On the Behavior of Solutions of Certain Differential Equations of the Fourth Order (*).

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Summary. - *In this paper three new results are obtained for equations of the form* (1.1). *Conditions are established which guarantee asymptotic stability, ultimate boundedness, and convergence of solutions of* (1.1).

1. - Introduction.

We shall be concerned here with equations of the form

$$x^{(4)} + f(\ddot{x})\dddot{x} + a_2\ddot{x} + g(\dot{x}) + h(x) = p(t,\ x,\ \dot{x},\ \ddot{x},\ \dddot{x}) \tag{1.1}$$

where a_2 is a positive constant and f, g, h, and p are continuous real-valued functions depending only on the arguments indicated.

We will find conditions on f, g, h, and p which will imply asymptotic stability, ultimate boundedness, and convergence of solutions of equation (1.1). The first result deals with conditions under which all solutions of (1.1) tend to zero as $t \to \infty$. The problem has been the subject of investigation by a number of authors for various forms of equation (1.1). An excellent account of the literature of stability and boundedness for fourth order equations can be found in [9, Chapter 6].

In particular, Ezeilo [4] and Harrow [6] established conditions under which the trivial solutions of the respective equations

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = 0 \tag{1.2}$$

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + g(\dot{x}) + h(x) = 0 \tag{1.3}$$

are asymptotically stable. Ezeilo [5] has generalized his result in [4] to the perturbed equation

$$x^{(4)} + g_1(\ddot{x})\dddot{x} + a_2\ddot{x} + g_3(\dot{x}) + a_4 x = p_2(t,\ x,\ \dot{x},\ \ddot{x},\ \dddot{x}). \tag{1.4}$$

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Here his results are combined with those of HARROW [6] and extended to the more general equation (1.1). We also show that the hypothesis on h required in [6] can be relaxed somewhat.

The second theorem is concerned with the ultimate boundedness of solutions of equation (1.1). EZEILO [4] and HARROW [7] investigated this property for solutions of

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t) \tag{1.5}$$

$$x^{(4)} + a\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t). \tag{1.6}$$

Both results depend on the initial conditions of the solutions and require the strong condition $\int_0^t |p(s)|\, ds \leq A < \infty$, $t \geq 0$. TEJUMOLA [14] obtained a similar result with the bound independent of the initial conditions for

$$x^{(4)} + f(\ddot{x})\dddot{x} + a_2\ddot{x} + g(\dot{x}) + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.7}$$

while significantly weakening EZEILO's conditions in [4] to $|p(t, x, y, z, w)| \leq \Delta + \Delta_0(y^2 + z^2 + w^2)^{\frac{1}{2}}$. Here similar results are obtained for the more general equation (1.1).

The last theorem deals with the convergence of solutions. The solutions of $\dfrac{dx}{dt} = F(t, x)$, where x is an n-vector and $F(t, x)$ is continuous on $[0, \infty) \times R^n$, are said to converge if each pair of solutions $x(t)$ and $y(t)$ satisfies $x(t) - y(t) \to 0$ as $t \to \infty$. To date there have been no convergence results for fourth order equations. EZEILO [2] has dealt with the second order case. In particular, SWICK [12] has developed several results for third order equations and his methods are extended here to the fourth order equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + g(\dot{x}) + a_4 x = p(t). \tag{1.8}$$

2. - Statement of Results.

THEOREM 1. - Assume that g and h are differentiable on R and that

(i) there exists $a_1 > 0$ such that

$$f(z) \geq a_1 \quad \text{all} \quad z,$$

(ii) $h(0) = 0$ and there are positive constants a_4 and A_4 such that

$$\frac{h(x)}{x} \geq a_4 \qquad x \neq 0$$

and for sufficiently small $\delta_1 > 0$

$$A_4 - \delta_1 \leq h'(x) \leq A_4 \quad \text{all} \quad x,$$

(iii) $g(0) = 0$ and there exist positive constants a_3 and $\triangle_0$ such that

$$g'(y) \geq a_3$$

$$[a_1 a_2 - g'(y)]a_3 - a_1 A_4 f(z) \geq \triangle_0 \tag{2.1}$$

for all y, z,

(iv) for all $y \neq 0$, $g'(y) - g(y)/y \leq \delta_2$ where δ_2 is a positive constant satisfying

$$\delta_2 < 2A_4\triangle_0/(a_1 a_3^2),$$

(v) $1/z \int_0^z f(s)ds - f(z) \leq \delta_3$ for all $z \neq 0$, where δ_3 satisfies

$$\delta_3 < 2\triangle_0/(a_1^2 a_3),$$

(vi) for all x, y, z, w

$$|p(t, x, y, z, w)| \leq \Theta_1(t) + \Theta_2(t)(y^2 + z^2 + w^2)^{\alpha/2} + \triangle_1(y^2 + z^2 + w^2)^{\frac{1}{2}} \tag{2.2}$$

where α, $\triangle_1$ are constants such that $0 \leq \alpha < 1$, $\triangle_1 \geq 0$, and the functions $\Theta_1 \geq 0$, $\Theta_2 \geq 0$ satisfy

$$\max_{t\geq 0} \Theta_i(t) < \infty \quad \text{and} \quad \int_0^\infty \Theta_i(t)dt < \infty \qquad (i = 1, 2). \tag{2.3}$$

Then for sufficiently small $\triangle_1$ every solution $x = x(t)$ of (1.1) satisfies

$$x \to 0, \ \dot{x} \to 0, \ \ddot{x} \to 0, \ \dddot{x} \to 0 \quad \text{as} \quad t \to \infty. \tag{2.4}$$

NOTE 1. - For the special cases (*i*) $h(x) = a_4 x$ and (ii) $h(x) = a_4 x$, $p(t, x, y, z, w) \equiv 0$ our result is the same as Theorem 3 in [5] and Theorem 1 in [4] respectively. When $f(z) \equiv a_1$, $p(t, x, y, z, w) \equiv 0$ our result reduces to Theorem 1 in [6] without the restriction that h satisfy $h(x)/x \leq (a_4 A_4)^{\frac{1}{2}}$.

THEOREM 2. - Suppose that in addition to conditions (i) — (v) of Theorem 1, h' and g' are continuous for all x, y and there are positive constants P_0 and P_1 such that

$$|p(t, x, y, z, w)| \leq P_0 + P_1(y^2 + z^2 + w^2)^{\frac{1}{2}}. \tag{2.5}$$

Then there is an $\alpha > 0$ such that if $P_1 \leq \alpha$ there exists a constant B, dependent only on equation (1.1), such that every solution $x = x(t)$ of (1.1) satisfies

$$|x(t)| + |\dot{x}(t)| + |\ddot{x}(t)| + |\dddot{x}(t)| \leq B \tag{2.6}$$

for all sufficiently large t.

