta,$$

and observe that, for $\xi \in (0,1)$,

$$\text{(4.9)} \qquad \begin{aligned} \int_1^\infty r(\xi,\eta)(F\gamma)(\eta)\,d\eta &= -\frac{1}{\pi\xi}\int_1^\infty \log\left(1-\frac{\xi^2}{\eta^2}\right)(F\gamma)(\eta)\,d\eta \\ &= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\xi^{2k-1}}{k}\int_1^\infty (F\gamma)(\eta)\eta^{-2k}\,d\eta, \end{aligned}$$

where the last integral is $O(k^{-1})$ as $k \to \infty$ because $(F\gamma)(\eta) = O(\log\eta)$ as $\eta \to \infty$.

For the first integral in (4.8), we apply Lemma 4.2 to the terms $B_j\eta^{\lambda_j}$ in (4.2), and Lemma 4.3 to the O-term there, recalling that $\mu + \beta_1$ is not an odd integer. The terms of form $c_k\xi^{2k-1}$ (where $k \in \mathbf{N}$ and $c_k = O(k^{-2})$ as $k \to \infty$) that result from the $B_j\eta^{\lambda_j}$ in (4.2), from the O-term there and from (4.9) are all collected in the first sum of (4.6) if $2k-1 < \mu + \beta_1$ (so that $k \leqslant P$) or in the O-term of (4.6) if $2k-1 > \mu+\beta_1$. In the second sum of (4.6) we have replaced $\Lambda_3(N)\setminus\{\text{odd}\}$ by $\Lambda_3(N)\setminus\mathbf{N}$ because (4.7) shows that $F_j = 0$ when λ_j is an even integer.

4.3. *The expansion of* γ. Lemma 4.4 provides a generalization of step (i) in the proof of Theorem 3.3, and we are now ready to generalize step (ii). To avoid a conceivable but highly improbable complication, we make

ASSUMPTION $A(N)$. *The sets* $\Lambda_3(N)$ *and* $B(N)$ *are disjoint.*

The reason for this will be explained after the proof of Theorem 4.5. We have verified Assumption $A(N)$ numerically for $N \leqslant 100$, and conjecture that it is true for all N.

THEOREM 4.5. *If the hypotheses of Lemma* 4.1 *and Assumption* $A(N)$ *hold, then*

$$\gamma(\xi) = \sum_{\lambda_j\in\Lambda_\beta(N)} A_j\xi^{\lambda_j} + O(\xi^\nu) \quad \text{as } \xi \to 0,$$

for some $\nu \in (\lambda_N, \lambda_{N+1})$. *Here a new coefficient* A_j, *with* j *such that* $\lambda_j \in \Lambda_\beta(N)\setminus\Lambda_\beta(n)$, *is determined by the previous coefficients* A_i, *with* i *such that* $\lambda_i \in \Lambda_\beta(n)$, *if and only if* $\lambda_j \notin B(N)$.

PROOF. Again we use the notation in the proof of Lemma 3.2. In view of (4.6), we define

$$h_1(\xi) = \begin{cases} \displaystyle\sum_{k=1}^{P} E_k\xi^{2k-1} + \sum_{\lambda_j\in\Lambda_3(N)\setminus\mathbf{N}} F_j\xi^{\lambda_j} \\ \displaystyle\qquad + \sum_{\lambda_j\in\Lambda_3(N)\cap\{\text{odd}\}} G_j\xi^{\lambda_j}\log\frac{1}{\xi} & \text{if } 0 \leqslant \xi \leqslant 1, \\ 0 \quad \text{if } 1 < \xi, \end{cases}$$

and $h_2 = h - h_1$. Accordingly, $h_2(\xi) = \rho_\gamma(\xi) - h_1(\xi)$ if $\xi \leqslant 1$, and $h_2(\xi) = O(\xi^{-1}\log\xi)$ as $\xi \to \infty$, by (3.9). The smallest exponent in $h_1(\xi)$ is 1; hence, for $\sigma > -1$,

