NBER WORKING PAPERS SERIES

ON THE BEHAVIOR OF COMMODITY PRICES

Angus Deaton

Guy Laroque

Working Paper No. 3439

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 1990

We should like to thank the World Bank for providing the data used in this study. We are grateful to the McDonnell Foundation, and the Lynde and Harry Bradley Foundation for financial support, and to John Campbell, Mark Gersovitz, Christopher Gilbert, Christina Paxson, and two referees for helpful discussions. We are particularly grateful to Brian Wright and Jeffrey Williams, who showed us early versions of their own work, and who identified important mistakes and misunderstandings in a previous version. This paper is part of NBER's research program in Economic Fluctuations. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Working Paper #3439
September 1990

ON THE BEHAVIOR OF COMMODITY PRICES

ABSTRACT

The classical theory of commodity price determination integrates myopic supply and demand on the one hand with competitive storage (speculation) under rational expectations on the other. Taking into account the fact that inventories must be non-negative, this paper derives from the theory testable implications on the behavior of prices, and makes a first attempt to confront these implications with the empirical evidence. The nonlinearities turn out to be a crucial ingredient in matching the stylized facts, particularity the asymmetries and the sharp upward flares that characterize many commodity prices. The model, simple as it is, goes a long way in reproducing the main features of the data for a range of commodities.

Angus Deaton
Woodrow Wilson School
Robertson Hall
Princeton University
Princeton, NJ 08544-1013

Guy Laroque
INSEE Direction Generale
18, Boulavard Adolphe-Pinard
75675 Paris, Cedex 14
FRANCE

0 Introduction

This paper is concerned with the theory and empirical behavior of commodity prices. The topic is of great practical importance, particularly for the formulation of economic policy in less developed countries, many of whom depend heavily on the exports of a small number of primary commodities. Commodity prices are *extremely* volatile. Figures 1 gives just one of many possible examples; it shows an index of the average annual dollar price of sugar from 1900 to 1987 deflated by the U.S. consumer price index. The series shows no obvious trend during this century, but there are several sharp peaks, most notably in 1920, in the late 1970's and early 1980's. Some of the price rises are very sharp indeed, and the price in 1975 was nearly ten times its value less than a decade before. For some commodities, there have been swings from trough to peak in just a few months. For countries whose export earnings and GNP are dependent on these commodities, such volatility poses major problems both of macroeconomic and microeconomic policy. An understanding of the stochastic processes governing these price movements is essential for macroeconomic management, for national consumption and saving policies, for agricultural pricing policies, and for the design of risk sharing mechanisms between farmers, resource holders, and government.

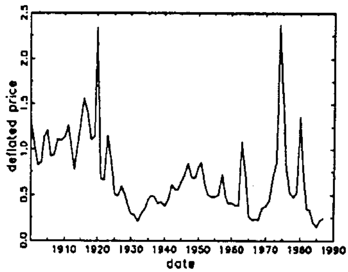

Figure 1: Annual sugar prices, 1900-1987

There exists a well developed theory of the determination of commodity prices, which will be discussed in the next section. However, there is little or no research that asks whether or not that theory is capable of explaining the actual behavior of prices, and it is the confrontation of theory and evidence that is our main task in this paper.

Before considering the theory, it is useful to have firmly in mind some of the stylized facts about commodity prices. The sugar prices in Figure 1 are among the most volatile, but the general features of this series are not atypical. Table 1 summarizes some of the facts about thirteen primary commodities over the same period, 1900 to 1987, facts that we shall need to refer to when we discuss the properties of the theoretical model. *All* of the commodities have first-order autocorrelation coefficients in excess of 0.6, with ten out of the thirteen greater than 0.8. The second-order coefficients are lower, but still substantial. The persistence measures in the third column can be interpreted as the fraction of a current innovation that will persist into the indefinite future, see Cochrane (1988). Apart from bananas, all of these measures are less than 0.3, so that the general picture is one where, in spite of the volatility, prices tend to revert to their mean or to a deterministic trend. The coefficients of variation show that sugar is the most volatile, but that there is substantial year to year variability in most of these prices. Finally, the last two columns measure skewness and kurtosis, with measures that are designed to be zero for a normal distribution. In Figure 1, there is substantial positive skewness for sugar, because there are few or no downward spikes to match the pronounced upward spikes. Similar skewness is found for several, but not all of the commodities; none show negative skewness. Sugar also displays substantial kurtosis, with tails much thicker than those of the normal distribution, a feature that, once again, appears for several other commodities.

Table 1 shows that commodity prices are highly autocorrelated, as well as displaying variability, skewness, and kurtosis. There are a number of strategies that have been used to account for these features. One is standard descriptive time-series

Table 1: Commodity Price Facts 1900-1987

	one year a-c	two year a-c	persist-ence	coeff of variation	skew-ness γ_1	kurt-osis γ_2
bananas	0.91	0.82	0.52	0.17	0.15	-0.98
cocoa	0.83	0.66	0.24	0.54	0.99	1.21
coffee	0.80	0.62	0.11	0.45	1.66	3.82
copper	0.84	0.64	0.22	0.38	1.02	0.86
cotton	0.88	0.68	0.13	0.35	0.35	0.07
jute	0.71	0.45	0.09	0.33	0.61	0.26
maize	0.76	0.53	0.10	0.38	1.18	2.48
palm oil	0.73	0.48	0.05	0.48	3.24	16.52
rice	0.83	0.61	0.08	0.36	0.55	0.03
sugar	0.62	0.39	0.06	0.60	1.49	3.08
tea	0.78	0.59	0.28	0.26	0.04	0.04
tin	0.90	0.76	0.18	0.42	1.66	3.05
wheat	0.86	0.68	0.11	0.38	0.87	0.61

Notes: One year and two-year a-c are first and second order autocorrelation coefficients respectively. Persistence is the normalized spectral density at zero, interpreted as a measure of the degree to which shocks persist; it uses a Bartlett window with a window width of 40 years. The skewness measure γ_1 is $\mu_3/(\mu_2)^{1.5}$ and the kurtosis measure γ_2 is (μ_4/μ_2^2)-3, where μ_r is the rth (central) moment.

Source: World Bank Commodities Division.

modelling. Given the high and slowly declining pattern of autocorrelations, many analysts would start by differencing the series, and fitting a low-order ARMA process to the first-differences, although note that non-standard assumptions are required about the innovations to reproduce the skewness and kurtosis. In practice, for several of these commodity prices, it is hard to reject the most parsimonious model, which is that the time series are random walks, see Cuddington and Urzua (1988) and Gersovitz and Paxson (1990), although note that the persistence measures in Table 1 are much lower than the figure of unity which is implied by a random walk. But, from an economist's point of view, the random walk hypothesis seems very implausible, at least for commodities where the weather plays a major role in price fluctuations; a random walk requires that *all* fluctuations in price be permanent. Nor would an LDC government be wise to treat commodity booms as permanent, although there are occasions when some appear to have done so.

A different, more theoretical modelling strategy is to begin with supply and demand, attributing fluctuations to supply shocks, magnified and distorted by inelastic and non-linear demand functions. Such an approach can produce skewness and kurtosis, even from symmetric and mesokurtic harvests. However, much of the work in this tradition, reviewed, for example, by Ghosh, Gilbert and Hughes Hallett (1987), assumes myopic demand and supply behavior, and uses essentially atheoretical distributed lags to account for autocorrelation. Our own approach is to follow the supply and demand tradition, but with explicit modelling of the behavior of competitive speculators who hold inventories of commodities in the expectation of making profits. Such models have been well-developed in the theoretical literature, and, as we shall see below, they provide a fairly natural way of accounting for many of the facts in the Table.

The paper proceeds as follows. In the next section, we restate the theory, together with a number of theoretical developments that are useful in interpreting the data. In particular, we set up the model in a way that allows for the standard functional forms that are used in applied work, and we develop a number of new comparative static results that are helpful in determining the properties of the model, and thus how it can be tailored to match reality. Section 2 attempts to do that matching. Since the model is complex, and there are few or no cases where there exist closed-form functional forms for commodity prices, direct full-information estimation would be computationally demanding and not necessarily very informative about the sources of model failure. Instead, we first follow Wright and Williams (1990) by simulating the model, guided by the comparative static results of Section 1, in order to discover the relationship between the inputs into the model, demand functions, harvest distributions, and storage technology, and its output, the time-series behavior of commodity prices. We also present the results of an econometric analysis of some of the implications of the model. Section 3 provides a brief summary and conclusion.

We find that the standard rational expectations competitive storage model of commodity price model is capable of explaining a number of the stylized facts, including skewness, and the existence of rare but violent explosions in prices, coupled with a high degree of price autocorrelation in more normal times. A central feature of the model is the explicit recognition of the fact that it is impossible for the market as a whole to carry negative inventories, and this introduces an essential nonlinearity which carries through into nonlinearity of the predicted commodity price series. For most of the thirteen commodity prices in Table 1, the behavior of prices from one year to the next conforms to the predictions of the theory about conditional expectations and conditional variances. However, given the nonlinearity both of the model and of the actual prices, such conformity is not enough to ensure that the theory yields a complete account of the data. In particular, our work so far does not yield a fully satisfactory explanation for the high autocorrelation coefficients observed in the Table.