NOTE 2. - For the special case $h(x) = a_4 x$, our result is the same as Theorem 2 in [14].

THEOREM 3. - Assume that

(i) there are positive constants A, B, C, and D such that every solution $x(t)$ of (1.8) satisfies ultimately

$$|x(t)| \leq A, \ |\dot{x}(t)| \leq B, \ |\ddot{x}(t)| \leq C, \quad \text{and} \quad |\dddot{x}(t)| \leq D,$$

(ii) the functions g, g', g'' exist on $[-B, B]$ and g and g' are continuous on $[-B, B]$,

(iii) there exist positive constants a_3, A_3, and Δ_0 such that

$$g'(y) \geq a_3$$

$$|g''(y)| \leq A_3$$

$$[a_1 a_2 - g'(y)] a_3 - a_1^2 a_4 \geq \Delta_0 \qquad |y| \leq B,$$

(iv) there exists $\varepsilon > 0$ such that

$$CA_3/a_3 < \varepsilon < \Delta_0/(a_1 a_3 D_0)$$

and

$$(a_3\varepsilon - CA_3)(\Delta/a_1 a_3 - D_0\varepsilon) > (\varepsilon + 1/a_1)^2 C^2 A_3^2$$

where $D_0 = a_1 a_2 + (a_2 a_3/a_4)$.

Then the solutions of (1.8) converge.

3. - Proof of Theorem 1.

Rather than deal with equation (1.1) directly we will consider the equivalent system

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = w$$

$$\dot{w} = -wf(z) - a_2 z - g(y) - h(x) + p(t, x, y, z, w). \tag{3.1}$$

It will suffice to show that every solution $(x(t), y(t), z(t), w(t))$ of (3.1) satisfies

$$x(t) \to 0, \ y(t) \to 0, \ z(t) \to 0, \ w(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.2}$$

The proof of Theorem 1 will depend upon the following Liapunov function, $V = V(x, y, z, w)$, obtained by an appropriate modification of that used by Ezeilo in [4].

$$\begin{aligned} 2V = {} & (a_2 d_1 - d_2)z^2 + 2\int_0^z sf(s)ds + d_1 w^2 + 2d_2 y \int_0^z f(s)ds \\ & + 2d_2 yw + 2zw + 2d_2 \int_0^x h(s)ds + (a_2 d_2 - A_4 d_1)y^2 \\ & + 2\int_0^y g(s)ds + 2d_1 z g(y) + 2yh(x) + 2d_1 z h(x), \end{aligned} \tag{3.3}$$

where

$$d_1 = \varepsilon + 1/a_1 \qquad d_2 = \varepsilon + A_4/a_3 \tag{3.4}$$

and $\varepsilon > 0$ is a constant whose value will be determined in the proofs of the following two lemmas.

Lemma 3.1. - Under the hypotheses of Theorem 1, $V(0, 0, 0, 0) = 0$ and there exist positive constants E_1, E_2, E_3, E_4 dependent only on a_1, a_2, a_3, a_4, A_4, Δ_0, and ε such that

$$V \geq E_1 x^2 + E_2 y^2 + E_3 z^2 + E_4 w^2 \tag{3.5}$$

for all x, y, z, w.

Proof. - Since $g(0) = h(0) = 0$ it is clear that $V(0, 0, 0, 0) = 0$ and thus it remains to establish (3.5). We shall deal first with the case $z = 0$. V becomes

$$\begin{aligned} 2V(x, y, 0, w) = {} & d_1 w^2 + 2d_2 yw + 2d_2 \int_0^x h(s)ds \\ & + (a_2 d_2 - A_4 d_1)y^2 + 2\int_0^y g(s)ds \\ & + 2yh(x) \end{aligned} \tag{3.6}$$

$$= d_1(w + d_2y/d_1)^2 + (a_2d_2 - A_4d_1 - d_2^2/d_1)y^2$$

$$+ 2\int_0^y g(s)ds + 2d_2\int_0^x h(s)ds + 2yh(x)$$

$$= V_1 + V_2 + V_3, \tag{3.7}$$

where

$$V_1 = d_1(w + d_2y/d_1)^2$$

$$V_2 = (a_2d_2 - A_4d_1 - d_2^2/d_1)y^2$$

$$V_3 = 2\int_0^y g(s)ds + 2d_2\int_0^x h(s)ds + 2yh(x).$$

To establish (3.5) we will need the following inequalities

$$[d_1 - 1/f(z)] \geq \varepsilon \quad \text{all} \quad z \tag{3.8}$$

$$[d_2 - A_4y/g(y)] \geq \varepsilon \quad y \neq 0 \tag{3.9}$$

$$a_2 - d_1g'(y) - d_2f(z) \geq (\triangle_0/a_1a_3) - D_0\varepsilon > 0 \quad \text{all} \quad y,\ z, \tag{3.10}$$

where $D_0 = a_1a_2 + (a_2a_3/A_4)$.

The first two inequalities follow immediately from hypotheses (i) and (iii) and (3.4), since we have

$$g(y)/y = g'(\Theta_1 y) \qquad 0 < \Theta_1 < 1. \tag{3.11}$$

Substituting the values for d_1 and d_2 from (3.4) into the left side of (3.10) we obtain

$$a_2 - d_1g'(y) - d_2f(z) = a_2 - (1/a_1)g'(y) - A_4f(z)/a_3$$

$$- [g'(y) + f(z)]\varepsilon$$

$$\geq (\triangle_0/a_1a_3) - [g'(y) + f(z)]\varepsilon.$$

Since (2.1) also implies that

$$g'(y) < a_1a_2 \quad \text{and} \quad f(z) < (a_2a_3/A_4) \quad \text{all} \quad y,\ z,$$

we have

$$a_2 - d_1 g'(y) - d_2 f(z) \geq (\Delta_0/a_1 a_3) - (a_1 a_2 + a_2 a_3/A_4)\varepsilon .$$

Assuming $\varepsilon < \Delta_0/(a_1 a_3 D_0)$, we have established (3.10).