$$\hat{h}_1(s) = \sum_{k=1}^{P}\frac{E_k}{s+2k-1} + \sum_{\lambda_j\in\Lambda_3(N)\setminus\mathbf{N}}\frac{F_j}{s+\lambda_j} + \sum_{\lambda_j\in\Lambda_3(N)\cap\{\text{odd}\}}\frac{G_j}{(s+\lambda_j)^2},$$

while $\hat{h}_2(s)$ exists and is analytic for $-\mu - \beta_1 < \sigma < 1$, and $\hat{h}_2(\sigma + i\,\cdot\,) \in L_{p'}(-\infty, \infty)$ for all $\sigma \in (-\mu - \beta_1, 1)$ and all $p' > 2$. The obvious analytic (more precisely, meromorphic) continuations of $\hat{h}_1$ and $\hat{h}$ into the strip of convergence of $\hat{h}_2$ will also be denoted by $\hat{h}_1$ and $\hat{h}$, respectively.

Let $\xi \in (0,1)$. We proceed from (3.14) and move the path of integration to $\sigma = -\nu$ for any $\nu \in (\lambda_N, \mu + \beta_1)$; the remarks, in the proof of Lemma 3.2, regarding the contribution of paths $t = \pm M$ remain valid when the l there is replaced by ν, and $\xi^{-s}Q(s)\hat{h}(s)$ has no singularities for $\sigma \in (-\mu - \beta_1, -\lambda_N)$. There are four sets of poles between the original path of integration ($\sigma = c$) and the new one ($\sigma = -\nu$); we consider one set at a time.

(a) The points $s = 1 - 2k$, $k \in \{1, 2, \ldots, P\}$, are simple poles of $\hat{h}_1$, and, since $Q(1 - 2k) = -1$,

$$\sum_{(a)} \text{residues} = -\sum_{k=1}^{P} E_k \xi^{2k-1}.$$

(b) The points $s = -\lambda_j$, where $\lambda_j \in \Lambda_3(N)\setminus\mathbf{N}$, are simple poles of $\hat{h}_1$ and are not poles of Q by Assumption $A(N)$, so that

$$\sum_{(b)} \text{residues} = \sum_{\lambda_j \in \Lambda_3(N)\setminus\mathbf{N}} F_j Q(-\lambda_j)\xi^{\lambda_j}.$$

(c) The points $s = -\lambda_j$, where $\lambda_j \in \Lambda_3(N) \cap \{\text{odd}\}$, are double poles of $\hat{h}_1$, and

$$\sum_{(c)} \text{residues} = -\sum_{\lambda_j \in \Lambda_3(N)\cap\{\text{odd}\}} G_j \left\{ \xi^{\lambda_j} \log\frac{1}{\xi} + \frac{\sqrt{3}\,\pi}{2}(1 + \lambda_j)\xi^{\lambda_j} \right\}.$$

(d) The points $s = -\lambda_j$, where $\lambda_j \in B(N)$, are simple poles of Q and distinct from those of $\hat{h}_1$ by Assumption $A(N)$, so that

$$\sum_{(d)} \text{residues} = \sum_{\lambda_j \in B(N)} H_j \xi^{\lambda_j},$$

where

$$H_j = \operatorname{res} Q(s)\,|_{s=-\lambda_j} \hat{h}(-\lambda_j). \tag{4.10}$$

Since $\xi \in (0,1)$, we have $g(\xi) = \gamma(\xi)$ and $h(\xi) = \rho_\gamma(\xi)$; equation (3.14) yields

$$\gamma(\xi) = \rho_\gamma(\xi) + \sum \text{residues} + \frac{1}{2\pi i}\int_{-\nu - i\infty}^{-\nu + i\infty} \xi^{-s} Q(s)\hat{h}(s)\,ds.$$

We substitute for $\rho_\gamma(\xi)$ from (4.6) and for the residues, noting that the terms $E_k\xi^{2k-1}$, and those containing $\log(1/\xi)$, in (4.6) are cancelled by the residues; there results