1. THE THEORY OF PRODUCTION WITH COMPETITIVE STORAGE

1.0 Introduction

This section contains the basic theory used in the rest of the paper. The theory originates with the pioneering work of Gustafson (1958), who studied the properties of the optimal demand for commodity stocks, and with the work of Muth (1961), who introduced the assumption of rational expectations. Unlike Muth however, Gustafson insisted on the impossibility of carrying forward negative inventories - crops cannot be consumed before they are grown - and the resulting non-negativity constraint is an important feature of the more recent literature. Over the past twenty years, the basic model has been elaborated in several different directions. Samuelson (1971) proved the optimality of competitive storage, and showed that the theory generates a non-linear first-order Markov process for prices, a point that we shall elaborate at some length below. Other important contributions are those of Danthine (1977), Schechtman and Escudero (1977), Kohn (1978), Newbery and Stiglitz (1981, 1982), Sheinkman and Schechtman (1983), Salant (1983), Wright and Williams (1982, 1984, 1990), and Hart and Kreps (1986).

We begin by reviewing the basic model, and by stating and proving Theorem 1 on the existence of a stationary rational expectations equilibrium. We repeat this more or less standard material partly in order to make the paper self-contained, but also because our approach is flexible enough to handle the frequently made assumption that demand functions are linear, something that is not true of the proofs currently available in the literature. We then state and discuss the implications of the theorem for the behavior of the price, and prove Theorem 2, which provides conditions under which the price follows a renewal process. Finally, Theorem 3 provides comparative static results that are useful for the empirical analysis, and which do not seem to have appeared previously in the literature.

1.1 Prices and inventories

Time is discrete. In period t, the price of the commodity relative to some unspecified numeraire is p_t. For clarity of exposition, we distinguish two categories of agents, producer-consumers whose excess demand for the commodity depends only on the current price p_t, and inventory holders, or speculators, who carry forward the commodity from one period to the next.

Start from the case where there are no inventories or inventory holders. Write $D(p_t)$ for the deterministic demand function, and z_t for the amount which is randomly but inelastically supplied; we shall frequently refer to z_t as the harvest. Equilibrium price is given by the equality $D(p_t) = z_t$. Note that it is also possible to allow demand to be stochastic, in which case z_t is interpreted as the difference between the harvest and the stochastic part of the demand function. An instantaneous supply response can also be accounted for by interpreting $D(p_t)$ as a non-stochastic *excess* demand function. Denote the inverse demand function $P(z) = D^{-1}(z)$, so that, with no inventories, the price at date t is $P(z_t)$. We also assume that z_t has compact support, with lower bound $\underline{z}$ and upper bound $\bar{z}$. In the simplest case, which is also the one studied in most of the paper, we take the harvest to be *i.i.d.* over time. Further, we require $\infty > P(\underline{z}) > 0$, so that, when the harvest takes its lowest possible value, price is positive and finite; no matter how bad the harvest, there is always a finite price that clears the market. We collect the assumptions of this paragraph into:

Assumption 1: The random variables z_t (the harvests), are *i.i.d.*, and have compact support with lower bound $\underline{z}$ and upper bound $\bar{z}$. $D(p)$, defined on the interval (p_0, p_1), is continuous and strictly decreasing, and $D(p)$ tends to $+\infty$ as p tends to p_0. The inverse demand function $P(z)$ satisfies $\infty > P(\underline{z}) > 0$.

For analytical tractability, it is sometimes assumed that $D(p)$ is linear, *e.g.* Gustafson (1958), Newbery and Stiglitz (1981, 1982), examples which we shall follow in part of Section 2 below. To encompass this case, we allow p_0 to be $-\infty$, and p_1 to be $+\infty$.

In the absence of inventories, and under the assumption that harvests are *i.i.d.*, so will the prices $p_t = P(z_t)$ be *i.i.d.* The introduction of inventories means that this will no longer be true. Suppose that inventory holders have access to a simple constant returns storage technology: one unit of commodity stored at t yields $(1-\delta)$ units at $t+1$. The model can be extended without difficulty to the case where there are also fixed storage costs per unit in terms of the numeraire commodity. We assume that inventory holders are risk-neutral and can borrow and lend from a perfect capital market where the rate of interest on the numeraire is r. Write $\beta=1/(1+r)$, and assume that there is a real cost to holding inventories, whether in deterioration or in interest charges, i.e.

Assumption 2: Inventories are costly: $\beta(1-\delta) = (1-\delta)/(1+r) < 1$.

Write $E_t p_{t+1}$ for the expected value of p_{t+1}, conditional on the available information at time t. An inventory holder who carries an inventory I_t into the next period expects a profit of $[\beta(1-\delta)E_t p_{t+1} - p_t]I_t$, so that profit maximization yields:

$$\begin{aligned} I_t = 0 \quad &\text{if} \quad \beta(1-\delta)E_t p_{t+1} < p_t \\ I_t \geq 0 \quad &\text{if} \quad \beta(1-\delta)E_t p_{t+1} = p_t. \end{aligned} \tag{1}$$

Inventories are zero if there is an expected loss from holding them, while if there is an expected profit from holding one unit when $I_t=0$, speculators will demand positive inventories, bidding up the price until current and expected future prices are equal, after due allowance for carrying costs.

In equilibrium, supply, including inventories from the previous period, must equal demand, including demand for inventories to carry forward into the next period, so that,

$$z_t + (1-\delta)I_{t-1} - I_t = D(p_t). \tag{2}$$

Hence, combining (1) and (2), we have

$$p_t = \max[\,\beta(1-\delta)E_t p_{t+1},\ P\{z_t + (1-\delta)I_{t-1}\}\,]. \tag{3}$$

Given price expectations, (3) determines the equilibrium price at date t, and (2) gives the corresponding level of inventories that will be carried forward from t to $t+1$.

To complete the model, it is necessary to define the information that is available to agents at time t, and which is used by them to form their expectations. We make the standard assumption that the agents know the "amount on hand," $z_t + (1-\delta)I_{t-1}$, the current harvest together with any inventories from the previous period. This is the state variable in the system and we denote it by x_t, *i.e.*

$$x_t = z_t + (1-\delta)I_{t-1}. \tag{4}$$

Since I_{t-1} is non-negative, x_t lies in the interval $X=[\underline{z}, \infty)$.

We are then led to the following:

Definition: A stationary rational expectations equilibrium *(SREE)* is a price function $f: X \to \Re$ which describes the current price p_t as a function of the state x_t, and satisfies for all x_t,

$$p_t = f(x_t) = \max[\,\beta(1-\delta)E_t f\{z_{t+1} + (1-\delta)I_t\},\ P(x_t)\,] \tag{5}$$

where

$$I_t = x_t - P^{-1}(p_t) = x_t - P^{-1}\{f(x_t)\} \tag{6}$$

Under the assumptions made, and in particular the assumption that the z_t's are *i.i.d.*, $f(x)$ is the solution to the functional equation

$$f(x) = \max[\,\beta(1-\delta)Ef\{z + (1-\delta)(x - P^{-1}\{f(x)\})\},\ P(x)\,], \tag{7}$$

where the expectation is taken with respect to the random variable z. The existence of the price function, as well as some of its important properties are given by the following:

Theorem 1: Under Assumptions 1 and 2, there is a unique SREE f in the class of non-negative continuous non-increasing functions. Furthermore, let $p^* = \beta(1-\delta)Ef(z)$. Then:

$$\begin{aligned} f(x) &> P(x) \quad \text{for} \quad P(x) < p^* \\ f(x) &= P(x) \quad \text{for} \quad P(x) \geq p^* \end{aligned} \tag{8}$$

f is strictly decreasing whenever it is strictly positive. The equilibrium level of inventories, $x - P^{-1}\{f(x)\}$, is strictly increasing whenever $P(x) < p^*$.

Proof: See Appendix.

The relationship between $f(x)$ and P(x) is illustrated in Figure 2 for the case where $P(x)=10/x$, the harvest is normally distributed (with truncation at five standard deviations), r is 3% and δ is 5%. Methods for calculating these functions will be described in the next section.