Using (3.10) the method used in [4] with α_4 replaced by A_4 yields the result

$$V_2 \geq (A_4/a_3)(\Delta_0/a_1 a_3 - D_0\varepsilon)y^2 . \tag{3.12}$$

Finally, from hypothesis (iii) and (3.11)

$$V_3 \geq a_3 y^2 + 2d_2 \int_0^x h(s)ds + 2yh(x)$$

$$= (1/a_3)[a_3 y + h(x)]^2 + 2d_2 \int_0^x h(s)ds - (1/a_3)[h(x)]^2$$

$$\geq 2 \int_0^x [d_2 - (1/a_3)h'(s)]h(s)ds ,$$

and from (3.4) and hypothesis (ii) we have

$$V_3 \geq a_4 \varepsilon x^2 . \tag{3.13}$$

Combining (3.12) and (3.13) into expression (3.7), we find

$$2V(x, y, 0, w) \geq d_1(w + d_2 y/d_1)^2 + (A_4/a_3)(\Delta_0/a_1 a_3 - D_0\varepsilon)y^2 + a_4 \varepsilon x^2 , \tag{3.14}$$

and clearly (3.5) is satisfied for $z = 0$.

For $z \neq 0$, defining $F(z) = \int_0^z f(s)ds$ and $\gamma = \gamma(y)$, $\beta = \beta(x)$ as

$$\gamma = \begin{cases} g(y)/y & y \neq 0 \\ g'(0) & y = 0 \end{cases} \qquad \beta = \begin{cases} h(x)/x & x \neq 0 \\ h'(0) & x = 0 \end{cases}$$

V becomes

$$2V(x, y, z, w) = [a_2 d_2 - A_4 d_1 - d_2^2 F(z)/z]y^2 + 2\int_0^y g(s)ds$$

$$- yg(y) + [a_2 d_1 - d_2 - d_1^2 \gamma(y)]z^2$$

$$+ 2\int_0^z sf(s)ds - zF(z) + 2d_2\int_0^x h(s)ds$$
$$- \{[\beta(x)]^2x^2/\gamma(y)\} + [d_1 - z/F(z)]w^2$$
$$+ [z/F(z)][w + F(z) + d_2yF(z)/z]^2$$
$$+ [1/\gamma(y)][\beta(x)x + y\gamma(y) + d_1z\gamma(y)]^2$$
$$\geq V_1 + V_2 + V_3 + [d_1 - z/F(z)]w^2, \tag{3.15}$$

where

$$V_1 = [a_2d_2 - A_4d_1 - d_2^2F(z)/z]y^2 + 2\int_0^y g(s)ds - yg(y)$$

$$V_2 = [a_2d_1 - d_2 - d_1^2\gamma(y)]z^2 + 2\int_0^z sf(s)ds - zF(z)$$

$$V_3 = 2d_2\int_0^x h(s)ds - \{[\beta(x)]^2x^2/\gamma(y)\}.$$

EZEILO's calculations in [4] with α_4 replaced by A_4 can be applied here to obtain the following inequalities for V_1 and V_2:

$$V_1 \geq \tfrac{1}{2}(\triangle_0A_4/a_1a_3^2 - \tfrac{1}{2}\delta_2)y^2 \tag{3.16}$$

provided $\varepsilon \leq \frac{1}{4}a_3(2\triangle_0A_4/a_1a_3^2 - \delta_2)/A_4D_0$,

$$V_2 \geq \tfrac{1}{4}(2\triangle_0/a_1^2a_3 - \delta_3)z^2 \tag{3.17}$$

provided $\varepsilon \leq (a_1/4D_0)(2\triangle_0/a_1^2a_3 - \delta_3)$.

Hypotheses (iv) and (v) insure that the coefficients of y^2 and z^2 are positive in (3.16) and (3.17).

From hypothesis (iii) and (3.11) we have $\gamma(y) \geq a_3$ for all y and thus

$$V_3 \geq 2d_2\int_0^x h(s)ds - (1/a_3)[\beta(x)]^2x^2$$
$$= 2\int_0^x \{[h(s)/s][d_2 - (1/a_3)h'(s)]s\}\, ds$$
$$\geq a_4\,\varepsilon\, x^2. \tag{3.18}$$

Finally, hypothesis (i) and the relation

$$F(z)/z = f(\Theta_2 z) \qquad 0 < \Theta_2 < 1 \tag{3.19}$$

insure that

$$[d_1 - z/F(z)]w^2 \geq \varepsilon w^2. \tag{3.20}$$

Combining (3.16), (3.17), (3.18), and (3.20) into expression (3.15) for V, we obtain (3.5) for $z \neq 0$. Thus the lemma has been established.

LEMMA 3.2. - Under the hypotheses of Theorem 1 there exist positive constants F_1, F_2, F_3, F_4 depending only on a_1, a_2, a_3, a_4, A_4, $\triangle_0$, and ε such that

$$\dot{V}_{(3.1)} \leq -(F_1 y^2 + F_2 z^2 + F_3 w^2) + F_4(y^2 + z^2 + w^2)^{\frac{1}{2}}[\Theta_1 + \Theta_2(y^2 + z^2 + w^2)^{\alpha/2}] \tag{3.21}$$

for all x, y, z, w when $\triangle_1$ is sufficiently small.

PROOF. - An easy calculation shows that

$$\begin{aligned} -\dot{V}_{(3.1)} = &-h'(x)y^2 + d_2 y g(y) + d_1[A_4 - h'(x)]yz \\ &+ \{[a_2 - d_1 g'(y)]z^2 - d_2 z F(z)\} \\ &+ [d_1 f(z) - 1]w^2 \\ &- (d_2 y + z + d_1 w)[p(t, x, y, z, w)]. \end{aligned} \tag{3.22}$$

From hypotheses (iii) and (ii) and (3.4) and (3.11)

$$-h'(x)y^2 + d_2 y g(y) \geq a_3\, \varepsilon\, y^2,$$

and from (3.10) and (3.19)

$$\begin{aligned} [a_2 - d_1 g'(y)]z^2 - d_2 z F(z) &= [a_2 - d_1 g'(y) - d_2 F(z)/z]z^2 \\ &\geq (\triangle_0/a_1 a_3 - D_0 \varepsilon)z^2. \end{aligned}$$

Thus we find that

$$\begin{aligned} -\dot{V} \geq &\, a_3\, \varepsilon\, y^2 + d_1[A_4 - h'(x)]yz + (\triangle_0/a_1 a_3 - D_0\varepsilon)z^2 \\ &+ a_1\, \varepsilon\, w^2 - (d_2 y + z + d_1 w)[p(t, x, y, z, w], \end{aligned} \tag{3.23}$$

and as long as δ_1 satisfies

$$\varepsilon a_3(\Delta_0/a_1a_3 - D_0\varepsilon) \geq \delta_1^2(\varepsilon + 1/a_1)^2/4,$$

hypothesis (ii) insures that there exists $\eta > 0$ such that

$$- \dot{V} \geq \eta(y^2 + z^2) + a_1 \varepsilon w^2 - (d_2y + z + d_1w)[p(t, x, y, z, w)]. \tag{3.24}$$

