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ON THE BEHAVIOR OF SOLUTIONS FOR LARGE $|x|$ OF PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

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1. Let R^n be the n-dimensional Euclidean space whose points x are represented by its coordinates $(x_1, \cdots, x_n)$ and let $\Omega_T \equiv R^n \times (0, T)$ $(T < +\infty)$ be a strip in the $(n+1)$-dimensional Euclidean half-space $R^n \times (0, \infty)$. Every point in Ω_T is denoted by (x, t), $x \in R^n$, $t \in (0, T)$.

We introduce a function space $E_\lambda(\Omega_T)(\lambda \in (0, 1])$ which is the totality of functions $W(x, t)$ such that

$$|W(x,t)| \leqq \mu \exp[\alpha(|x|^2 + 1)^\lambda]$$

in the closure $\overline{\Omega}_T$ of Ω_T for some positive constants μ and α.

Consider a parabolic differential equation

$$(1) \qquad Lu \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu - \frac{\partial u}{\partial t} = 0$$

with variable coefficients a_{ij} $(= a_{ij})$, b_i and c defined in $\overline{\Omega}_T$, where $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j > 0$ in $\overline{\Omega}_T$ for every non-zero real vector $\xi = (\xi_1, \cdots, \xi_n)$. We assume that there exist positive constants K_1, K_2, K_3 and $\lambda \in (0,1]$ such that in $\overline{\Omega}_T$

$$(2) \qquad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)^{1-\lambda}|\xi|^2,$$

$$(3) \qquad |b_i| \leqq K_2(|x|^2+1)^{1/2}, \quad (1 \leqq i \leqq n),$$

$$(4) \qquad c \leqq K_3(|x|^2+1)^\lambda .$$

Under these assumptions the equation (1) was treated by many authors, Krzyżański, Bodanko, Aronson, Besala and others. In particular, Bodanko [2]

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proved the existence and the uniqueness of solutions $u(x,t)\in E_\lambda(\Omega_T)$ of the Cauchy problem for (1). Aronson-Besala [1] showed the existence of a fundamental solution of (1) in some strip $R^n\times(0,T')$, where $T'\leqq T$.

In this paper, we shall deal with the behavior of solutions of the Cauchy problem of (1) for large $|x|$.

2. In the later discussion, the existence of positive function $H(x,t)$ such that $LH\leqq 0$ in Ω_T, plays an important role. The following lemma shows the existence of such a function.

LEMMA 1. *Suppose that all the coefficients of the differential operator L in* (1) *satisfy* (2), (3) *and* (4). *Let ρ be a number greater than* 1. *Then the function*

$$H_\alpha=H_\alpha(x,t)=\exp[-\alpha(|x|^2+1)^\lambda\rho^{\beta(\alpha)t}] \tag{5}$$

satisfies $LH_\alpha\leqq 0$ in $\bar{\Omega}_{T_\alpha}\equiv R^n\times[0,T_\alpha]$, where $\alpha>0$, $\beta(\alpha)=-[4\alpha\lambda^2K_1-4\lambda(\lambda-1)K_1+2\lambda K_2 n+\frac{K_3}{\alpha}\rho](\log\rho)^{-1}$ and $T_\alpha=\min(T,|\beta(\alpha)|^{-1})$.