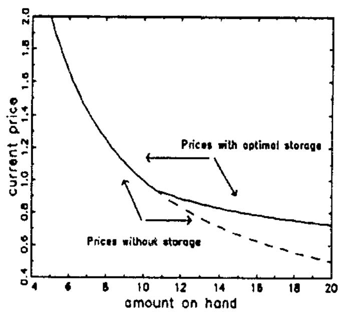

Figure 2: Prices with and without storage

The implications of the theorem for the price process are worth describing in some detail since they are important for assessing the empirical evidence. There are two possibilities or regimes depending on whether or not there are positive inventories:

(*i*) If $p_t \geq p^*$, $I_t = 0$, $p_t = P(x_t)$ is the price at which current (use) demand is equated to current supply (including left over inventories), and p_{t+1} is independent of p_t and depends only on z_{t+1}, so that

$$p_{t+1} = f(z_{t+1}), \quad p_t \geq p^*. \tag{9.1}$$

(*ii*) If $p_t < p^*$, $I_t = x_t - P^{-1}\{f(x_t)\} > 0$, , then total demand is more than the demand for current consumption, $p_t = \beta(1-\delta)E_t p_{t+1}$ is larger than $P(x_t)$ and

$$p_{t+1} = [\beta(1-\delta)]^{-1} p_t + \eta_{t+1}, \quad p_t < p^*. \tag{9.2}$$

where $\eta_{t+1} = f(x_{t+1}) - p_t/\beta(1-\delta) = f[z_{t+1} + (1-\delta)\{x_t - P^{-1}f(x_t)\}] - p_t/\beta(1-\delta)$ is an innovation, *i.e.* $E_t(\eta_{t+1})=0$. The cutoff p^* is equal to $\beta(1-\delta)E\{f(z)\}$, the current price at which, with no inventory demand, a unit held into the next period would make zero expected profit. The autoregression function of p_t, $E(p_{t+1} | p_t)$, is therefore given by

$$E(p_{t+1} | p_t) = \min(p^*, p_t)/[\beta(1-\delta)]. \tag{10}$$

It is also useful for empirical purposes to be able to characterize the conditional variance of prices, $V(p_{t+1} | p_t)$. We have not been able to derive any expression corresponding to (10), but in the Appendix, lemma 4, we prove that, given convexity of $P(x)$,

$$\partial V(p_{t+1} | p_t)/\partial p_t \geq 0. \tag{11}$$

At higher levels of prices, there are fewer inventories on hand before the next period's harvest, so that next period's price will be more volatile, although once p_t is above p^*, there are no inventories, and the conditional variance is constant. Like the autoregression (10), the conditional variance function rises with p until $p = p^*$, after which it is constant.

What are the long-run dynamics of prices? A priori, there are three possibilities: the price might stay in one or other of the regimes forever, or it might oscillate between the two. If the distribution of z is very concentrated, if the price function $P(x)$ is not very convex, or if storage and interest costs are high, so that almost surely $P(z) \geq \beta(1-\delta)E\{P(z)\}$, then storage will never be profitable, the price stays above p^*, and (9.1) always prevails. On the other hand, if δ were negative, so that inventories are "productive," and one unit today yields $(1-\delta) > 1$ tomorrow, the level of inventories may increase without limit, with the price tending to zero. The case of interest is, of course, the third one, where the price follows a renewal process. When prices are low, inventories are positive, but the depreciation rate $\delta > 0$ ensures that in finite time there will be a stockout, and the process will revert to regime (9.1). The long run invariant stationary distribution of prices oscillates between the two regimes.

Theorem 2: Assume that $\delta > 0$ and $P(\bar{z}) < \beta(1-\delta)EP(z)$. Then the limit distribution of inventories has a compact support, and the price follows a renewal process.

Proof: See Appendix.

Remark: The existence of an invariant distribution has been thoroughly investigated in the literature. Following the line initiated by Schechtman and Escudero (1977), Theorem 2 can be extended to the case where $\delta = 0$, provided that p_0 is nonnegative. When $\delta < 0$, there are cases where the level of inventories tends to infinity, see again Schechtman and Escudero.

Our final theorem concerns comparative statics, and in particular the effects on the equilibrium of the convexity of the price function $P(x)$.

Theorem 3 (Comparative statics): the equilibrium price function f, the associated cut-off price p^* and the inventory demand $I(x) = x - P^{-1}\{f(x)\}$ are increasing in the discount factor β. They decrease when there is a first-order stochastic increase in the distribution of supply shocks. Moreover, if $P(x)$ is convex, $f(x)$ is convex, and both $f(x)$ and $I(x)$ increase when the distribution of supply shocks is modified through a mean preserving spread.

Proof: See Appendix.

Remark: A large part of the previous literature takes an approach which is dual to ours, whereby $P(x)$ is interpreted as the derivative of the instantaneous utility of an infinite horizon consumer with intertemporally separable tastes and discount factor β, see e.g. Danthine (1977), Schechtman and Escudero (1977), Scheinkman and Schechtman (1983). As already noted, our more flexible approach to the domain of definition of the demand functions allows for the case of linear demands. More importantly, as in Hart and Kreps (1986), the existence of the policy function $f(x)$ can be studied without reference to the limiting distribution of the state variable in situations where the optimal intertemporal utility level of the representative consumer would be unbounded. Finally, note that in the presence of risk averse speculators, the equilibrium no longer corresponds to a representative consumer problem, so that such cases could only be approached, as in this paper, by directly modelling the formation of prices in the market.

2. MATCHING THE THEORY AND THE DATA

2.1 Simulation of the theoretical model

The first step in assessing the empirical usefulness of the theory in Section 1, is to gain a better understanding of its implications for the behavior of prices. While the theorems characterize several important aspects of price dynamics, and how they respond to changes in harvests, demands, and storage costs, a precise characterization requires knowledge of the equilibrium price function. Given this, we can check the correspondence between the data in Table 1 and the time series properties of the associated stationary price distribution.

The equilibrium price function is the stationary rational expectations equilibrium characterized by equation (7). Unfortunately, as has been recognized since the work of Gustafson (1958), there is no simple analytic form for the equilibrium price function, so that it is necessary to resort to numerical approximation. The proof of Theorem 1 in the Appendix studies the mapping T which associates with a function f_s the function f_{s+1} defined by

$$f_{s+1}(x) = \max[\beta(1-\delta)Ef_s[z+(1-\delta)(x-P^{-1}\{f_{s+1}(x)\}], P(x)], \tag{12}$$

and shows that, given a choice of some suitable f_0, for example $f_0 = P$, the sequence f_0, $f_1 = Tf_0, \ldots, f_{s+1} = Tf_s$, converges to the SREE f. By Theorem 2, we know that for $\delta > 0$, the invariant distribution of the state variable x is bounded, so that we can calculate these successive approximations on a grid over the range of x. In practice, this is an inconveniently slow algorithm, since it requires, at each iteration, a set of subsidiary iterations to solve for the next function, which is itself going to be modified at the next step. We have found that removing the subsidiary iterations does not prevent convergence, so that our algorithm is given by (12), but with the $f_{s+1}(x)$ on the right hand side replaced by $f_s(x)$. All of the results reported here use a grid of 100 points over the range of x_t, although on some occasions it was necessary to run trials to discover the range. Starting from $P(x)$, at each iteration the new function was calculated at each point on the grid, using numerical integration for the expectation in (10), and linearly interpolating between the grid points for values of x not on the grid. Convergence typically required between five and two hundred iterations, depending on the specification, with each iteration requiring about a minute on a 386-based machine using GAUSS; much of the computation time is spent on the numerical integration.

Other authors, e.g. Newbery and Stiglitz (1982) and Wright and Williams (1988), Williams and Wright (1990) use piecewise linear or polynomial approximations for the segment of $f(x)$ over which inventories are held. Such techniques may well be faster than those employed here, but given the moderate costs of our calculations, we did not experiment with these alternatives. See Wright and Williams (1990, Chapter 3) for a comparative evaluation of the relative merits of the various algorithms, and Taylor and Uhlig (1990) and the accompanying papers for a more general survey of computational techniques for evaluating policy functions.

Once the equilibrium price function is known, and an initial level of x_t given, a series of prices can be simulated by randomly drawing a series of harvests z_t, and calculating prices according to

$$p_t = f(x_t); \quad x_{t+1} = (1-\delta)[x_t - P^{-1}f(x_t)] + z_{t+1}. \tag{13}$$

Before calculating statistics, it is necessary to check that the results do indeed come from the invariant distribution. We try to ensure this by setting initial inventories to zero, so that the first value of x is known to come from the invariant distribution. Invariance is checked by splitting the simulated time-series into two, and using a two-sample Wilcoxon statistic to test that the two half series come from the same distribution.