Letting $d_3 = \max(d_1, d_2, 1)$ and using (2.2),

$$\begin{aligned}\dot{V} &\leq - \eta(y^2 + z^2) - a_1 \varepsilon w^2 \\ &+ d_3(|y| + |z| + |w|)[\Theta_1(t) + \Theta_2(t)(y^2 + z^2 + w^2)^{\alpha/2} \\ &\qquad + \Delta_1(y^2 + z^2 + w^2)^{\frac{1}{2}}] \\ &\leq (- \eta + 3d_3\Delta_1)y^2 + (- \eta + 3d_3\Delta_1)z^2 + (- a_1\varepsilon + 3d_3\Delta_1)w^2 \\ &+ 3d_3(y^2 + z^2 + w^2)^{\frac{1}{2}}[\Theta_1(t) + \Theta_2(t)(y^2 + z^2 + w^2)^{\alpha/2}]\end{aligned}$$

$$\begin{aligned}&\leq - \frac{\eta}{2} y - \frac{\eta}{2} z^2 - \frac{a_1\varepsilon}{2} w^2 \\ &+ 3d_3(y^2 + z^2 + w^2)^{\frac{1}{2}}[\Theta_1 + \Theta_2(y^2 + z^2 + w^2)^{\alpha/2}]\end{aligned} \tag{3.25}$$

provided

$$\Delta_1 \leq \Gamma = \min(\eta/6d_3, a_1\varepsilon/6d_3).$$

Hence we have (3.21) and the lemma is established.

The remainder of the proof of Theorem 1 is similar to the end of the proof of Theorem 1 in [5] and only the main steps will be outlined here.

From (3.5) and (3.21) we can show that there exist positive constants F_5 and F_6 such that

$$\begin{aligned}\dot{V} &\leq - F_5[y^2 + z^2 + w^2] \\ &+ F_6(y^2 + z^2 + w^2)^{(1-r/2)}[\Theta_1^r + \Theta_2^{r(1-\alpha)^{-1}}],\end{aligned} \tag{3.26}$$

where r is any constant in the range $1 \leq r \leq 2$.

From (3.5) with $V(t) = V(x(t), y(t), z(t), w(t))$

$$y^2 + z^2 + w^2 \leq DV(t),$$

and hence

$$\dot{V} \leq DF_6[V(t)]^{(1-r/2)}[\Theta_1^r + \Theta_2^{r(1-\alpha)^{-1}}].$$

By substituting $V = W^{(2/r)}$ and integrating the resulting inequality, we obtain

$$(3.27) \qquad [V(t)]^{r/2} \leq [V(t_0)]^{r/2} + F_7 \int_t^{t_0} \{[\Theta_1(s)]^r + [\Theta_2(s)]^{r(1-\alpha)^{-1}}\} \, ds$$

for $t \geq t_0$, where F_7 is a positive constant.

Integrating the inequality (3.26) from 0 to t we have

$$V(t) \leq V(0) - F_5 \int_0^t [y^2(s) + z^2(s) + w^2(s)] ds + F_6 \int_0^t \{[y^2(s) + z^2(s) + w^2(s)^{(1-r/2)}][\Theta_1^r(s) + \Theta_2^{r(1-\alpha)^{-1}}(s)\} \, ds,$$

and this holds for all $t \geq 0$. But by (3.5) $V(t) \geq 0$ for all t. Hence

$$(3.28) \qquad F_5 \int_0^t [y^2(s) + z^2(s) + w^2(s)] ds \leq V(0) + F_6 \int_0^t \{[y^2(s) + z^2(s) + w^2(s)]^{(1-r/2)}[\Theta_1^r(s) + \Theta_2^{r(1-\alpha)^{-1}}(s)]\} \, ds.$$

Now from (3.5) and (3.27) and the further assumption from (2.3) that

$$(3.29) \qquad \int_0^\infty [\Theta_1(t)]^r \, dt < \infty \quad \text{and} \quad \int_0^\infty [\Theta_2(t)]^{r(1-\alpha)^{-1}} \, dt < \infty$$

there exists a constant L, $0 < L < \infty$, whose value depends only on a_1, a_2, a_3, a_4, A_4, $\triangle_0$, ε, $V(0)$ such that

$$(3.30) \qquad x^2(t) + y^2(t) + z^2(t) + w^2(t) \leq L^2, \qquad t \geq 0.$$

Using this inequality in (3.28) yields

$$F_5 \int_0^t [y^2(s) + z^2(s) + w^2(s)]ds \leq V(0) + F_6 L^{(2-r)} \int_0^t [\Theta_1^r(s) + \Theta_2^{r(1-\alpha)^{-1}}(s)]ds\,,$$

which with (3.29) implies that every solution $(x(t), y(t), z(t), w(t))$ of (3.1) satisfies

$$\int_0^\infty [y^2(s) + z^2(s) + w^2(s)]ds < \infty. \tag{3.31}$$

Now from (3.27) and (3.31) it is easy to verify that

$$y^2(t) + z^2(t) + w^2(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.32}$$

Then by integrating both sides of the equality

$$x^{(4)} = -wf(z) - a\,z - g(y) - h(x) + p(t, x, y, z, w)$$

from t to $t+1$ and utilizing (3.27) and (3.32) we are able to show that

$$h(x(t)) \to 0 \quad \text{as} \quad t \to \infty,$$

which implies that

$$x(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{3.33}$$

and hence we have (3.2) and the proof of Theorem 1 is complete.

4. - Proof of Theorem 2.

The proof of Theorem 2 relies upon the following lemma.

Consider a system of differential equations

$$\frac{dx}{dt} = F(t, x), \tag{4.1}$$

where x is an n-vector and $F(t, x)$ is continuous on $[0, \infty) \times R^n$.

LEMMA 4.1 [15]. - Suppose that there exists a LIAPUNOV function $V(t, x)$ defined on $0 \leq t < \infty$, $\|x\| \geq H$, where H may be large, which satisfies the following conditions:

(i) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$, where $a(\tau) \in CI$ (i.e. continuous and increasing), $a(\tau) \to \infty$ as $\tau \to \infty$, and $b(\tau) \in CI$,

(ii) $\dot V_{(4.1)}(t, x) \leq -c(\|x\|)$, where $c(\tau)$ is positive and continuous.

Then the solutions of (4.1) are uniform-ultimately bounded.

It is assumed here that $V(t, x)$ is continuous in t and satisfies a local LIPSCHITZ condition with respect to x and that $\dot V_{(4.1)}$ is defined as

$$\dot V_{(4.1)}(t, x) = \overline{\lim_{h \to 0}}\,(1/h)\{V[t+h,\ x+hF(t, x)] - V(t, x)\}.$$

The following definition will prove helpful.

