Analysis and Mathematical Physics (2021) 11:77
https://doi.org/10.1007/s13324-021-00506-3

On the behaviour of analytic representation of the generalized Pascal snail

S. Kanas[1] · **V. S. Masih**[2]

Received: 4 November 2020 / Revised: 20 January 2021 / Accepted: 12 February 2021 /
Published online: 9 March 2021

Abstract
We find an unifying approach to the analytic representation of the domain bounded by a generalized Pascal snail. Special cases as the Pascal snail, Both leminiscate, conchoid of the Sluze and a disc are included. The behaviour of functions related to generalized Pascal snail is studied.

Keywords Domain bounded by the generalized Pascal snail · Booth leminiscate · Conchoid of the Sluze · Pascal snail · Univalent functions · Applied mathematics · Starlike and convex functions

Mathematics Subject Classification 30C80 · 30C45

1 The analytic representation of the Pascal snail

For $-1 \leq \alpha \leq 1$, $-1 \leq \beta \leq 1$, $\alpha\beta \neq \pm 1$, and $0 \leq \gamma < 1$ let $\mathfrak{L}_{\alpha,\beta,\gamma}$ denote the complex valued mapping

$$\mathfrak{L}_{\alpha,\beta,\gamma}(z) = \frac{(2-2\gamma)z}{(1-\alpha z)(1-\beta z)} = \sum_{n=1}^{\infty} B_n z^n = \begin{cases} (2-2\gamma)\displaystyle\sum_{n=1}^{\infty}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)z^n, & \alpha \neq \beta; \\ (2-2\gamma)\displaystyle\sum_{n=1}^{\infty} n\alpha^{n-1}z^n, & \alpha = \beta, \end{cases} \tag{1.1}$$

✉ S. Kanas
skanas@ur.edu.pl

V. S. Masih
masihvali@gmail.com; v_soltani@pnu.ac.ir

1 University of Rzeszow, Al. Rejtana 16c, 35-959 Rzeszów, Poland

2 Department of Mathematics, Payame Noor University, Tehran, Iran

where $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We note that $\mathfrak{L}_{\alpha,\beta,\gamma}$ maps $\mathbb{D}$ onto a domain $\mathfrak{D}(\alpha, \beta, \gamma)$ whose boundary is a given by

$$\partial\mathfrak{D}(\alpha,\beta,\gamma) = \left\{ w = u + iv : \frac{\big(2(1-\gamma)u + (\alpha+\beta)(u^2+v^2)\big)^2}{(1+\alpha\beta)^2} \right. \\ \left. + \frac{4(1-\gamma)^2 v^2}{(1-\alpha\beta)^2} - (u^2+v^2)^2 = 0 \right\}.$$

Indeed, for $z = e^{i\theta}$, with $\theta \in [0, 2\pi)$, we obtain

$$\mathfrak{L}(z) := \frac{z}{(1-\alpha z)(1-\beta z)} = \frac{e^{i\theta}\left(1-\alpha e^{-i\theta}\right)\left(1-\beta e^{-i\theta}\right)}{\left|1-\alpha e^{i\theta}\right|^2 \left|1-\beta e^{i\theta}\right|^2} \\ = \frac{(1+\alpha\beta)\cos\theta - (\alpha+\beta) + i(1-\alpha\beta)\sin\theta}{(1+\alpha^2-2\alpha\cos\theta)\big(1+\beta^2-2\beta\cos\theta\big)}, \quad -1 \le \alpha, \beta \le 1. \tag{1.2}$$

Let $u = u(\theta) = \Re\left\{\mathfrak{L}\big(e^{i\theta}\big)\right\}$ and $v = v(\theta) = \Im\left\{\mathfrak{L}\big(e^{i\theta}\big)\right\}$. Then

$$u = \frac{(1+\alpha\beta)\cos\theta - (\alpha+\beta)}{(1+\alpha^2-2\alpha\cos\theta)\big(1+\beta^2-2\beta\cos\theta\big)}, \\ v = \frac{(1-\alpha\beta)\sin\theta}{(1+\alpha^2-2\alpha\cos\theta)\big(1+\beta^2-2\beta\cos\theta\big)}. \tag{1.3}$$

Hence, u, v satisfy the equation

$$\frac{\big(u + (\alpha+\beta)(u^2+v^2)\big)^2}{(1+\alpha\beta)^2} + \frac{v^2}{(1-\alpha\beta)^2} - (u^2+v^2)^2 = 0 \quad (\alpha\beta \neq \pm 1). \tag{1.4}$$