While simulation is essential to obtain a visual impression of the implications of the theory for time-series behavior of the prices, the nonlinearity of the model means that it is sometimes necessary to generate *very* large samples in order to obtain accurate estimates of the invariant distribution. An alternative procedure is to use the policy function $f(x)$ to calculate the invariant distribution directly. Since the distribution of harvests z is known, $F(z)$, say, equation (13) implies that the invariant distribution of x, $G(x)$ must satisfy:

$$G(x) = \int F\{x-(1-\delta)[s-P^{-1}f(s)]\}dG(s) \tag{14}$$

In practice, it is easier not to use (14) for the calculations. Instead, recall that $f(x)$ is calculated over a grid of 100 points, xx_i, $i=1,\dots,100$, so that if these points are a distance are κ apart, we can define a transition matrix T, by

$$\begin{aligned} t_{ij} &= pr\{x_{t+1} = xx_i \mid x_t = xx_j\} \\ &= pr\{xx_i + \kappa/2 \geq (1-\delta)[xx_j - P^{-1}f(xx_j)] + z \geq xx_i - \kappa/2\} \end{aligned} \tag{15}$$

This matrix T can be calculated directly from the distribution function $F(z)$ once the *SREE* $f(x)$ is known. The eigenvector of T corresponding to the unit eigenvalue then gives the masses in the invariant distribution on the points of the grid. These masses are also the masses corresponding to the points on a price grid $pp=f(xx)$, and this invariant distribution of prices can be used, together with the transition probabilities are used to calculate autocorrelations, measures of skewness and kurtosis, and the other statistics reported below.

In selecting specifications for examination, we are guided by the theorems, and by the stylized facts in Table 1. In a world without inventories, positive skewness of prices could be generated even from symmetric harvests, provided the demand function is convex. The inverse demand function with inventories is more convex than that without, see Figure 2, so that it is of interest to discover, with a linear demand function and symmetric harvests, how much skewness in the price distribution can be delivered by the competitive storage model. Given *i.i.d.* harvests, and no storage, prices would also be *i.i.d.*, so that to account for the high positive autocorrelation in the data, the model must be one in which stockouts are rare. Theorem 3 tells us that, at a given level of the carry-over, stockouts will be less probable the wider the spread in the harvest distribution if the inverse demand function is convex. While this does not necessarily imply that stockouts will be rarer with more convex demands, or with more uncertain harvests, the parameters that determine convexity and uncertainty are natural candidates to consider in designing the experiments. We have also found it useful to control for what would be the distribution of prices if there were no storage; it is the spread of the pre-storage distribution of prices that determines the opportunities for speculators to profitably arbitrage commodity stocks between periods. Since the relationship between harvest variability and price variability depends on the slope of the demand function, the probability of stockouts can be kept low by keeping high the dispersion of pre-storage prices, either by ensuring high variability of the harvest or by choosing inelastic demand functions.

Table 2 shows the results of five specifications, all with convex demand functions. In the first two cases, the demands are linear, and in the last three cases, isoelastic and strictly convex; for the linear demands, the harvest is assumed to be normal, while for the isoelastic case, harvests are drawn from a lognormal distribution. Even though, in theory, these distributions have an unbounded support, in practice we truncate the distributions at five standard deviations from the mean, so that the theorems of the previous section can be directly applied. For both linear and isoelastic models, we consider two cases, corresponding to coefficients of variations in the no-storage price distribution of 0.10, the low variance case, and 0.50, the high variance case. For the linear model, we hold the distribution of harvests constant over the low and high variance cases, but change the slope and intercept of the demand function so as to quintuple the coefficient of variation while holding constant the value of price when the harvest takes its mean value of 100. The second case corresponds exactly to one of the cases previously simulated by Wright and Williams (1988), and the fact that the results correspond provides a useful check on our different computational procedures, theirs based on simulation and ours, on calculation of the invariant distribution. For an

TABLE 2: DISTRIBUTIONS OF SIMULATED PRICES

Demand Shocks	$P(x) = a+bx$ z is $N(100,\sigma=10)$		$P(x) = x^{-\rho}$ lnz is $N(0,\sigma^2)$		
Parameters	r=.05/.9 δ=.05 a=200 b=-1	r=.05 δ=0 a=600 b=-5	r=.05/.9 δ=.05 ρ=1 σ=0.10	r=.05/.9 δ=0.05 ρ=1 σ=0.50	r=.05/.9 δ=0.05 ρ=5 σ=0.10
Coeff of variation(1)	0.10	0.50	0.10	0.53	0.53
Probability of stockouts	0.82	0.16	0.82	0.24	0.24
First-order a.c.(2)	0.08	0.48	0.08	0.33	0.29
Second order a.c.(2)	0.01	0.31	0.01	0.16	0.14
Third order a.c.(2)	0.00	0.22	0.00	0.09	0.08
Skewness γ_1(3)	0.43	1.99	0.61	3.41	3.15
Kurtosis γ_2(3)	-0.29	5.50	0.17	24.22	16.43
Wilcoxon two-sample pooling test(4)	0.59 1.44 2.30 0.16 -1.08	1.27 -3.27 -3.27 -1.08 4.37	1.43 -0.40 -0.76 -0.05 -0.55	-0.05 0.05 2.33 -0.20 0.33	0.19 0.29 -2.57 -0.30 -1.07

Notes: (1) Coefficient of variation of the price distribution in the *absence* of inventories.
(2) Autocorrelations of prices
(3) $\gamma_1 = \mu_3/(\mu_2)^{1.5}$, $\gamma_2 = (\mu_4/\mu_2^2) - 3$, where μ_r is the *r*th central moment.
(4) Test statistics as defined, e.g. in Kendall and Stuart (1973, pp 510-6), computed from two independent drawings of a 2000 sample of prices; the results for five separate such experiments are shown.

isoelastic demand function, $P(x) = x^{-\rho}$, with the log harvest distributed as $N(0,\sigma^2)$, the coefficient of variation of the no-storage price distribution is approximately $\rho\sigma$, so that, in the low variance case, we choose $\rho = 1$ and $\sigma = 0.10$, while for high variance, we choose the two pairs of (1,0.5) and (5,0.10) for the couple (ρ,σ).

For the low variance cases, where the coefficient of variation is 0.10, there is little incentive to store, and inventories are carried forward in only one year in five, with little difference between the linear and isoelastic models. As a result, storage does little to smooth the price series, although enough to generate small positive first-order autocorrelations. There is some positive skewness, even in these low variance cases, but little evidence of induced leptokurtosis.

When we move to the high variance case, the situation is quite different. In the linear case, stocks are carried forward 84% of the time, there is a great deal of skewness and kurtosis, well within the range of the figures in Table 1, and although the autocorrelations are far from being as large as those in Table 1, at 0.48, 0.31, and 0.22, they are substantial. Clearly, with enough variance in the pre-storage distribution, even linear demand functions are capable of generating skewness, kurtosis, and autocorrelation. The isoelastic case generates rather similar results, and it is clear from the last two columns of Table 2 that it makes relatively little difference whether the source of variability is uncertain harvests or inelastic demand. In either cases, stocks are usually held, although stockouts, with probabilities of 24% for both cases, are a good deal more common than in the linear case. Autocorrelation is also substantial, but again lower than for linear demands. By contrast, the isoelastic specification generates more skewness and kurtosis, presumably because the strict convexity of the $P(x)$ function ensures much sharper spiking of prices when there are stockouts and supply is short.

The mechanisms that generate the asymmetries in the price distribution can be seen clearly in Figures 3 and 4, which show the invariant distributions with and without storage for the first and fourth columns of Table 1. Inventory holders buy when the price is low, thereby "thinning out" the left tail of the distribution. When prices are high, more precisely when prices are higher than p^*, they sell any inventories that they hold. This also tends to reduce the right tail of the distribution, since what would otherwise be high prices are reduced by the held over inventories. Of course, if stockouts are

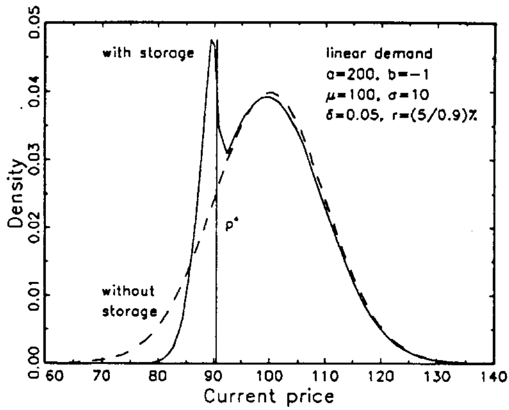

Figure 3: Prices with and without storage: linear case

common, and inventories typically low or zero, as in Figure 3, there will be little or no effect on the right tail. When stockouts are rare, as in Figure 4, the thinning effect is much more marked. Even in this case, however, there will still be years when there is a bad harvest and when inventories have already been exhausted, so that prices have to reflect the full extent of the shortfall. Particularly when the inverse demand is convex, such situations can produce prices that are extremely high relative to normal, and show up as sharp spikes on time-series plots. Note that these spikes are no higher than the highest prices that would occur without storage; indeed Theorem 1 immediately implies that with or without storage, there are maximum prices, and that they are identical. What produces the appearance of a spike is that these high prices become relatively rare, because there are usually inventories to act as buffers. The upper tail of the distribution in Figure 4 is thin, but there is positive density all the way up to the maximum.