DEFINITION [15]. - The solutions of (4.1) are uniform-ultimately bounded for bound B if they are uniform-bounded and if there exists a $B > 0$ and a $T > 0$ such that for every solution $x(t;\ x_0,\ t_0)$ of (4.1), $\|x(t;\ x_0,\ t_0\| < B$ for all $t \geq t_0 + T$, where B is independent of the particular solution while T may depend on each solution.

To apply Lemma 4.1, consider the system used in the previous section

$$
\begin{aligned}
&\dot x = y \\
&\dot y = z \\
&\dot z = w \\
&\dot w = -wf(z) - a_2 z - g(y) - h(x) + p(t, x, y, z, w).
\end{aligned}
\tag{4.2}
$$

We will modify the LIAPUNOV function used in the proof of Theorem 1 so as to make the time derivative negative outside of an arbitrarily large but fixed neighborhood of the origin. Define $V = V(x, y, z, w)$ as $2V = 2V_1 + 2V_2$, where V_1 is defined by equation (3.3) and V_2 is as follows

$$
V_2 = \begin{cases}
0 & (x, y, z, w) \in R_1 = \{w \geq M\} \\
\delta w - \delta M & (x, y, z, w) \in R_2 = \{|w| \leq M,\ x \geq N\} \\
-2\delta M & (x, y, z, w) \in R_3 = \{w \leq -M,\ x \geq N\} \\
-2\delta(M/N)x & (x, y, z, w) \in R_4 = \{w \leq -M,\ |x| \leq N\} \\
2\delta M & (x, y, z, w) \in R_5 = \{w \leq -M,\ x \leq -N\} \\
-\delta w + \delta M & (x, y, z, w) \in R_6 = \{|w| \leq M,\ x \leq -N\} \\
0 & (x, y, z, w) \in R_7 = \{0 \leq w \leq M,\ |x| \leq (N/M)w\} \\
\delta w - \delta(M/N)x & (x, y, z, w) \in R_8 = \{0 \leq x \leq N,\ |w| \leq (M/N)x\} \\
-2\delta(M/N)x & (x, y, z, w) \in R_9 = \{-M \leq w \leq 0,\ |x| \leq -(N/M)w\} \\
-\delta w - \delta(M/N)x & (x, y, z, w) \in R_{10} = \{-N \leq x \leq 0,\ |w| \leq -(M/N)x\}
\end{cases}
\tag{4.3}
$$

where $\delta > 0$ is an arbitrary positive constant and M and N are selected to satisfy Lemma 4.3 with $2M \leq N$. Since all the partial derivatives of V_1 are continuous, it satisfies a local LIPSCHITZ condition with respect to (x, y, z, w). V_2 is either linear or constant in each of the ten regions and V is continuous; hence V satisfies a local LIPSCHITZ condition. The partial derivatives $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$, $\frac{\partial V}{\partial z}$, and $\frac{\partial V}{\partial w}$ exist and are continuous for all values of x, y, z, w except along the planes:

$$S_1 : w = \pm M;\ S_2 : x = \pm N,\ w \leq M;\ S_3 : w = \pm (M/N)x,\ |x| < N.$$

All the upper and lower partial derivatives exist along these planes and thus $\dot{V}$ exists for all x, y, z, w, and t.

The proof of Theorem 2 will depend upon the following two lemmas.

LEMMA 4.2. - Under the hypotheses of Theorem 2 there exists a constant $H_1 > 0$ and functions $a(\tau)$ and $b(\tau)$ in CI such that $a(\tau) \to \infty$ as $\tau \to \infty$, $a(\tau) > 0$ for $\tau > H_1$ and

$$a(x^2 + y^2 + z^2 + w^2) \leq V(x, y, z, w) \leq b(x^2 + y^2 + z^2 + w^2) \tag{4.4}$$

for $x^2 + y^2 + z^2 + w^2 \geq H_1$.

PROOF. - Under the hypotheses of Theorem 2, it was established in Lemma 3 1 that there exist positive constants E_1, E_2, E_3, E_4 such that

$$V_1 \geq E_1 x^2 + E_2 y^2 + E_3 z^2 + E_4 w^2 \quad \text{for all} \quad x, y, z, w.$$

Since V_1 is continuous and independent of t and $V_1 \to \infty$ as $x^2 + y^2 + z^2 + w^2 \to \infty$, it follows that there exist functions $a_1(\tau), b_1(\tau) \in CI$ such that

$$a_1(x^2 + y^2 + z^2 + w^2) \leq V_1(x, y, z, w) \leq b_1(x^2 + y^2 + z^2 + w^2)$$

and $a_1(\tau) \to \infty$ as $\tau \to \infty$.

From the definition of V_2 we have

$$-2\delta M \leq V_2 \leq 2\delta M \quad \text{for all} \quad (x, y, z, w) \in R^4,$$

and hence if we define $a(\tau) = a_1(\tau) - 2\delta M$ and $b(\tau) = b_1(\tau) + 2\delta M$, then

$$a(x^2 + y^2 + z^2 + w^2) \leq V(x, y, z, w) \leq b(x^2 + y^2 + z^2 + w^2),$$

$a(\tau) \to \infty$ as $\tau \to \infty$, and there is an $H_1 > 0$ such that $a(\tau) > 0$ for $\tau \geq H_1 > 0$.

LEMMA 4.3. - Under the hypotheses of Theorem 2 there exists an $\alpha > 0$ such that if $P_1 \leq \alpha$, then there is an $H_2 > 0$ such that

$$\dot{V}_{(4.2)}(t, x, y, z, w) < -1 \tag{4.5}$$

for $x^2 + y^2 + z^2 + w^2 \geq H_2$ and $t \geq 0$.

PROOF. - The lemma will be established by considering each of the ten regions in the definition of V_2. $\dot{V}_2$ exists within each of the regions and will be estimated from both possible values at each boundary point as in [10].