PROOF. It is easy to see that

$$\begin{aligned}\frac{LH_\alpha}{H_\alpha}&=[4\alpha^2\lambda^2(|x|^2+1)^{2\lambda-2}\rho^{2\beta(\alpha)t}-4\alpha\lambda(\lambda-1)(|x|^2+1)^{\lambda-2}\rho^{\beta(\alpha)t}]\sum_{i,j=1}^n a_{ij}x_ix_j\\&\quad-2\alpha\lambda(|x|^2+1)^{\lambda-1}\rho^{\beta(\alpha)t}\sum_{i=1}^n(a_{ii}+b_ix_i)+c+\alpha(|x|^2+1)^\lambda\rho^{\beta(\alpha)t}\beta(\alpha)\log\rho\\&\leqq 4\alpha^2\lambda^2\rho^{2\beta(\alpha)t}(|x|^2+1)^\lambda K_1-4\alpha\lambda(\lambda-1)\rho^{\beta(\alpha)t}K_1+2\alpha\lambda\rho^{\beta(\alpha)t}(|x|^2+1)^\lambda K_2 n\\&\quad+K_3(|x|^2+1)^\lambda+\alpha(|x|^2+1)^\lambda\rho^{\beta(\alpha)t}\beta(\alpha)\log\rho\\&\leqq\alpha(|x|^2+1)^\lambda\rho^{\beta(\alpha)t}[4\alpha\lambda^2K_1\rho^{\beta(\alpha)t}-4\lambda(\lambda-1)K_1+2\lambda K_2 n\\&\qquad+\frac{K_3}{\alpha}\rho^{-\beta(\alpha)t}+\beta(\alpha)\log\rho].\end{aligned}$$

So, if (x,t) is in Ω_{T_α}, then the term in the bracket of the last side of the above is non-positive. Thus we have the lemma.

The following maximum principle due to Bodanko [2] will be important in the later treatment.

LEMMA 2. *Suppose that coefficients of L in* (1) *satisfy* (2), (3) *and*

$c \leqq 0$ *in* $\bar{\Omega}_T$. *If a usual solution* $u(x, t) \in E_\lambda(\Omega_T)$ *of the equation* (1) *fulfills* $|u(x, 0)| \leqq \mu_0$ *for a constant* μ_0, *then* $|u(x, t)| \leqq \mu_0$ *throughout* $\bar{\Omega}_T$.

3. Now we consider a usual solution $u(x, t) \in E_\lambda(\Omega_T)$ of (1). Here we assume that all the coefficients of (1) satisfy (2), (3) and (4). Let us suppose that $|u(x, 0)| \leqq \mu_0 \exp[-\alpha_0(|x|^2+1)^\lambda]$ for some positive constants μ_0 and α_0. We put

$$u(x, t) = v(x, t) H_{\alpha_0}(x, t),$$

where $H_{\alpha_0}(x, t)$ is obtained by putting $\alpha = \alpha_0$ in (5). Then it is obvious that

$$\sum_{i,j=1}^n a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^* \frac{\partial v}{\partial x_i} + \frac{LH_{\alpha_0}}{H_{\alpha_0}} v - \frac{\partial v}{\partial t} = 0,$$

where $b_i^* = b_i - 4\alpha_0 \lambda \rho^{\beta(\alpha_0)t}(|x|^2+1)^{\lambda-1} \sum_{j=1}^n a_{ij} x_j$. Lemma 1 implies that $\frac{LH_{\alpha_0}}{H_{\alpha_0}} \leqq 0$ in $\bar{\Omega}_{T_{\alpha_0}}$, where $T_{\alpha_0} = \min(T, |\beta(\alpha_0)|^{-1})$ and $\beta(\alpha_0) = -[4\alpha_0\lambda^2 K_1 - 4\lambda(\lambda-1)K_1 + 2\lambda K_2 n + \frac{K_3}{\alpha_0}\rho](\log \rho)^{-1}$.

Further in $\bar{\Omega}_{T_{\alpha_0}}$ we have $|b_i^*| \leqq K_2'(|x|^2+1)^{1/2}$ for some positive constant K_2' which is independent of t. Clearly $|v(x, 0)| = \frac{|u(x,0)|}{|H_{\alpha_0}(x, 0)|} \leqq \mu_0$. Hence we see by Lemma 2 that $|v(x, t)| \leqq \mu_0$ in $\bar{\Omega}_{T_{\alpha_0}}$.

Therefore it holds that

$$|u(x, t)| \leqq \mu_0 \exp[-\alpha_0(|x|^2+1)^\lambda \rho^{\beta(\alpha_0)t}]$$

in $\bar{\Omega}_{T_{\alpha_0}}$.