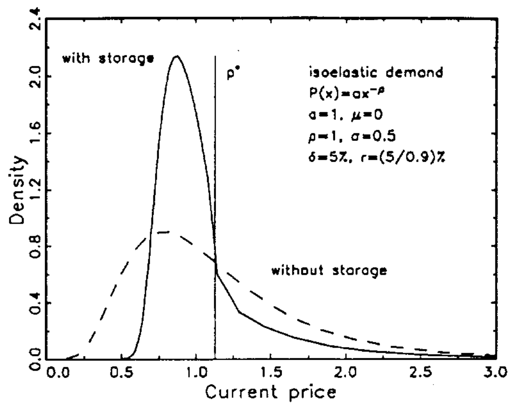

Figure 4: Prices with and without storage: isoelastic case

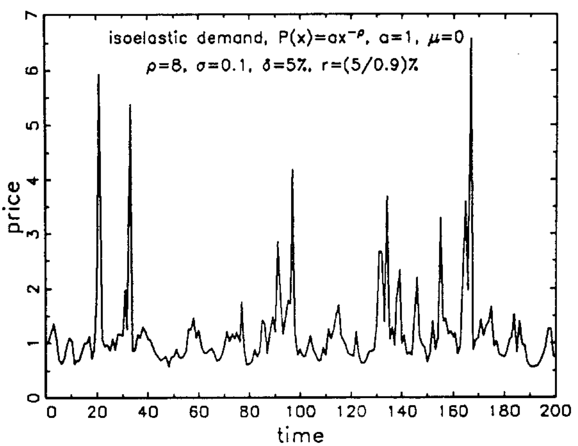

Figure 5: 200 period simulation of prices

It is also instructive to look at the behavior of prices in a time-series plot, such as Figure 5, which corresponds to an isoelastic model with a standard deviation of 0.1 for the logarithm of the harvest, but with inelastic demand and a value for ρ of 8. This simulation, even more than those corresponding to any of the cases in Table 2, shows clearly the occasional very sharp spikes, together with the more normal "doldrums" type behavior when prices are low, something that is a direct consequence of the behavior of the conditional variance (11) when the underlying demand function is convex. There are marked resemblances between the behavior in Figure 5 and the actual behavior of several commodity prices, see Figure 1 for the case of sugar.

Note finally the Wilcoxon tests in Table 2, which fail for three out of the five cases for the high variance linear model, and in two of the simulations for the isoelastic case. By Theorem 2, we know that the price distributions are stationary, and we are comparing samples of 2,000 observations, so that it is somewhat puzzling to discover that the samples are in some way different. However, the high variance linear case is one in which stockouts are rare, and where the simulated price series shows rare but sharp spikes, spikes that occur when there is a bad harvest and when there are no inventories. Such spikes are a good deal rarer than stockouts, which themselves occur only 16% of the time. In consequence, two successive simulations, even of 2,000 observations, can look very different from one another, and certainly different enough to cause a rejection by the pooling test. The implication is that, when stockouts are rare, 2,000 observations are insufficient to establish the properties of the series. But stockouts must be rare in practice to generate the high degree of autocorrelation that we actually observe. And if 2,000 observations are too few, inference in real life situations is going to be extremely difficult.

2.2 Estimation

In this paper, we confine ourselves to the analysis of the price data, and make no attempt to incorporate data on levels of commodity stocks. Although there are at least partial data for stocks of some commodities, they are of notoriously low quality, and there are serious conceptual problems in using them. As argued by Wright and Williams (1989), stocks are heterogenous, and commodities ready for sale in the markets of London or New York are not the same thing as partially processed commodities warehoused with dealers or manufacturers scattered around the globe. In particular, we focus on equation (10), i.e.

$$E(p_{t+1} \mid p_t) = \min(p_t, p^*)/[\beta(1-\delta)]. \tag{16}$$

At this point, we prefer not to embark on an attempt to estimate the model by maximum likelihood or other full information techniques. These techniques require calculation of the equilibrium price *function* in order to evaluation whatever criterion is being maximized in the estimation procedure. In consequence, each iteration of the estimation procedure will itself require the many further iterations required to compute the SREE. Even so, computational expense is not the most serious issue. The competitive storage model has many free parameters, from the demand function, the distribution of harvests, and the storage technology, and it is dubious whether all of these are identifiable from the distribution of prices alone.

We estimate (11) using a generalized methods of moments (GMM) technique. Let $\gamma = 1/\{\beta(1-\delta)\}$ and write $\theta = (\gamma, p^*)$ for the parameters to be estimated. Define the series u_t by

$$u_t = p_t - \gamma \min(p_{t-1}, p^*). \tag{17}$$

Under the truth of the model, the time series u_t is stationary, and although heteroskedastic, it is a series of innovations, so that each u_t is uncorrelated with previous values, as well as with prices dated t-1 and earlier. Hence, we define the (Tx4) matrix of instruments W with typical row given by

$$w_t = (1, p_{t-1}, p_{t-2}, p_{t-3}). \tag{18}$$

Let u be the (Tx1) vector of u_t's. The GMM estimates $\hat{\theta} = (\hat{\gamma}, \hat{p}^*)$ of the model are the values of the parameters that minimize $u'W(W'W)^{-1}W'u$ as a function of the parameters; note that because of the non-differentiability in (10), this criterion is not differentiable. We minimized the criterion using a Newton type algorithm, where the gradient is $2u'W(W'W)^{-1}W'\partial u/\partial\theta$ which is almost everywhere differentiable, and the second derivative is replaced by the expression $4(\partial u'/\partial\theta)W(W'W)^{-1}W'(\partial u/\partial\theta')$, to which it is asymptotically equivalent. The arguments in Laroque and Salanié (1990), there applied to the canonical disequilibrium model, also apply here, and $\hat{\theta}$ is consistent and asymptotically normal. Given that the model predicts heteroskedasticity, we use the consistent estimate of the variance covariance matrix Ω defined by:

$$\Omega = \left(\frac{\partial\hat{u}'}{\partial\theta}W(W'W)^{-1}W'\frac{\partial\hat{u}}{\partial\theta'}\right)^{-1}\frac{\partial\hat{u}'}{\partial\theta}W(W'W)^{-1}W'\hat{D}W(W'W)^{-1}W'\frac{\partial\hat{u}}{\partial\theta}\left(\frac{\partial\hat{u}'}{\partial\theta}W'(W'W)^{-1}W'\frac{\partial\hat{u}}{\partial\theta'}\right) \tag{19}$$

where $\hat{D}$ is the (TxT) diagonal matrix whose tth diagonal element is equal to $\hat{u}_t^2$, i.e. the squared value of u_t evaluated at the GMM estimates. If the basic model is true, all the instruments are orthogonal to the innovations, and we may test the overidentifying restrictions in the usual way, by computing

$$OIT = \hat{u}'W(W'\hat{D}W)^{-1}W'\hat{u}, \tag{20}$$

which, under the null, is asymptotically distributed as χ^2 with 2 degrees of freedom. While the overidentifying test is a valuable diagnostic of model adequacy in general, it is here specifically motivated by the fact that, in the simulations reported in Table 2, we had some difficulty in reproducing the high autocorrelations in the actual data. If the model (11) also conflicts with the autocorrelations in the data, the residuals are likely to be correlated with the additional lags of price, and the OID test will fail. For the same reasons, we compute and report the Durbin-Watson statistics associated with $\hat{u}$.

Before applying these procedures to the actual data, it seemed wise to check their properties on artificial data, and particularly to make sure that the estimates are well-behaved in the ideal case when the data are generated by the model. Quite apart from the general issues of the accuracy of asymptotic approximation, this model, like disequilibrium econometric models in general, has regions where the gradient is zero, but which do not contain the minima that we seek. In particular, when p^* is greater than $\max\{p_t\}$, the gradient with respect to p^* is zero. For these preliminary experiments, we chose the model underlying the second column of Table 2, the high variance linear demand case. Table 2 shows that the probability of a stockout is only 16%, so that we can expect only 16% of the price observations to lie above the cutoff, something that is likely to make it difficult to obtain accurate estimates of p^*. We drew 500 samples of 100 observations each, using the previously calculated equilibrium price function, and for each of samples calculated the GMM estimates, the standard errors from (14), the Durbin-Watson test, and the overidentifying test statistic.

The main results are summarized in Table 3. The GMM technique performs very well under these ideal conditions. The means of the empirical distributions of the two parameters are close to their true values, and the averages of the robust standard errors compare well with the dispersion of the empirical estimates. For the cut-off price p^*, the average of the calculated standard errors (13.0) is higher than the standard deviation of the empirical distribution (8.8), a problem that is probably associated with the non-differentiability of the criterion function with respect to this parameter. However, the

Table 3: Monte Carlo GMM Estimation of Commodity Price Model

Parameter or statistic	Theoretical asymptotic value or mean	Average value from 500 replications	Empirical dispersion	Average estimated standard error
γ	1.050	1.055	0.022	0.023
p^*	114.2	112.6	8.79	13.00
OIT	2.00	2.64	2.25	-
D.W.	2.00	1.99	0.14	-

estimator of $\theta_1 = [\beta(1-\delta)]^{-1}$ performs well in all respects, and a normal distribution with the true mean and standard deviation equal to the average of the calculated standard errors is almost exactly equal to the density of the empirical estimates. Note that, while the Durbin Watson statistic behaves as predicted by theory, both the mean and the standard deviation of the overidentifying test are larger than the values of 2 predicted by the theory. As a consequence, the null of orthogonality would be rejected too frequently on these data; 9.8% of the time using a test with a nominal size of 5%, although at 1%, the actual and nominal sizes are identical.