For $(x, y, z, w) \in R_1 \cup R_3 \cup R_5$, we have that

$$\begin{aligned} \dot{V} = \dot{V}_1 = h'(x)y^2 &- d_2 y g(y) - d_1[A_4 - h'(x)]yz \\ &- \{[a_2 - d_1 g'(y)]z^2 - d_2 z F(z)\} \\ &- [d_1 f(z) - 1]w^2 \\ &+ (d_2 y + z + d_1 w)[p(t, x, y, z, w)]. \end{aligned} \tag{4.6}$$

From (3.24) and (2.5) and again letting $d_3 = \max(d_1, d_2, 1)$

$$\begin{aligned} \dot{V} &\leq -\eta(y^2 + z^2) - a_1 \varepsilon w^2 \\ &\quad + d_3(|y| + |z| + |w|)[P_0 + P_1(y^2 + z^2 + w^2)^{\frac{1}{2}}] \\ &\leq (-\eta + 3d_3P_1)y^2 + (-\eta + 3d_3P_1)z^2 + (-a_1\varepsilon + 3d_3P_1)w^2 \\ &\quad + d_3P_0(|y| + |z| + |w|) \\ &\leq -\frac{\eta}{2}y^2 + d_3P_0|y| - \frac{\eta}{2}z^2 + d_3P_0|z| - \frac{a_1\varepsilon}{2}w^2 + d_3P_0|w| \end{aligned} \tag{4.7}$$

provided

$$P_1 \leq \alpha = \min(\eta/6d_3,\ a_1\varepsilon/6d_3).$$

Let $D = \max\left(d_3P_0 + \delta,\ d_3P_0 + \delta a_1 a_2 + \delta P_1 + \frac{\delta}{2},\right.$

$$\left. d_3P_0 + \delta a_2 + \delta P_1,\ d_3P_0 + \delta\frac{a_2 a_3}{A_4} + \delta P_1\right).$$

It can be easily shown that there exist positive constants G_1, G_2, and G_3 such that

$$-\frac{\eta}{2}y^2 + D|y| \leq G_1 \quad \text{all} \quad y \tag{4.8}$$

(4.9) $$-\frac{\eta}{2}z^2+D|z|\leq G_2 \quad \text{all} \quad z$$

(4.10) $$-\frac{a_1\varepsilon}{2}w^2+D|w|\leq G_3 \quad \text{all} \quad w$$

so that we have

$$\dot{V}=\dot{V}_1\leq G_1+G \;-\frac{a_1\varepsilon}{2}w^2+D|w|.$$

Clearly an $M_1>0$ can be found such that

(4.11) $$G_1+G_2-\frac{a_1\varepsilon}{2}w^2+D|w|<-1 \quad \text{for} \quad |w|\geq M_1.$$

Hence if $M\geq M_1$, then

(4.12) $$\dot{V}<-1 \quad \text{for} \quad (x,\ y,\ z,\ w)\in R_1\cup R_3\cup R_5.$$

On R_4, $\dot{V}=\dot{V}_1-2\delta\frac{M}{N}y$

$$\leq-\frac{\eta}{2}y^2+(d_3P_0+\delta)|y|-\frac{\eta}{2}z^2+d_3P_0|z|$$

$$-\frac{a_1\varepsilon}{2}w^2+d_3P_0|w|$$

$$\leq G_1+G_2-\frac{a_1\varepsilon}{2}w^2+D|w|.$$

Thus from (4.11) and (4.12) we have that if $M\geq M_1$,

(4.13) $$\dot{V}<-1 \quad \text{for} \quad (x,\ y,\ z,\ w)\in R_1\cup R_3\cup R_4\cup R_5.$$

On R_2, $\dot{V}=\dot{V}_1-\delta wf(z)-\delta a_2z-\delta g(y)-\delta h(x)+\delta p(t,\ x,\ y,\ z,\ w)$ so

$$\dot{V}\leq-\frac{\eta}{2}y^2+(d_3P_0+\delta a_1a_2)|y|-\frac{\eta}{2}z^2+(d_3P_0+\delta a_2)|z|$$

$$-\frac{a_1\varepsilon}{2}w^2+\left(d_3P_0+\delta\frac{a_2a_3}{A_4}\right)|w|-\delta a_4x$$

$$+\delta[P_0+P_1(y^2+z^2+w^2)^{\frac{1}{2}}]$$

$$\leq-\frac{\eta}{2}y^2+D|y|-\frac{\eta}{2}z^2+D|z|-\frac{a_1\varepsilon}{2}w^2+D|w|+\delta P_0-\delta a_4x$$

$$\leq G_1+G_2+G_3+\delta P_0-\delta a_4x.$$

An N_1 can certainly be found such that

$$G_1 + G_2 + G_3 + \delta P_0 - \delta a_4 x < -1 \quad \text{for} \quad x \geq N_1$$

so if $N \geq N_1$, then

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z, w) \in R_2. \tag{4.14}$$

Similarly for $(x, y, z, w) \in R_6$,

$$\dot{V} = \dot{V}_1 + \delta w f(z) + \delta a_2 z + \delta g(y) + \delta h(x) - \delta p(t, x, y, z, w)$$
$$\leq G_1 + G_2 + G_3 + \delta P_0 + \delta a_4 x$$

and thus if $N \geq N_1$,

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z, w) \in R_2 \cup R_6. \tag{4.15}$$

On R_8, $\dot{V} = \dot{V}_1 - \delta w f(z) - \delta a_2 z - \delta g(y) - \delta h(x) + \delta p(t, x, y, z, w) - \delta \dfrac{M}{N} y$

$$\leq -\frac{\eta}{2} y^2 + \left(d_3 P_0 + \delta a_1 a_2 + \delta P_1 + \frac{\delta}{2}\right)|y|$$
$$-\frac{\eta}{2} z^2 + (d_3 P_0 + \delta a_2 + \delta P_1)|z|$$
$$-\frac{a_1 \varepsilon}{2} w^2 + \left(d_3 P_0 + \delta \frac{a_2 a_3}{A_4} + \delta P_1\right)|w| + \delta P_0 - \delta a_4 x$$
$$\leq -\frac{\eta}{2} y^2 + D|y| + G_2 + G_3 + \delta P_0.$$

It is possible to find a $K_1 > 0$ such that if $|y| \geq K_1$,

$$-\frac{\eta}{2} y^2 + D|y| + G_2 + G_3 + \delta P_0 < -1.$$

On R_8, it is also true that

$$\dot{V} \leq G_1 + G_3 + \delta P_0 - \frac{\eta}{2} z^2 + D|z|,$$

and we can also find an $L_1 > 0$ such that if $|z| \geq L_1$,

$$G_1 + G_3 + \delta P_0 - \frac{\eta}{2} z^2 + D|z| < -1.$$

Thus,

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z, w) \in R_8 \tag{4.16}$$

provided $|y| \geq K_1$ or $|z| \geq L_1$. In the same way it is possible to choose positive constants K_2 and L_2 such that

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z, w) \in R_7 \cup R_9 \cup R_{10} \tag{4.17}$$

provided $|y| \geq K_2$ or $|z| \geq L_2$.

Let $K = \max(K_1, K_2)$ and $L = \max(L_1, L_2)$. Then

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z, w) \in \bigcup_{j=7}^{10} R_j \tag{4.18}$$

provided $|y| \geq K$ or $|z| \geq L$.