Therefore $\mathfrak{L}_{\alpha,\beta,\gamma}$ maps the unit circle onto a curve (cf. [4,5])

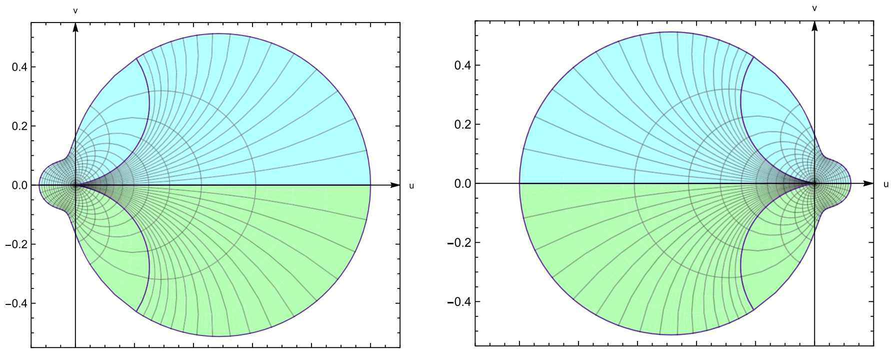
$$\frac{\big(2(1-\gamma)u + (\alpha+\beta)(u^2+v^2)\big)^2}{(1+\alpha\beta)^2} + \frac{4(1-\gamma)^2}{(1-\alpha\beta)^2}v^2 = (u^2+v^2)^2 \quad (\alpha\beta \neq \pm 1), \tag{1.5}$$

or

$$\left(u^2+v^2 - \frac{2(1-\gamma)(\alpha+\beta)}{(1-\alpha^2)(1-\beta^2)}u\right)^2 = \frac{4(1-\gamma)^2(1+\alpha\beta)^2}{(1-\alpha^2)^2(1-\beta^2)^2}u^2 \\ + \frac{4(1-\gamma)^2(1+\alpha\beta)^2}{(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)^2}v^2, \tag{1.6}$$

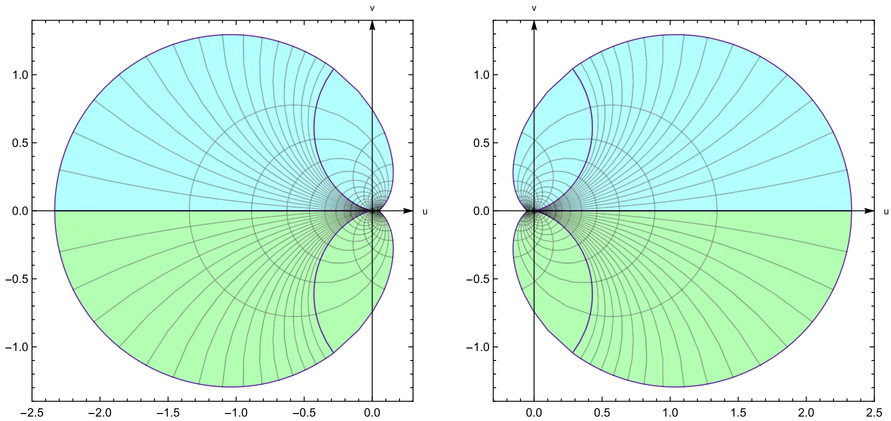
that is generalization of *the Pascal snail* (see Figs. 1 and 2).

The wide applications of the Pascal snail have been known since their description; the newest ones rely on the application to figure the path of airflow around object



(A) $\alpha = -0.4, \beta = 0.9, \gamma = 0.93$ **(B)** $\alpha = -0.9, \beta = 0.4, \gamma = 0.93$

Fig. 1 The image of $\mathbb{D}$ under $\mathfrak{L}_{\alpha,\beta,\gamma}(z)$



(A) $\alpha = -0.4, \beta = -0.9, \gamma = 0.93$ **(B)** $\alpha = 0.9, \beta = 0.4, \gamma = 0.93$

Fig. 2 The image of $\mathbb{D}$ under $\mathfrak{L}_{\alpha,\beta,\gamma}(z)$

like plane wings, in the design of race and train tracks but also in cryptography for selecting the points of the curve (ellipse, leminiscate, etc.) over the prime fields. Also, the leminiscates are used in the construction of grids on irregular regions in the development of software for numerically solving partial differential equations. Very recently a method based on leminiscates is applied for meander like regions and rely on covering the region with sectors bounded by two confocal leminiscate and two arcs orthogonal to the Pascal snail (cf. [7]).