The GMM procedures therefore seem well suited to the problem at hand, and we apply them to the deflated commodity price described in Table 1. The results are presented in Table 4, which, together with the estimates of the parameters, their standard errors, the overidentifying test statistic, and the Durbin Watson statistic, shows the fraction of observations on each series that are above the estimated cutoff, and where, if the model is correct, stockouts are occurring. As is to be expected given the high autocorrelation in the data, the fraction of stockouts is quite small for all of these commodities. For four commodities, palm oil, tin, maize, and copper, the fraction of stockouts is 7% or less, so that the parameter p^* is being identified by six points or less. The standard errors for these commodities are correspondingly large, particularly in the extreme case of palm oil where a single high spike in the series drives the estimation procedure to what may well be a spurious maximum.

With the exception of palm oil, where the coefficient is 0.995, all the slope coefficients are estimated to be slightly larger than one, although none are significantly different from one. While discount rates are notoriously hard to estimate, and while it is satisfactory that 12 out of the 13 coefficients are greater than unity, the implied estimates of $r+\delta$, are all implausibly small, the largest being 9% for sugar. Although the standard errors are consistent with much larger values, it is disappointing that none of the estimates in the Table are significantly larger than unity, particularly given the encouraging results from the Monte Carlo experiments. Contrary to our expectations, the *OID* tests do not contain a great deal of evidence of residual autocorrelation. The 5% critical value of a χ^2 with 2 degrees of freedom is 5.89, and for only three of the commodities, cotton, jute, and rice, are the test statistics greater than this value, which, in the light of the Monte Carlo evidence, is likely to be too low. For cotton and rice, the Durbin Watson statistics are also low, although for the other commodities, the Durbin Watson and the overidentifying test statistics do not always agree.

At first sight, the results in Table 4 may look consistent with the hypothesis, discussed in the introduction, that these prices follow random walks. The coefficients on the lagged prices are insignificantly different from unity, and the cutoff parameters, which (in addition to heteroskedasticity) are what separate the current model from a random walk, are not well determined. However, this model becomes a martingale, not when the cutoff parameter is zero, but when it becomes very large, or, in practice, larger than the largest sample observation. Even in a simple random walk, the presence of a

Table 4: GMM estimates of commodity price models

	$\theta^1=[\beta(1-\delta)]^{-1}$	s.e.	p^*	s.e.	stock-outs	*OID*	d.w.
bananas	1.004	0.02	0.694	0.07	0.16	1.17	1.88
cocoa	1.025	0.14	0.345	0.25	0.08	2.31	1.63
coffee	1.040	0.07	0.378	0.11	0.10	2.54	1.45
copper	1.015	0.05	0.792	0.43	0.07	5.35	1.58
cotton	1.013	0.05	0.939	0.30	0.08	10.87	1.21
jute	1.046	0.04	0.763	0.10	0.19	8.17	1.74
maize	1.015	0.08	1.076	0.74	0.06	0.81	1.79
palm oil	0.995	0.14	1.426	2.81	0.01	2.26	1.84
rice	1.012	0.06	0.927	0.34	0.08	6.37	1.43
sugar	1.090	0.14	1.029	0.28	0.23	4.50	1.90
tea	1.009	0.04	0.637	0.15	0.16	0.65	1.77
tin	1.013	0.07	0.449	0.23	0.05	2.71	1.62
wheat	1.005	0.07	1.103	0.47	0.08	5.80	1.36

Notes: The parameters are those given in equation (11). The column labelled stockouts shows the fraction of observations above the estimated cutoff p^*. *OID* is the overidentifying test statistic, and d.w. is the Durbin Watson test.

finite cutoff radically alters the behavior of the series. Indeed, the artificial time series underlying the Monte Carlo experiments in Table 3 mostly generate slope coefficients greater than, but close to one, and several do not have very well determined cutoff parameters. But these series have first order autocorrelations that are so low that they would not be readily mistaken for random walks. Conversely, it would be hard to have random walks produce the results in Table 4. Attempts to fit (14) by GMM to real random walks frequently led to computational breakdown as the cutoff became larger than the largest observation and identification failed. The evidence in Table 4 is therefore legitimate evidence in favor of the competitive storage model against the random walk alternative.

However, it is still far from clear that the competitive storage model gives a fully adequate account of the data. Indeed, there are many stochastic processes that satisfy the autoregression (14), but which could not be generated from the commodity price model. For example, the model

$$p_{t+1} = [\beta(1-\delta)]^{-1}\min(p_t, p^*) + \varepsilon_t \tag{21}$$

with ε_t iid $N(0,\sigma^2)$, and with $\beta(1-\delta)$ corresponding to the second column of Table 2, generates a time series that bears little resemblance to prices simulated from the model, even though the latter satisfy (10). The higher order moments of the series are also important, in particular, the conditional variance $V(p_{t+1} | p_t)$, which, by (11) is increasing in p_t if $P(x)$ is convex. When prices are low, next period's price is expected to be close by, and thus typically also low, which helps explain the typically long periods of low prices when very little seems to happen.

For each commodity, we checked for heteroskedasticity by calculating Lagrange multiplier tests by regressing the squared residuals, normalized by their average, on the lagged prices. The R^2-statistics of these regressions, multiplied by the sample size, are distributed as χ^2 under the null of homoskedasticity. Not surprisingly, the statistics for all commodities are very large, and there is no doubt of the *existence* of heteroskedasticity. In order to learn more about its shape, we also regressed the logarithm of the absolute value of u_t on the logarithm of lagged price p_{t-1}, and again, in all cases, we found a strong positive relationship, with a coefficient that in most cases is larger than unity, and which scatter diagrams typically reveal to be close to linearity. We found no obvious evidence that the conditional variance is constant when price is above the cutoff, but as already noted, there are typically rather few such points, so that the phenomenon, even if present, would be hard to detect. While the competitive storage model is perhaps not the only way of explaining these findings, the positive heteroskedasticity is predicted by the model, and is strongly evident in the data.

These results are encouraging; apart from the cases already noted, the actual commodity prices display conditional means and conditional variances that conform to the predictions of the theory of competitive storage. If the model were linear, this would be enough, but since it is not, there is no guarantee that other important characteristics of the model are matched in the data. In particular, although the model (21), with the correct form of heteroskedasticity, seems to fit the data for most of the commodities, it may again be because there are processes satisfying (21), and with the correct conditional heteroskedasticity, but which could not have been generated by the competitive model. Indeed, the models displayed in Table 2 all predict conditional skewness and kurtosis. These higher order properties may or may not match the actual data, and with a non-linear time series model, failure to match higher order moments may be linked to a failure to match autocorrelations.

We do not press these issues further in this paper. Further progress will require the use of full-information techniques; these require more computational development than we have yet undertaken, and we feel that this and the development of adequate diagnostics for the results of such estimation is a major project in its own right.

3 SUMMARY AND CONCLUSIONS

This paper has made a first attempt to explain commodity prices in terms of a simple theory based on competitive storage. We have used the simplest possible form of the theory, with random harvests that are independent and identically distributed over time, and that do not respond to current or expected future prices. We have also assumed that neither consumers nor speculators have advance information about harvests, and know only the amount on hand immediately after the harvest. There is only one market each season, held immediately after the harvest, in which a price is set for that season. Such a simplified model is clearly unrealistic, more so for some commodities than others, but it serves to focus attention on the role that storage can play in transferring commodities from relatively plentiful times to relatively scarce times, and on the effects on the behavior of price. It is the natural starting point for a more sophisticated analysis, for which the basic insights are a prerequisite.

The simple model also delivers a number of predictions, about autoregression functions, and about conditional variances, predictions that bear up reasonably well against the annual commodity price data for the thirteen commodities analyzed here. Simulations of the theoretical model are also capable of replicating many of the features of the actual data. Because the variance of next period's price is a non-decreasing function of current price, prices spend long periods in the "doldrums," showing little movement but high autocorrelation from year to year. However, once a high price is established, there exists the possibility of further high prices, and the theoretical results replicate the occasional extreme spikes that are characteristic of actual commodity prices. Simulated prices can also replicate the skewness and kurtosis that is displayed by many actual prices.