If $T_{\alpha_0} < T$, then we consider $u(x, T_{\alpha_0})$ to be the initial condition of $u(x, t)$ in $R^n \times (T_{\alpha_0}, T)$ and repeat the same procedure as the above. Since

$$|u(x, T_{\alpha_0})| \leqq \mu_0 \exp[-\alpha_0 \rho^{-1}(|x|^2+1)^\lambda],$$

we get

$$|u(x, t)| \leqq \mu_0 \exp[-\alpha_0 \rho^{-1}(|x|^2+1)^\lambda \rho^{\beta(\alpha_0\rho^{-1})t}]$$

in $R^n \times [T_{\alpha_0}, T_{\alpha_0}+T_{\alpha_1}]$, where $T_{\alpha_1} = \min(T-T_{\alpha_0}, |\beta(\alpha_0\rho^{-1})|^{-1})$.

In general, if $T_{\alpha_0} + \cdots + T_{\alpha_k} < T$, then by the argument used above, we can conclude that

$$(6) \qquad |u(x, t)| \leqq \mu_0 \exp[-\alpha_0 \rho^{-(k+1)}(|x|^2+1)^\lambda \rho^{\beta(\alpha_0\rho^{-(k+1)})t}]$$

in $R^n \times [T_{\alpha_0} + \cdots + T_{\alpha_k}, T_{\alpha_0} + \cdots + T_{\alpha_k} + T_{\alpha_{k+1}}]$, where

$$T_{\alpha_{k+1}} = \min(T-(T_{\alpha_0} + \cdots + T_{\alpha_k}), |\beta(\alpha_0 \rho^{-(k+1)})|^{-1}) > 0 .$$

We consider the convergent series

$$(7)\quad \sum_{k=0}^{\infty} |\beta(\alpha_0 \rho^{-k})|^{-1} = \log\rho \sum_{k=0}^{\infty} \left[4\alpha_0 \lambda^2 K_1 \rho^{-k} - 4\lambda(\lambda-1)K_1 + 2\lambda K_2 n + \frac{K_3}{\alpha_0} \rho^{k+1} \right]^{-1} .$$

For simplicity we put $f = 4\alpha_0 \lambda^2 K_1$, $g = -4\lambda(\lambda-1)K_1 + 2\lambda K_2 n$, and $h = K_3 \alpha_0^{-1}$. Assume now $4fh - g^2 > 0$. The function $[f\rho^{-\tau} + g + h\rho^{\tau+1}]^{-1}$ of the real variable $\tau \in (-\infty, \infty)$ has its maximum at $\tau = \tau_0 = (1/2)\log_\rho(f/h\rho)$.

First suppose that $f > h$. Then we can find ρ_0 (>1) so that if $\rho_0 > \rho > 1$, then $f/h\rho > 1$, that is, $\tau_0 > 0$. Let p be the non-negative integer such that $p < \tau_0 \leqq p+1$. Then we see easily from $4fh\rho - g^2 > 0$ that

$$(8)\quad \sum_{k=0}^{\infty} |\beta(\alpha_0 \rho^{-k})|^{-1} \geqq \log\rho \int_1^p \frac{d\tau}{f\rho^{-\tau} + g + h\rho^{\tau+1}} + \log\rho \int_{p+1}^{\infty} \frac{d\tau}{f\rho^{-\tau} + g + h\rho^{\tau+1}}$$

$$= \frac{2}{\sqrt{4h\rho f - g^2}} \times \tan^{-1} \frac{\sqrt{4h\rho f - g^2}\,[4h\rho f - g^2 + (2h\rho^{p+1} + g)(2h\rho + g) + 2h\rho(\rho^p - 1)(2h\rho^{p+2} + g)]}{(2h\rho^{p+2} + g)[4h\rho f - g^2 + (2h\rho^{p+1} + g)(2h\rho + g)] - (4h\rho f - g^2)\, 2h\rho(\rho^p - 1)} .$$

The last term of the above will be denoted by $T^*(\rho)$, which is continuous in $\rho \in [1, \infty)$.

In the case when $f \leqq h$, we see that $f \leqq h\rho$, $\tau_0 \leqq 0$ and that

$$(9)\quad \sum_{k=0}^{\infty} |\beta(\alpha_0 \rho^{-k})|^{-1} \geqq \log\rho \int_1^{\infty} \frac{d\tau}{f\rho^{-\tau} + g + h\rho^{\tau+1}} = \frac{2}{\sqrt{4h\rho f - g^2}} \tan^{-1} \frac{\sqrt{4h\rho f - g^2}}{2h\rho + g} .$$

The right hand side of (9) will be denoted by $T^{**}(\rho)$, which is also continuous in $[1, \infty)$.