In this paper we will deal with the Pascal snail (1.5) or (1.6) and its analytical representation. Also, we will discuss the special cases of (1.5) or (1.6) which give some interesting curves.

Let us consider individual cases separately. By a symmetry, from now on we make the assumption: $\beta \geq \alpha$, unless otherwise stated.

1.1 Circular domains

A circular domain is obtained from (1.5) in the case, when one of the parameter α or β is zero, and the second is in the interval $(-1, 1)$. Let $\alpha = 0 < \beta < 1$. Then $\mathfrak{L}_{0,\beta,\gamma}$ has the form

$$\mathfrak{L}_{0,\beta,\gamma}(z) = \frac{2(1-\gamma)z}{1-\beta z},$$

and $\mathfrak{L}_{0,\beta,\gamma}(\mathbb{D})$ is a circular domain

$$\mathfrak{D}(0, \beta, \gamma) = \left\{ w \in \mathbb{C} : \left| \frac{w}{2(1-\gamma)} - \frac{\beta}{1-\beta^2} \right| < \frac{1}{1-\beta^2} \right\}.$$

For the case $\alpha = \beta = 0$ a curve $\partial\mathfrak{D}(0, 0, \gamma)$ is a circle $|w| < 2(1-\gamma)$.

1.2 Halfplane

For the case when $\alpha = 0$ and $\beta = 1$ the domain $\mathfrak{L}(\mathbb{D})$ is the halfplane $\Re w > \gamma - 1$. The case $\beta = 0, \alpha = -1$ gives a halfplane $\Re w < 1 - \gamma$ which is not the ones of interest to us.

1.3 Pascal snail regions

In the case $\beta = \alpha \in (-1, 1)\backslash\{0\}$ the function $\mathfrak{L}_{\alpha,\alpha,\gamma}$ becomes

$$\mathfrak{L}_{\alpha,\alpha,\gamma}(z) = \frac{2(1-\gamma)z}{(1-\alpha z)^2}, \tag{1.7}$$

with $0 \le \gamma < 1$, that maps the unit disk onto simply connected and bounded region, which can be described as

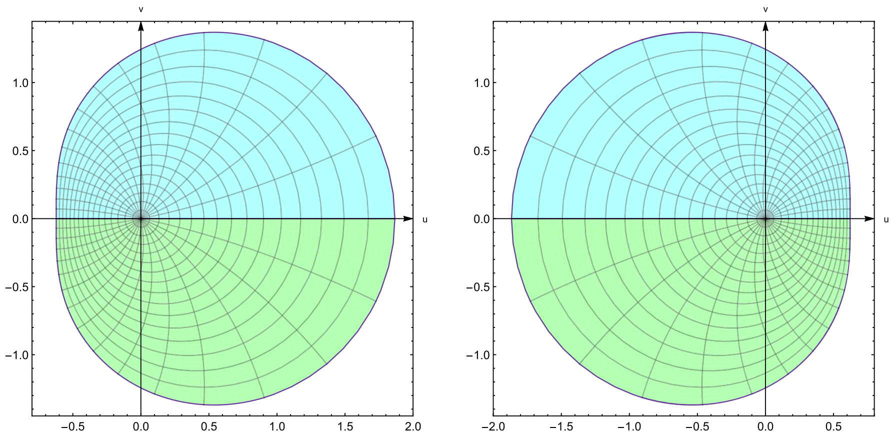
$$(u^2 + v^2 - eau)^2 = a^2(u^2 + v^2), \tag{1.8}$$

where

$$e = \frac{2\alpha}{1+\alpha^2}, \quad a = \frac{2(1-\gamma)(1+\alpha^2)}{(1-\alpha^2)^2}.$$

The equation (1.8) can be rewritten in a polar equation

$$\mathfrak{L}_{\alpha,\alpha,\gamma}(e^{it}) = \left\{ \rho e^{i\varphi} : \ \rho = \Theta(\alpha, \varphi), \ -\pi < \varphi \le \pi \right\}, \tag{1.9}$$



(A) $\alpha = 2 - \sqrt{3}, \gamma = 0.5$ **(B)** $\alpha = -2 + \sqrt{3}, \gamma = 0.5$

Fig. 3 The image of $\mathbb{D}$ under $\mathfrak{L}_{\alpha,\alpha,\gamma}(z)$

where

$$\Theta(\alpha, \varphi) = \frac{2(1-\gamma)|\alpha|}{\alpha} \frac{1 + \alpha^2 + 2|\alpha| \cos \varphi}{\left(1 - \alpha^2\right)^2}.$$

The boundary curve, known as Pascal snail (limaçon of Pascal), is a bicircular rational plane algebraic curve of degree 4 which belongs to the family of curves called centered trochoids or epitrochoids (cf. Fig. 3. Certainly $\mathfrak{L}_{0,0,\gamma}(\mathbb{D})$ is a disk). Pascal snail is the inversion of conic sections with respect to a focus.