Whether autocorrelation patterns also conform to the theory remains an open question. While speculative storage generates price autocorrelation over and above what would exist without storage, we do not know whether there are plausible theoretical specifications that generate the degree of autocorrelation observed in the actual time-series. The theoretical work in this paper tells us a good deal about the conditions needed to generate autocorrelation, but we have not been able to generate models that reproduce the autocorrelation in most of the actual commodity prices. It is possible that we have not looked sufficiently hard or cleverly, but it also possible that high autocorrelation reflects phenomena not discussed here, such as autocorrelation in harvests, as would be expected for many tree crops, or the smoothing of price that is a plausible consequence of consumers and speculators having advance information about the harvest. Such questions remain for further analysis; unfortunately, it is an understanding of autocorrelation that is perhaps the most vital for the conduct of policy.

List of works cited:-

Cochrane, J.H., (1988), "How big is the random walk in GNP?" *Journal of Political Economy*, **96**, 893-920.

Cuddington, J.T., and C.M. Urzua, (1987), "Trends and cycles in primary commodity prices," Economics Department, Georgetown University, Washington, D.C., processed.

Danthine, J.-P., (1977), "Martingale, market efficiency and commodity prices," *European Economic Review*, **10**, 1-17.

Gersovitz, M., and C.H. Paxson, (1990), "The economies of Africa and the prices of their exports," Princeton University, Princeton Studies in International Finance, forthcoming.

Ghosh, S., C.L. Gilbert, and A.J. Hughes Hallett, (1987), *Stabilizing speculative commodity markets*, Oxford. Clarendon.

Gustafson, R.L., (1958), *Carryover levels for grains*, US Department of Agriculture, Technical Bulletin 1178.

Hart, O.D., and D.M. Kreps, (1986), "Price destabilizing speculation." *Journal of Pollical Economy*, **94**, 927-52.

Kendall, M., and A. Stuart, (1973), *The advanced theory of statistics, Vol 2: Inference and relationship*, 3rd edition, London. Charles Griffin.

Kohn, M., (1978), "Competitive speculation," *Econometrica*, **46**, 1061-76.

Laroque, G., and B. Salanié, (1990), "Estimating the canonical disequilibrium model: asymptotic theory and finite sample properties," Discussion Paper # 9005, INSEE, Paris.

Muth, J.F., (1961), "Rational expectations and the theory of price movements," *Econometrica*, **29**, 315-35.

Newbery, D.M.G., and J.E.Stiglitz, (1981), *The theory of commodity price stabilization: a study in the economics of risk*, Oxford. Oxford University Press.

Newbery, D.M.G., and J.E. Stiglitz, (1982), "Optimal commodity stockpiling rules," *Oxford Economic Papers*, **34**, 403-27.

Salant, S.W. (1983), "The vulnerability of price stabilization schemes to speculative attack," *Journal of Political Economy*, **91**, 1-38.

Samuelson, P.A., (1971), "Stochastic speculative price," *Proceedings of the National Academy of Science*, **68**, 335-7.

Schechtman, J., and V. Escudero, (1977), "Some results on 'An income fluctuation problem'," *Journal of Economic Theory*, **16**, 151-66.

Scheinkman, J.A., and J. Schechtman, (1983), "A simple competitive model with production and storage," *Review of Economic Studies*, **50**, 427-41.

Taylor, J., and H. Uhlig, (1990), "Solving nonlinear stochastic growth models: a comparison of alternative solution methods," *Journal of Business and Economic Statistics*, **8**, 1-17.

Williams, J.C., and B.D. Wright, (1990), *Storage and commodity markets*, Cambridge University Press, forthcoming.

Wright, B.D., and J.C. Williams, (1982), "The economic role of commodity storage," *Economic Journal*, **92**, 596-614.

Wright, B.D., and J.C. Williams, (1984), "The welfare effects of the introduction of storage," *Quarterly Journal of Economics*, **98**, 169-192.

Wright, B.D., and J.C. Williams, (1989), "A theory of negative prices for storage," *Journal of Futures Markets*, **9**, 1-13.

APPENDIX

The proof of Theorem 1 proceeds through a series of lemmata. First, we collect and state formally the assumptions of the text:

Assumption A.1:

(i) $0 < \beta(1-\delta) < 1$.
(ii) The real random variable z has a compact support with lower bound $\underline{z}$ *and upper bound* $\bar{z}$.
(iii) The function $P^{-1}: (p_0, p_1) \to \Re$ *is continuous, strictly decreasing, and satisfies:*

$$\lim_{p \to p_0} P^{-1}(p) = +\infty$$

$\underline{z}$ *belongs to the range of* P^{-1} *and* $\infty > P(\underline{z}) > 0$

To solve the functional equation (7), imagine that the equilibrium at date $t+1$ is expected to be given by a function g of the state variable x at that date. Then the equilibrium at date t is a function f that satisfies:

$$f(x) = \max[\beta(1-\delta)Eg\{z+(1-\delta)(x-P^{-1}(f(x))\}, P(x)]. \tag{a.1}$$

A *stationary rational expectations equilibrium (SREE)* is a function g such that $f = g$.

Recall that the state variable (the amount on hand) takes its values in a set $X = \{ x \text{ in } \Re, x \geq \underline{z} \}$ so that g and f are defined on X. Let $Y = \{ (p,x) \mid x \text{ in } X, P(x) \leq p < p_1\}$. It is useful to introduce the function $G: Y \to \Re$, defined by:

$$G(q,x) = \beta(1-\delta)Eg\{z+(1-\delta)(x-P^{-1}(q))\}. \tag{a.2}$$

$G(q, x)$ is the expected discounted present value of tomorrow's price given that today's amount on hand is x and today's price is q. For a *SREE* today's price must be consistent with today's x, and the function linking them must be the same today as tomorrow, so that G can be used to rewrite (a.1) as

$$f(x) = \max[G(f(x), x), P(x)]. \tag{a.3}$$

We shall denote by T the operator that assigns to a function g the function f that is the solution of (a.1). Correspondingly, for (a.3) we also write TG to denote f as the solution of (a.3)

Lemma 1: *Assume that* $g : X \to (p_0, p_1)$ *is continuous non-increasing. Then the corresponding function* $G: Y \to \Re$ *is continuous and non-increasing in both its arguments. Furthermore:*

$$G(P(x), x) = \beta(1-\delta)Eg(z),$$

$$\lim_{q \to p_1} G(q,x) \leq \beta(1-\delta)p_1 < p_1.$$

Proof: trivial.

Lemma 2: *Assume that G satisfies the properties of lemma 1. Then:*

(i) There is a unique function $f = TG$ *which is the solution of (a.3).* $f : X \to (p_0, p_1)$ *is continuous, non-increasing and:*

$$f(x) = P(x) \quad \textit{whenever} \quad P(x) \geq \beta(1-\delta)Eg(z)$$
$$f(x) = G(f(x), x) \quad \textit{whenever} \quad P(x) < \beta(1-\delta)Eg(z).$$

(ii) Furthermore, $G_1(q, x) \geq G_2(q, x)$ *for all (q, x) implies* $TG_1 \geq TG_2$.

Proof: For given x in X, $f(x)$ is equal to the solution in unknown q of:

$$\max[G(q,x)-q, P(x)-q] = 0.$$

It follows that $q \geq P(x)$. By lemma 1, the function $G(q,x)-q$ is continuous strictly decreasing in q, and varies between $\beta(1-\delta)Eg(z) - P(x)$ and some negative number when q varies between $P(x)$ and p_1. It therefore has a unique zero when $P(x) \leq \beta(1-\delta)Eg(z)$, and is always negative when $P(x) > \beta(1-\delta)Eg(z)$. This proves the uniqueness of the solution. Continuity and monotonicity follow from the continuity and monotonicity of $\{G(q,x)-q\}$. Finally, if $G_1 \geq G_2$, $G_1(q_1,x)-q_1 = 0$ implies $G_2(q_1,x)-q_1 \leq 0$, and the root q_2 of $G_2(q_2,x)-q_2 = 0$ belongs to $[P(x),q_1]$, which shows that $TG_1 \geq TG_2$. ■

Lemma 3 (upper range of the price function):

(i) If f is a SREE, and f non-increasing in x, then $f(\underline{z}) = P(\underline{z})$.
(ii) If g satisfies the assumptions of lemma 1 and $g(\underline{z}) = P(\underline{z})$, then $Tg(\underline{z}) = P(\underline{z})$.

Proof: Since f is non-increasing:

$$Ef\{z+(1-\delta)(x-P^{-1}(f(x))\} \leq Ef(z) \leq f(\underline{z}).$$

so that, taking $x=\underline{z}$,

$$\beta(1-\delta)Ef\{z+(1-\delta)(\underline{z}-P^{-1}f(\underline{z}))\} \leq \beta(1-\delta)f(\underline{z}).$$

Evaluating (a.1) at $x=\underline{z}$, and using the previous inequality, we have

$$f(\underline{z}) \leq \max[\beta(1-\delta)f(\underline{z}), P(\underline{z})].$$

If $\beta(1-\delta)f(\underline{z}) \geq P(\underline{z})$, $f(\underline{z}) \leq \beta(1-\delta)f(\underline{z})$, which can only be true if $f(\underline{z}) < 0$. But, by lemma 2, $P(\underline{z}) \leq f(\underline{z}) < 0$, contradicting Assumption A.1 (iii). Hence, $P(\underline{z}) \geq \beta(1-\delta)f(\underline{z})$, and $f(\underline{z}) \leq P(\underline{z})$. But, by lemma 2, the reverse weak inequality holds, so that $f(\underline{z}) = P(\underline{z})$, which is (i). A closely parallel argument establishes (ii). ■

Proof of Theorem 1: We sketch the argument which is a standard one.