(4.11)

$$\gamma(\xi) = \sum_{\lambda_j \in \Lambda_3(N)\setminus\mathbf{N}} F_j\{1 + Q(-\lambda_j)\}\xi^{\lambda_j} - \frac{\sqrt{3}\,\pi}{2}\sum_{\lambda_j \in \Lambda_3(N)\cap\{\text{odd}\}} G_j(1 + \lambda_j)\xi^{\lambda_j}$$
$$+ \sum_{\lambda_j \in B(N)} H_j \xi^{\lambda_j} + \frac{\xi^\nu}{2\pi}\int_{-\infty}^{\infty} \xi^{-it} Q(-\nu + it)\hat{h}(-\nu + it)\,dt + O(\xi^{\mu + \beta_1}),$$

where $Q(-\nu + i\,\cdot\,) \in L_{3/2}(-\infty, \infty)$, because $Q(-\nu + it) = O(t^{-1})$ as $t \to \pm\infty$, and $\hat{h}(-\nu + i\,\cdot\,) \in L_3(-\infty, \infty)$, so that the integral (without the factor $\xi^{\nu}/2\pi$) is bounded, by Hölder's inequality.

We claim that all the exponents λ_j in (4.11) belong to $\Lambda_\beta(N)$. Indeed, if $\lambda_j \in \Lambda \setminus \Lambda_\beta$, then $\lambda_j \in \{2, 4, 6, \dots\}$, and this is impossible in (4.11) because there $\lambda_j \notin \mathbf{N}$ or $\lambda_j \in \{\text{odd}\}$ or $\lambda_j \in B$. Since in (4.11) each $\lambda_j \leqslant \lambda_N$, it follows that $\lambda_j \in \Lambda_\beta(N)$.

Comparison with (4.1) shows that in (4.11) the coefficient of ξ^{λ_j} must equal A_j whenever $\lambda_j \in \Lambda_\beta(n)$. Consider the coefficient of ξ^{λ_j} when $\lambda_j \in \Lambda_\beta(N) \setminus \Lambda_\beta(n)$; if $\lambda_j \notin B(N)$, then (4.7) and Lemma 4.1 show that the coefficient is determined by the previous A_i; if $\lambda_j \in B(N)$, then (4.10) shows that the coefficient is not so determined. Of course, we relabel this new coefficient A_j in either case.

Remark. Assume that the hypotheses of Lemma 4.1 hold but that Assumption $A(N)$ is false, for some particular n and $N(n, \mu)$. Then there exists an exponent $\lambda_m \in \Lambda_3(N) \cap B(N)$, and $Q(s)\hat{h}(s)$ has a double pole at $s = -\lambda_m$. This causes the term with $j = m$ in (4.11) to be replaced by $F_m \xi^{\lambda_m}(a + b \log 1/\xi)$, where a, b are constants and $b \neq 0$. At the Nth stage (and perhaps earlier), the hypothesis (4.1), with n replaced by $N(n, \mu)$, must be modified to include the logarithmic term; at still later stages, higher and higher powers of logarithms accrue in the process of expanding $F\gamma - \sqrt{3}\gamma$.

Corollary 4.6. *If ψ is a solution of* (2.1) *and $\Lambda_3 \cap B = \varnothing$, then*

$$\psi(\xi) \sim \frac{\pi}{6} + \sum_{\lambda_j \in \Lambda_\beta} A_j \xi^{\lambda_j} \quad \text{as } \xi \to 0 \tag{4.12}$$

in the sense of an asymptotic expansion:

$$\psi(\xi) - \frac{\pi}{6} - \sum_{\lambda_j \in \Lambda_\beta(n)} A_j \xi^{\lambda_j} = o\left(\xi^{\lambda_n}\right) \quad \text{as } \xi \to 0,$$

for all $n \in \mathbf{N}$.

Corollary 4.7. *If $\Lambda_3 \cap B = \varnothing$, then the complex coordinate $z = x + iy$ is related to the complex potential $\chi = \Phi + i\Psi$ by*

$$z(\chi) \sim -ig^{-1/3}\left(\frac{3}{2}i\chi\right)^{2/3}\left\{1 + \sum_{\lambda_j \in \Lambda_\beta} K_j (i\chi)^{\lambda_j}\right\} \quad \text{as } \chi \to 0, \tag{4.13}$$

where the constant g is the gravitational acceleration, $\arg(i\chi) \in [-\pi/2, \pi/2]$, *and the coefficients K_j are real. If the set $\{1\} \cup B$ is linearly independent over the rationals, then the sum in* (4.13) *is only over exponents in*

$$\Lambda_{\beta,0} = \{m_1\beta_1 + \cdots + m_r\beta_r \colon m_1 + \cdots + m_r \geqslant 1\};$$

in other words, $K_j = 0$ whenever $\lambda_j \in \Lambda_\beta \setminus \Lambda_{\beta,0}$.