We note that $|e| < 1$ for $\alpha \neq \pm 1$. In this case the snail is elliptic which is inverse of an ellipse with respect to its focus. In the case, when $|e| < 1/2$, that is $\alpha \in \left(-2 + \sqrt{3}, 2 - \sqrt{3}\right)$, the domain bounded by the Pascal snail (1.8) is convex, and tends to the circle when $\alpha \to 0$. For $|e| = 1/2$ the snail has a flattened segment of the boundary and when $|e| > 1/2$, that is for $\alpha \in \left(-1, -2 + \sqrt{3}\right) \cup \left(2 - \sqrt{3}, 1\right)$, the curve has a shape of a bean. The case when the Pascal snail has a loop does not hold, because it is equivalent to the inequality $(1-\alpha)^2 < 0$. Summarizing, the domain bounded by the Pascal snail is bounded, convex for $\alpha \in \left[-2 + \sqrt{3}, 2 - \sqrt{3}\right]$, concave for $\alpha \in \left(-1, -2 + \sqrt{3}\right) \cup \left(2 - \sqrt{3}, 1\right)$, and symmetric with respect to real axis.

The function $\mathfrak{L}_{\alpha,\alpha,\gamma}(z)$ with $\alpha \neq 0$ can also be written as a composition of two analytic univalent functions, that is,

$$\mathfrak{L}_{\alpha,\alpha,\gamma}(z) = (h_2 \circ h_1)(z) = \frac{1-\gamma}{2\alpha} \left[\left(\frac{1 + \alpha z}{1 - \alpha z} \right)^2 - 1 \right],$$

where

$$h_1(z) = \frac{1+\alpha z}{1-\alpha z} \quad \text{and} \quad h_2(z) = \frac{1-\gamma}{2\alpha}\left(z^2-1\right).$$

The function h_1 is univalent in $\mathbb{D}$ and h_2 is univalent in $h_1(\mathbb{D}) = \left\{w \in \mathbb{C}\colon \left|\frac{w-1}{w+1}\right| < |\alpha|\right\}$.

1.4 Conchoid of the Sluze

In the fourth special case we set $\beta = 1$, $\alpha \in (-1, 1)\backslash\{0\}$ or $\alpha = -1$, $\beta \in (-1, 1)\backslash\{0\}$. Let us consider $\beta = 1$, $\alpha \in (-1, 1)$. Thus $\mathfrak{L}_{\alpha,1,\gamma}$ has a form

$$\mathfrak{L}_{\alpha,1,\gamma}(z) = \frac{2(1-\gamma)z}{(1-\alpha z)(1-z)}, \tag{1.10}$$

where $0 \le \gamma < 1$, that maps the unit disk onto simply connected region with boundary that is a curve

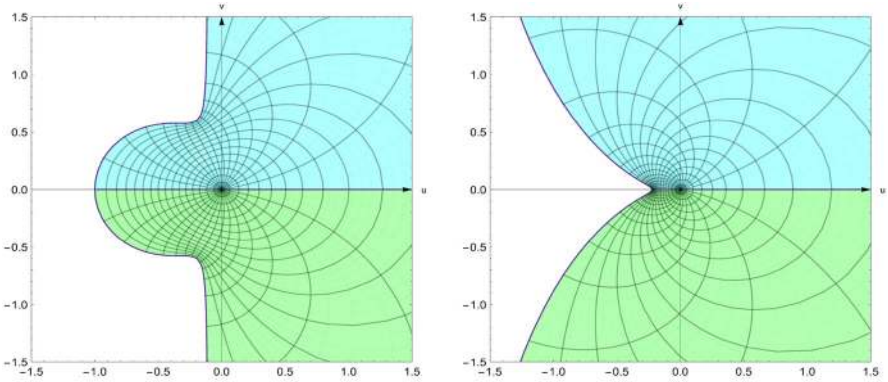
$$\begin{aligned} \partial\mathfrak{D}(\alpha, 1, \gamma) &= \left\{u + \mathrm{i}v\colon \left(u + \frac{(1+\alpha)(1-\gamma)}{(1-\alpha)^2}\right)(u^2+v^2) \right. \\ &\qquad \left. - \frac{4\alpha(1-\gamma)}{(1-\alpha)^2(1+\alpha)}u^2 = 0\right\} \\ &= \left\{u + iv\colon \frac{\left[2(1-\gamma)u + (1+\alpha)(u^2+v^2)\right]^2}{(1+\alpha)^2} + \frac{4(1-\gamma)^2v^2}{(1-\alpha)^2} \right. \\ &\qquad \left. = \left[u^2+v^2\right]^2\right\} \end{aligned}$$