(i) *Consider two functions g_1 and g_2 satisfying the assumptions of lemma 1, such that there exists a positive scalar a with $g_2(x) \leq g_1(x) \leq g_2(x)+a$ for all x in* X. *Then:*

$$Tg_2 \leq Tg_1 \leq Tg_2+\beta(1-\delta)a.$$

This follows from lemma 2(ii). and the fact that, by (a.1):

$$T(g_2+a) \leq \beta(1-\delta)a+\max[\beta(1-\delta)Eg_2\{z+(1-\delta)(x-P^{-1}(f(x))\}, P(x)] \leq Tg_2+\beta(1-\delta)a.$$

(ii) Let G be the space of non-negative functions defined on X that satisfy the assumptions of lemma 1, and such that $g(\underline{z}) = P(\underline{z})$. By lemmata 2 and 3, T maps G into itself. Furthermore, for any two g_1 and g_2 in G,

$$d(g_1,g_2) = \sup_{x\in X}|g_1(x)-g_2(x)|$$

is well-defined (less than or equal to $P(\underline{z})$), so that letting $a = d(g_1,g_2)$ in (i) above, we have immediately that

$$d(Tg_1,Tg_2) \leq \beta(1-\delta)d(g_1,g_2),$$

so that T is a contraction mapping. Take any g_0 in G and let $g_1 = Tg_0, \ldots, g_t = Tg_{t-1}, \ldots$ We have $d(g_{t-1}, g_t) \leq \beta^{t-1}(1-\delta)^{t-1}d(g_0, g_1)$. The sequence of continuous functions g_t converges uniformly, so that the limit $f(x)$ of $g_t(x)$ for each x in X defines a continuous function on X. Since $d(f, Tf) = 0$, this is the desired equilibrium. The equilibrium is unique. Otherwise, if f_1 and f_2 were two distinct equilibria with $d(f_1,f_2) > 0$, then $d(f_1,f_2) = d(Tf_1,Tf_2) \leq \beta(1-\delta)d(f_1,f_2)$ which is a contradiction. Lemma 2(i) gives $p^* = \beta(1-\delta)Ef(z)$ and (8) follows.

(iii) *f is strictly decreasing whenever it is strictly positive, and $x - P^{-1}\{f(x)\}$ is strictly increasing when $f(x) > P(x)$.*

We know that f is non-increasing on X. Let $[x', x'')$ be the first interval on which f is constant, i.e. such that f is strictly decreasing on $[\underline{z}, x']$, and let k be the value of f on this interval. Since P is strictly decreasing, it follows from (a.1) that for all x in $[x', x'')$:

$$f(x) = k = \beta(1-\delta)Ef\{z+(1-\delta)(x-P^{-1}(k))\}.$$

The right hand side of the above expression is constant, and since f is monotone non-increasing, we must have $f\{\underline{z}+(1-\delta)(x-P^{-1}(k))\}$ constant on $[x', x'')$, and, by the choice of $[x', x'')$, less than or equal to k. It follows that $k \leq \beta(1-\delta)k$, so that $k = 0$.

Finally, when $f(x) > P(x)$, letting $I(x) = x-P^{-1}\{f(x)\}$, we have:

$$f(x) = \beta(1-\delta)Ef\{z+(1-\delta)I(x)\}.$$

Since $f(x)$ is non-increasing, $I(x)$ is non-decreasing. If f is strictly decreasing on some interval, $I(x)$ strictly increases. If $f(x) = 0$, $I(x) = x - P^{-1}(0)$ is also strictly increasing.

■

Proof of Theorem 2: The evolution of inventories is described by the stochastic difference equation:

$$I_t = (1-\delta)I_{t-1} - X(p_t) + z_t.$$

Since $p_t \leq P(\underline{z})$ by lemma 3 and Theorem 1, $X(p_t) \geq \underline{z}$, so that

$$I_t \leq (1-\delta)I_{t-1} + \bar{z} - \underline{z},$$

which, since $0 < \delta < 1$, implies that, in the long run, $0 \leq I_t \leq (\bar{z} - \underline{z})/\delta$. Suppose now that I_0 is strictly positive, which is guaranteed for some date by $P(\bar{z}) < \beta(1-\delta)EP(z)$, which precludes $I_t = 0$ for all t. We show that, with probability 1, the inventory level I_t will be equal to zero in finite time, so that the price level follows a renewal process. If not, with some positive probability, we should have $p_t < p^*$ for all t. Then $X(p_t) \geq X(p^*)$ for all t. Given $J_0 = I_0$, define J_t by:

$$J_t = (1-\delta)J_{t-1} - X(p^*) + z_t.$$

We have $J_t \geq I_t$ for all t, and J_t has the same limiting distribution as $\{z - X(p^*)\}/\delta$. But

$$p^* = \beta(1-\delta)Ef(z) \leq \beta(1-\delta)f(\underline{z}) = \beta(1-\delta)P(\underline{z}) < P(\underline{z}),$$

since f is non-increasing, and by lemma 3, so that, inverting, $X(p^*) > \underline{z}$. Hence,

$$Pr[(z-X(p^*))/\delta \leq 0] > 0.$$

Hence, by standard arguments, the probability that J_t (and thus I_t) stays strictly positive for all t is equal to zero.

■

Proof of Theorem 3:

(i) *f is convex when P is convex.*

Suppose that when g is convex, we can show that Tg is also convex. Then, if in the argument in part (ii) of the proof of Theorem 1 above, we take g_0 convex, g_t is convex for all t, and thus so is f. Hence, all that is required to prove f convex is that g convex implies Tg convex. First note that g convex and non-increasing, and P convex decreasing imply that G is a convex function in the couple (q, x). This follows from two well-known properties of convex functions: if P is convex decreasing, P^{-1} is convex; if g is convex decreasing and h is concave, then $g \circ h$ is convex. By construction, see (a.2), G is therefore convex in (q, x). Recall from lemma 2 that Tg is obtained as the solution in q of the equation:

$$H(q,x) = \max[G(q,x)-q, P(x)-q] = 0.$$

$H(q, x)$ is the maximum of two convex functions, and is therefore convex in (q, x). Take any two x', x'' in X and λ in [0,1]. Let $q' = Tg(x')$ and $q'' = Tg(x'')$. We have:

$$H(q',x') = H(q'',x'') = 0,$$

and therefore:

$$H(\lambda q'+(1-\lambda)q'', \lambda x'+(1-\lambda)x'') \leq 0.$$

Since H is decreasing in q, the desired result follows.

(ii) All the comparative statics properties follow from essentially the same argument, which is sketched here for the case of a mean preserving spread in the distribution of shocks. Let G_1 be the initial function defined by (a.2) and G_2 that obtained after the change in distribution. If g is convex, $G_2 \geq G_1$ and by lemma 2(ii), $TG_2 \geq TG_1$. Then apply the argument of (i) above, starting from the same convex g_0, to get the desired result.

■

Finally, we prove (11), which follows from the following lemma:

Lemma 4: Assume $f: X \rightarrow \Re_+$ is convex decreasing, and z a random variable with bounded support in X. Then, for all $I \geq 0$, $V[f(z+(1-\delta)I)]$ is a decreasing function of I.

Proof: By definition:

$$V[f(z+(1-\delta)I)] = Ef^2-(Ef)^2.$$

Since f is convex, it is continuously differentiable almost everywhere, and we can write:

$$(1-\delta)^{-1}\partial V[f(z+(1-\delta)I)]/\partial I = 2[Eff'-EfEf'] = 2E(f-Ef)f'$$

Let a be the certainty equivalent of $z + (1-\delta)I$ defined by $f(a) = Ef$, and, for brevity, write s for $(1-\delta)I$. Now:

$$E(f-Ef)f' = \int_{z+s\leq a} [f(z+s)-Ef)f'(z+s)dF(z) + \int_{z+s>a} [f(z+s)-Ef)f'(z+s)dF(z).$$

For $z + s \leq a$, $f(z+s) \geq Ef$ and $f'(z+s) < f'(a)$, by convexity, so that:

$$[f(z+s)-Ef]f'(z+s) \leq f'(a)[f(z+s)-Ef].$$

Similarly, for $z + s > a$, $f(z+s) \leq Ef$ and $f'(z+s) > f'(a)$, so that:

$$[f(z+s)-Ef]f'(z+s) \leq f'(a)[f(z+s)-Ef].$$

It follows that:

$$E(f-Ef)f' \leq f'(a)E(f-Ef) = 0.$$

■