Now define $H_2 = 4 \max (M, N, K, L, 1)$. If $x^2 + y^2 + z^2 + w^2 \geq H_2$, then one of the following must be true:

(i) $|w| \geq M$

(ii) $|w| < M$, $|x| \geq N$

(iii) $|w| < M$, $|x| < N$, $|y| \geq K$

(iv) $|w| < M$, $|x| < N$, $|y| < K$, $|z| \geq L$.

In each case $\dot{V} < -1$ from (4.13), (4.15), and (4.18), and thus the lemma has been established.

Let $H = \max (H_1, H_2)$. Then Lemmas 4.1, 4.2, and 4.3 show that there exists a $B_1 > 0$ such that if $(x(t), y(t), z(t), w(t))$ is any solution of (4.2), then

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) \leq B_1 .$$

From the relationship between (4.2) and (1.1) we have (2.6) and Theorem 2 is thus established.

5. - Proof of Theorem 3.

Consider the system of differential equations

$$\frac{dx}{dt} = f(x) + e(t), \tag{5.1}$$

where $f: R^n \to R^n$, $e: R^1 \to R^n$, and f and e are continuous for $x \in R^n$ and $t \in R^1$ respectively. Since we are concerned with the behavior of a pair of solutions it is useful to introduce the product system

$$\begin{cases} \dot{x} = f(x) + e(t) \\ \dot{y} = f(y) + e(t) \end{cases} \tag{5.2}$$

where $\dot{V}_{(5.2)}$ is defined as

$$\dot{V}_{(5.2)}(t,\ x,\ y) = \overline{\lim_{h \to 0+}}\ (1/h)\{V[x + hf(x) + he(t),\ y + hf(y) + he(t)] - V(x,\ y)\}.$$

Before proceeding with the proof of Theorem 3 we will need the following definition.

DEFINITION [12]. - If Ω is a closed subset of R^n, then we will say that a scalar function $W: U \times U \to R^1$ is positive definite with respect to Ω if $W(x,\ y) = 0$ when $(x,\ y) \in U \times U$ and $x - y \in \Omega$, and if corresponding to each $\varepsilon > 0$ and each compact set $K \subset U$ there is a positive number δ such that $W(x,\ y) \geq \delta$ when $(x,\ y) \in K \times K$ and $x - y \notin N(\varepsilon,\ \Omega)$. We define $N(\varepsilon,\ \Omega) = \{x \in R^n \mid d(x,\ \Omega) < \varepsilon\}$.

The proof of Theorem 3 will rely upon the following result established by SWICK [12].

LEMMA 5.1 [12]. - Assume that the solutions of equation (5.1) are ultimately bounded by $B > 0$ and that there exists a non-negative LIAPUNOV function $V(x,\ y)$ defined on $S_B \times S_B$ for which $-\dot{V}_{(5.2)}(t,\ x,\ y) \geq W(x,\ y)$, where $W(x,\ y)$ is positive definite with respect to a closed subset Ω of R^n. Suppose further that if $(x,\ y) \in S_B \times S_B$, then $f(x) - f(y) = f(x - y)$ whenever $x - y \in \Omega$. If S is the largest semi-invariant set contained in Ω of the equation

$$\dot{x} = f(x) \qquad x \in \Omega\ ,$$

then every pair of solutions $\theta(t)$ and $\varphi(t)$ of equation (5.1) satisfy

$$\theta(t) - \varphi(t) \to S \quad \text{as} \quad t \to \infty.$$

To apply Lemma 5.1, consider the equivalent system of (1.8)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= -a_1 w - a_2 z - g(y) - a_4 x + p(t) \end{aligned} \tag{5.3}$$

and the associated system

$$\dot{x} = y \qquad \dot{u} = v$$

$$\dot{y} = z \qquad \dot{v} = r$$

$$\dot{z} = w \qquad \dot{r} = s$$

$$(5.4) \quad \dot{w} = -a_1 w - a_2 z - g(y) - a_4 x + p(t) \qquad \dot{s} = -a_1 s - a_2 r - g(v) - a_4 u + p(t)$$

To prove Theorem 3 all that is necessary is to construct a suitable Liapunov function with $S = \{(0, 0, 0, 0)\}$.

Define $V = V(x, y, z, w, u, v, r, s)$ by

$$(5.5) \quad 2V = a_4 d_2 X^2 + (a_2 d_2 - a_4 d_1) Y^2 + \psi(y, v) Y^2 + (a_1 + a_2 d_1 - d_2) Z^2$$
$$+ d_1 W^2 + 2a_4 XY + 2a_4 d_1 XZ + 2a_1 d_2 YZ + 2d_1 \psi(y, v) YZ$$
$$+ 2d_2 YW + 2ZW,$$

where

$$X = x - u, \qquad Y = y - v, \qquad Z = z - r, \qquad W = w - s,$$

$$\psi(y, v) = \begin{cases} \dfrac{g(y) - g(v)}{y - v} & y \neq v \\ g'(y) & y = v \end{cases}$$

$$d_1 = \varepsilon + 1/a_1 \qquad d_2 = \varepsilon + a_4/a_3$$

and where $\varepsilon > 0$ is a constant which satisfies (iv) above. The Liapunov function is dependent on this $\varepsilon > 0$ which will be used to establish the following two lemmas.

Lemma 5.2. - Under the hypotheses of Theorem 3 there exist positive constants D_1, D_2, D_3, D_4 dependent only on $a_1, a_2, a_3, a_4, \Delta_0$, and ε such that

$$(5.6) \qquad V \geq D_1 X^2 + D_2 Y^2 + D_3 Z^2 + D_4 W^2$$

for $|x|, |u| \leq A$; $|y|, |v| \leq B$; $|z|, |r| \leq C$; $|w|, |s| \leq D$.

PROOF. - $2V$ can be rewritten as

$$\begin{aligned} 2V = & (a_4 d_2 - a_4^2/\psi)X^2 + (a_2 d_2 - a_4 d_1 - d_2^2 a_1)Y^2 \\ & + (a_2 d_1 - d_2 - d_1^2 \psi)Z^2 + (d_1 - 1/a_1)W^2 \\ & + (1/a_1)(W + a_1 Z + a_1 d_2 Y)^2 \\ & + (1/\psi)(a_4 X + \psi Y + d_1 \psi Z)^2 \\ \geq & (a_4 d_2 - a_4^2/\psi)X^2 + (d_1 - 1/a_1)W^2 + V_1 + V_2, \end{aligned}$$

where

$$V_1 = (a_2 d_2 - a_4 d_1 - d_2^2 a_1)Y^2$$

$$V_2 = (a_2 d_1 - d_2 - d_1^2 \psi)Z^2.$$

Paralleling the proof of Theorem 1, we use the inequalities

$$(5.7) \qquad (d_2 - a_4/\psi) \geq \varepsilon$$

$$(5.8) \qquad (a_2 - d_1\psi - a_1 d_2) \geq (\triangle_0/a_1 a_3) - D_0\varepsilon \qquad |y|,\ |v| \leq B$$

to obtain

$$V_1 \geq (a_4/a_3)(\triangle_0/a_1 a_3 - D_0\varepsilon)Y^2$$

$$V_2 \geq (1/a_1)(\triangle_0/a_1 a_3 - D_0\varepsilon)Z^2.$$

Combining results, we find

$$\begin{aligned} 2V \geq & a_4\, \varepsilon X^2 + (a_4/a_3)(\triangle_0/a_1 a_3 - D_0\varepsilon)Y^2 \\ & + (1/a_1)(\triangle_0/a_1 a_3 - D_0\varepsilon)Z^2 + \varepsilon W^2, \end{aligned}$$

and hence we have (5.6).