Proof. On the image $\Psi = 0$ of the free boundary, we have

$$\xi = -a\Phi\left\{1 + \sum_{k=1}^{\infty} c_k \Phi^{2k}\right\}, \qquad a > 0, \tag{4.14}$$

where a and c_k are constants, and the series converges for sufficiently small values of $|\Phi|$. Define $\theta(\Phi) = \psi(\xi(\Phi))$; it is a basic hypothesis in the derivation of (1.1) that x and θ are odd functions of Φ, while y is even. Accordingly, (4.12) and (4.14) imply that

$$\theta(\Phi) \sim -\operatorname{sgn}\Phi\left\{\frac{\pi}{6} + \sum_{\lambda_j \in \Lambda_\beta} B_j|\Phi|^{\lambda_j}\right\} \quad \text{as } \Phi \to 0, \tag{4.15}$$

for certain coefficients B_j (which are not those in (4.2)). Moreover, it follows from the boundary condition of the basic problem (or, equivalently, from (1.1), from the fact that $\theta(\Phi)$ is the boundary value of $\operatorname{Im}\log(dz/d\chi)$, and from suitable choice of the additive constant in $\operatorname{Re}\log(dz/d\chi)$) that

$$\frac{\partial z}{\partial \Phi} = \left(-3g\int_0^\Phi \sin\theta\right)^{-1/3} e^{i\theta(\Phi)}, \qquad \Phi \in [-c, c]\setminus\{0\},\ \Psi = 0, \tag{4.16}$$

for some constant $c > 0$. Using (4.15) in this formula, we obtain (4.13) for $z(\Phi)$ (that is, for $\arg(i\chi) = -\pi/2$ or $\pi/2$); that the K_j are real follows from the symmetry.

To prove that the asymptotic series for $z(\chi)$ in (4.13) is the only appropriate extension (into the half-plane $\Psi < 0$) of the series for $z(\Phi)$, we let h denote the difference between two such extensions to *finitely* many terms. It then suffices to prove the following: if $h(\chi)$ is analytic and bounded in $D = \{\chi\colon\ 0 < |\chi| \leqslant c, -\pi/2 \leqslant \arg(i\chi) \leqslant \pi/2\}$, and if, for some constant $\mu > 0$, we have $h(\Phi) = O(|\Phi|^\mu)$ as $\Phi \to 0$ (with $\arg(i\Phi) = -\pi/2$ or $\pi/2$), then $h(\chi) = O(|\chi|^\mu)$ as $\chi \to 0$ in D. Now this follows from application of the Phragmén-Lindelöf theorem [**11**, p. 176] to the function f defined by $f(\chi) = \chi^{-\mu}h(\chi)$; a suitable auxiliary function is $\omega(\chi) = \exp\{-(i\chi)^{-1/2}\}$.

It remains to prove that, if the set $\{1\} \cup B$ is linearly independent over the rationals, then $K_j = 0$ whenever $\lambda_j \in \Lambda_\beta \setminus \Lambda_{\beta,0}$. Suppose that

$$z(\Phi) = -ig^{-1/3}\left(\tfrac{3}{2}i\Phi\right)^{2/3}\left\{1 + b_1(i\Phi)^{\alpha_1} + \cdots + b_n(i\Phi)^{\alpha_n} + k(i\Phi)^\lambda + O\left(|\Phi|^\mu\right)\right\},$$

where $0 < \alpha_1 < \cdots < \alpha_n < \lambda < \mu$, each $\alpha_j \in \Lambda_{\beta,0}$ and $\lambda \in \Lambda_\beta \setminus \Lambda_{\beta,0}$. We know, from our construction by way of (4.16), that this approximation may be differentiated term by term, and that the exact function $z(\Phi)$ satisfies the boundary condition of the basic problem. Then a slight variant of a calculation by Norman [**9**, p. 262] shows that, under the foregoing hypothesis, the coefficient $k = 0$. Repeated application of this argument proves the result.