We put

$$(10)\quad \widetilde{T}(\rho) = \begin{cases} T^*(\rho), & (f > h) \\ T^{**}(\rho), & (f \leqq h). \end{cases}$$

Now we can prove the following

THEOREM 1. *Suppose that the parabolic operator L in* (1) *satisfies the conditions* (2), (3) *and* (4) *in* $\bar{\Omega}_T$ *and that the constants* K_1, K_2, K_3 *appeared in* (2), (3) *and* (4) *satisfy* $D = 4\lambda^2[(K_2 n - 2(\lambda-1)K_1)^2 - 4K_1K_3] < 0$. *Let* $u(x, t) \in E_\lambda(\Omega_T)$ $(\lambda \in (0, 1])$ *be a usual solution of* $Lu = 0$ *in* $\bar{\Omega}_T$. *Put*

$$T_0 = \min\left(T, \frac{2}{\sqrt{-D}} \tan^{-1} \frac{\sqrt{-D}}{2\lambda K_2 n - 4\lambda(\lambda-1)K_1 + 2K_3\alpha_0^{-1}}\right).$$

If

$$|u(x, 0)| \leqq \mu_0 \exp[-\alpha_0(|x|^2+1)^\lambda]$$

for some positive constants μ_0 *and* α_0, *then for any* t *in the closed interval* $[0, T']$ *contained in* $[0, T_0)$ *there exists a positive constant* α' *such that*

$$|u(x, t)| \leqq \mu_0 \exp[-\alpha'(|x|^2+1)^\lambda]$$

for any $x \in R^n$.

PROOF. We see easily from the continuity of $\tilde{T}(\rho)$ in $[1, \infty)$ that there exist a positive integer N and a positive number ρ (>1) such that

$$T' \leqq \sum_{k=0}^{N} |\beta(\alpha_0\rho^{-k})|^{-1}.$$

Therefore, for $\alpha' = \max_{0 \leqq k \leqq N}(\alpha_0 \rho^{-k+\beta(\alpha_0\rho^{-k})t})$, we have $|u(x, t)| \leqq \mu_0 \exp[-\alpha'(|x|^2 +1)^\lambda]$ at every point $(x, t) \in R^n \times [0, T']$, which proves the theorem.

4. Example. We consider a particular parabolic equation

$$\Delta u + k^2(|x|^2+1)u - \frac{\partial u}{\partial t} = 0, \quad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right), \tag{11}$$

where k (>0) is a constant. Krzyżański [3] proved the existence of the solution

(12)
$$u(x, t) = \left(\frac{k}{2\alpha_0 \sin 2kt + k \cos 2kt}\right)^{n/2} \exp\left[-\frac{k(2\alpha_0 \cos 2kt - k \sin 2kt)}{2(2\alpha_0 \sin 2kt + k \cos 2kt)}|x|^2 + k^2 t\right]$$

of the above equation (11) in $R^n \times (0, \pi/4k)$ with the Cauchy data $u(x, 0) = e^{-\alpha_0|x|^2}$ by using the fundamental solution, which was constructed in [4].

The solution $u(x, t)$ decays exponentially as $|x| \to \infty$ if $t < (1/2k)\tan^{-1}(2\alpha_0/k)$.

If we put $K_1=1$, $K_2=0$, $K_3=k^2$ and $\lambda=1$ in our Theorem 1, then we get the result stated above.

As is easily seen, the solution $u(x, t)$ in (12) grows exponentially as $|x| \to \infty$ provided that $t > (1/2k)\tan^{-1}(2\alpha_0/k)$.