LEMMA 5.3. - Under the hypotheses of Theorem 3 there exist positive constants D_5, D_6, D_7 depending only on a_1, a_2, a_3, a_4, $\triangle_0$, and ε such that

$$(5.9) \qquad -\dot{V}_{(5.4)} \geq D_5 Y^2 + D_6 Z^2 + D_7 W^2$$

for $|x|,\ |u| \leq A$; $|y|,\ |v| \leq B$; $|z|,\ |r| \leq C$; $|w|,\ |s| \leq D$.

PROOF. - Defining $\Psi = \dfrac{\partial\psi}{\partial y}\dot{y} + \dfrac{\partial\psi}{\partial v}\dot{v}$ an easy calculation shows that

$$\begin{aligned} -\dot{V}_{(5.4)} = & (d_2\psi - a_4 - \Psi/2)Y^2 - d_1 \Psi YZ \\ & (a_2 - a_1 d_2 - d_1\psi)Z^2 + (a_1 d_1 - 1)W^2. \end{aligned}$$

We can rewrite the first three terms in the form

$$(d_2\psi - a_4 - \Psi/2)Y^2 - d_1\Psi YZ + (a_2 - a_1 d_2 - d_1\psi)Z^2 =$$

$$(Y,\ Z)\begin{pmatrix} d_2\psi - a_4 - \Psi/2 & -d_1\Psi/2 \\ -d_1\Psi/2 & a_2 - a_1 d_2 - d_1\psi \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix}.$$

$|\dot{y}| \leq C$, $|\dot{v}| \leq C$, and hypothesis (iii) imply that $|\Psi/2| \leq CA_3$ and hence $d_2\psi - a_4 - \Psi/2 \geq a_3\varepsilon - CA_3$. This plus (5.8) and hypothesis (iv), i.e.

$$(a_3\varepsilon - CA_3)(\Delta_0/a_1a_3 - D_0\varepsilon) > (\varepsilon + 1/a_1)^2 C^2 A_3^2,$$

insure that there exists a $\sigma > 0$ such that

$$-\dot{V}_{(5.4)} \geq \sigma(Y^2 + Z^2) + a_1\,\varepsilon\, W^2.$$

Thus the lemma is established and $-\dot{V}_{(5.4)}$ will be positive definite with respect to

$$\Omega = \{(x,\ y,\ z,\ w) \in R^4 \,|\, y = z = w = 0\}.$$

It follows now from Lemma 5.1 that if $x(t)$ and $y(t)$ are any two solutions of (5.3), then $x(t) - y(t) \to S$ as $t \to \infty$, where S is the largest semi-invariant set contained in Ω of the system

$$\dot{x} = 0$$
$$\dot{y} = 0$$
$$\dot{z} = 0$$
$$\dot{w} = -a_4 x.$$

The solutions of this system are

$$x = c_1$$
$$y = c_2$$
$$z = c_3$$
$$w = -a_4 c_1(t - t_0) + c_4.$$

To remain in Ω we must have $c_1 = c_2 = c_3 = c_4 = 0$. Thus $S = \{(0,\ 0,\ 0,\ 0)\}$ and the proof of Theorem 3 is complete due to the relation between (1.8) and (5.3).

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REFERENCES

[1] M. L. CARTWRIGHT, *On the Stability of Solutions of Certain Differential Equations of the Fourth Order*, Quart. J. Mech. and Applied Math., 9, (1966), pp. 185-194.

[2] J. O. C. EZEILO, *A Note on the Convergence of Solutions of Certain 2nd Order Differential Equations*, Portugal. Math. 24 (1965), pp. 49-58.

[3] — —, *A Stability Result for Solutions of a Certain Fourth Order Differential Equation*, Journal London Math. Soc., 37 (1962), pp. 28-32.

[4] — —, *On the Boundedness and the Stability of Solutions of Some Differential Equations of the Fourth Order*, Journal of Math. Analysis and Appl. 5 (1962), pp. 136-146.

[5] — —, *Stability Results for the Solutions of Some Third and Fourth Order Differential Equations*, Ann. Math. Pura Appl., 66 (1964), pp. 233-249.

[6] M. HARROW, *A Stability Result for Solutions of Certain Fourth Order Homogeneous Differential Equations*, Journal London Math. Soc., 42 (1967), pp. 51-56.

[7] — —, *Further Results on the Boundedness and the Stability of Solutions of Some Differential Equations of the Fourth Order*, SIAM Journal Math. Anal., 1 (1970), pp. 189-194.

[8] J. P. LA SALLE, *Stability Theory for Ordinary Differential Equations*, Journal of Diff. Equs., 4, pp. 57-65.

[9] R. REISSIG, G. SANSONE, R. CONTI, *Nichtlineare Differentialgleichungen Höherer Ordnung*, Consiglio Nazionale delle Ricerche, Monografie Matematiche, No. 16 (1969), pp. 462-502.

[10] K. E. SWICK, *A Boundedness Result for the Solutions of Certain Third Order Differential Equations*, Ann. Math. Pura Appl., (to appear).

[11] — —, *Asymptotic Behavior of the Solutions of Certain Third Order Differential Equations*, SIAM J. of Appl. Math., 19 (1970), pp. 96-102.

[12] — —, *Invariant Sets and Convergence of Solutions of Nonlinear Differential Differential Equations*, Journal of Diff. Equs., (to appear).

[13] — —, *On the Boundedness and the Stability of Solutions of Some Non-Autonomous Differential Equations of the Third Order*, Journal London Math. Soc., 44 (1966), pp. 347-359.

[14] H. O. TEJUMOLA, *Boundedness and Periodicity of Solutions of Certain Fourth Order Differential Equations*, Ann. Math. Pura Appl., 14, 80 (1968), pp. 177-196.

[15] T. YOSHIZAWA, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, 1966.