Appendix. On the numbers β_j. We recall that β_j denotes the jth positive root of $\sqrt{3}(1 + \beta) = \tan(\beta\pi/2)$; its simplest properties were noted in Lemma 3.1. We now probe a little deeper.

LEMMA A.1. *Each number β_j is transcendental.*

PROOF. We abbreviate β_j to β for any fixed $j \in \mathbf{N}$. It was shown by Grant [**4**, p. 260], albeit somewhat tersely, that β is irrational. To prove it transcendental, assume the contrary: that β is an algebraic irrational number. The equation of which β is a root may be written

$$e^{i\pi\beta} = \frac{1 + i\sqrt{3}\,(1+\beta)}{1 - i\sqrt{3}\,(1+\beta)}. \tag{A.1}$$

Here the right-hand side is an algebraic number (because β is one, and the algebraic numbers form a field), so, therefore, is the left-hand side. In other words, $(-1)^{\beta}$ is an algebraic number. But this contradicts the Gelfond-Schneider theorem [**10**, p. 76], which states that a^b is transcendental whenever a is algebraic (and neither 0 nor 1) and b is algebraic and irrational.

REMARK. In Corollary 4.6 we assumed that the sets

$$\Lambda_3 = \{2\} \cup \{m_1\beta_1 + \cdots + m_r\beta_r + m_{r+1}2\colon m_1 + \cdots + m_r + m_{r+1} \geqslant 2\}$$

and $B = \{\beta_1, \beta_2, \beta_3, \ldots\}$ are disjoint. It is clear that the truth of this assumption would be implied by the truth of the following, very natural

CONJECTURE. *The set $\{1\} \cup B$ is linearly independent over the rationals.*

(That is, if $x_1, \ldots, x_k$ are distinct elements of $\{1\} \cup B$, and $\alpha_1, \ldots, \alpha_k$ are rational numbers, then $\sum_{n=1}^{k}\alpha_n x_n = 0$ implies that $\alpha_1 = \cdots = \alpha_k = 0$.) Unfortunately, our only result in this direction is the following.

THEOREM A.2. *If $j \neq k$, then the set $\{1, \beta_j, \beta_k\}$ is linearly independent over the rationals.*

PROOF. Assume the contrary: then there exist integers p, q, and r, not all zero, such that $p\beta_j + q\beta_k = r$. Neither p nor q can be zero; we may suppose that $r \geqslant 0$; then at least one of p and q must be positive, say $p > 0$. Now

$$(-1)^r = e^{i\pi r} = \left(e^{i\pi\beta_j}\right)^p \left(e^{i\pi\beta_k}\right)^q = \left\{\frac{1 + i\sqrt{3}\,(1+\beta_j)}{1 - i\sqrt{3}\,(1+\beta_j)}\right\}^p \left\{\frac{1 + i\sqrt{3}\,(1+\beta_k)}{1 - i\sqrt{3}\,(1+\beta_k)}\right\}^q$$

by (A.1); since $\beta_k = (r - p\beta_j)/q$, we have

$$\begin{aligned}(-1)^r\left\{1 - i\sqrt{3}\,(1+\beta_j)\right\}^p &\left\{1 - i\sqrt{3}\left(1 + \frac{r - p\beta_j}{q}\right)\right\}^q \\ &= \left\{1 + i\sqrt{3}\,(1+\beta_j)\right\}^p \left\{1 + i\sqrt{3}\left(1 + \frac{r - p\beta_j}{q}\right)\right\}^q.\end{aligned} \tag{A.2}$$

Suppose that $q > 0$; expanding both sides of (A.2), and rearranging the result, we obtain

$$i\sqrt{3}\,P_1(\beta_j) = P_2(\beta_j),$$

where P_1 and P_2 are polynomials with (real) rational coefficients, and β_j is real. Hence each polynomial is zero, which makes β_j an algebraic number and thus contradicts Lemma A.1.

If $q < 0$, we rewrite (A.2) to have positive exponents p and $-q$ on both sides, and argue as before.

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