5. Recently Kusano [5] discussed the decay of solutions of the Cauchy problem of (1) for large $|x|$ under the assumptions (2), (3) and $c \leqq K_3'$ for a positive constant K_3' in $\bar{\Omega}_T$. Here we show that Kusano's result can be derived from the discussion stated above. First we prove the following:

LEMMA 3. *Let* $u(x, t) \in E_\lambda(\Omega_T)$ ($\lambda \in (0, 1]$) *be a usual solution of the parabolic equation* (1) *and the operator* L *in* (1) *satisfy the conditions* (2), (3), *and* $c \leqq 0$ *in* $\bar{\Omega}_T$. *If for some positive constants* μ_0, α_0 *and* $\lambda \in (0, 1]$

$$|u(x, 0)| \leqq \mu_0 \exp[-\alpha_0(|x|^2+1)^\lambda],$$

then there exists a positive constant $\tilde{\alpha} = \tilde{\alpha}(\alpha_0, T)$ *for which*

$$|u(x, t)| \leqq \mu_0 \exp[-\tilde{\alpha}(|x|^2+1)^\lambda]$$

in $\bar{\Omega}_T$.

PROOF. We put $K_3=0$ in (3). Then we get the divergent series

$$\sum_{k=0}^{\infty} |\beta(\alpha_0 \rho^{-k})|^{-1} = \log \rho \sum_{k=0}^{\infty} (4\alpha_0 \lambda^2 K_1 \rho^{-k} - 4\lambda(\lambda-1)K_1 + 2\lambda K_2 n)^{-1}$$

instead of the convergent series (7).

So we can easily conclude the existence of a positive constant $\tilde{\alpha}$ in our lemma.

Now we can prove Kusano's result.

THEOREM 2. (Kusano [5]) *Assume that the parabolic operator* L *in* (1) *satisfies the conditions* (2), (3) *and* $c \leqq K_3'$ *for a positive constant* K_3' *in* $\bar{\Omega}_T$. *Let* $u(x, t) \in E_\lambda(\Omega_T)$ ($\lambda \in (0, 1]$) *be a usual solution of* $Lu=0$ *in* $\bar{\Omega}_T$. *If*

$$|u(x, 0)| \leqq \mu_0 \exp[-\alpha_0(|x|^2+1)^\lambda]$$

for some positive constants μ_0 *and* α_0, *then* $u(x, t)$ *decays exponentially as* $|x|$ *tends to* ∞ *for any* $t \in [0, T]$.

PROOF. We put $v(x, t)=u(x, t)e^{-K_3't}$. Then $v(x, t)$ satisfies

$$\sum_{ij=1}^{n} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + (c-K_3')v - \frac{\partial v}{\partial t} = 0 .$$

Lemma 3 implies the existence of a positive constant $\tilde{\alpha}$ such that $|v(x, t)| \leqq \mu_0 \exp[-\tilde{\alpha}(|x|^2+1)^\lambda]$ in $\bar{\Omega}_T$. Thus we see $|u(x, t)| \leqq \mu_0 \exp[-\tilde{\alpha}(|x|^2+1)^\lambda + K_3't]$, which proves our theorem.

6. By the similar argument to that used in §3, we can prove the following whose proof is omitted.

THEOREM 3. *Assume that the parabolic operator L in* (1) *satisfies the conditions* (2), (3) *and*

$$(4') \qquad c \leqq K_3'' \log(|x|^2+1) + K_3', \qquad (K_3', K_3'' > 0)$$

in $\bar{\Omega}_T$. *Let* $u(x, t) \in E_\lambda(\Omega_T)$ $(\lambda \in (0, 1])$ *be a usual solution of* $Lu=0$ *in* $\bar{\Omega}_T$. *If*

$$|u(x, 0)| \leqq \mu_0 \exp[-\alpha_0(|x|^2+1)^\lambda]$$

for some positive constants μ_0 *and* α_0, *then there exist positive constants* $\tilde{\mu}$ *and* $\tilde{\alpha}$ *for which*

$$|u(x, t)| \leqq \tilde{\mu}(|x|^2+1)^{k_3''t} \exp[-\tilde{\alpha}(|x|^2+1)^\lambda]$$

in $\bar{\Omega}_T$.

REMARK. If $K_3' = 0$ in Theorem 3, then Theorem 3 also reduces to Kusano's result, Theorem 2.

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