known as the Conchoid of de Sluze, see Fig. 4. We note that the special case $\beta = 1$ and $-1 < \alpha < 0$ was also considered in [10].

In the case when $\alpha = -1$, $\beta \in (-1, 1)\backslash\{0\}$ we obtain the conchoid of de Sluze (see Fig. 5) of the form

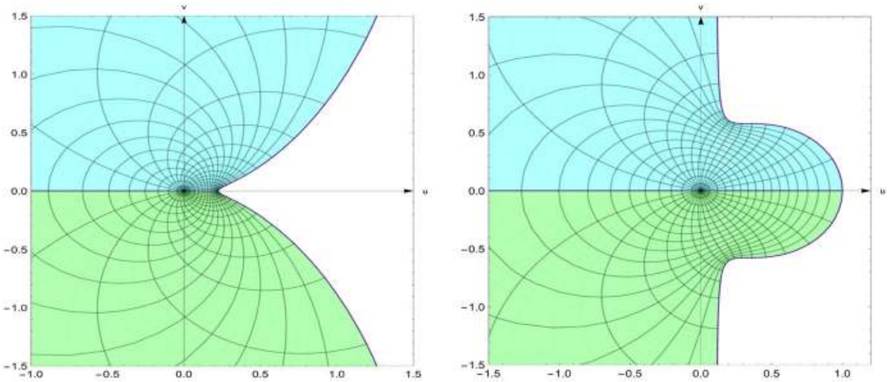
$$\begin{aligned} \partial\mathfrak{D}(-1, \beta, \gamma) &= \left\{u + \mathrm{i}v\colon \left(u - \frac{(1-\gamma)(1-\beta)}{(1+\beta)^2}\right)(u^2+v^2) \right. \\ &\qquad \left. - \frac{4\beta(1-\gamma)}{(1+\beta)^2(1-\beta)}u^2 = 0\right\} \\ &= \left\{u + iv\colon \frac{\left[2(1-\gamma)u - (1-\beta)(u^2+v^2)\right]^2}{(1-\beta)^2} \right. \\ &\qquad \left. + \frac{4(1-\gamma)^2v^2}{(1+\beta)^2} = \left[u^2+v^2\right]^2\right\}, \end{aligned}$$

symmetric to the $\partial\mathfrak{D}(\alpha, 1, \gamma)$ with respect to the imaginary axis.



(A) $\alpha = -0.5, \gamma = 0.5$ **(B)** $\alpha = 0.5, \gamma = 2/3$

Fig. 4 The image of $\mathbb{D}$ under $\mathfrak{L}_{\alpha,1,\gamma}$



(A) $\beta = -0.5, \gamma = 2/3$ **(B)** $\beta = \gamma = 0.5$

Fig. 5 The image of $\mathbb{D}$ under $\mathfrak{L}_{-1,\beta,\gamma}$

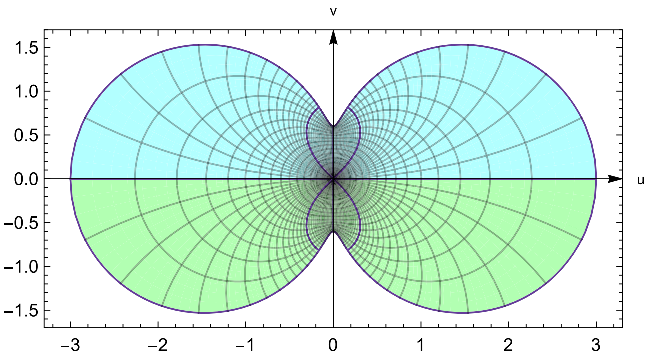
1.5 Hippopede. Leminiscate of Booth

Here we let $\beta = -\alpha$, $\alpha \in (-1, 0)$. In this case $\mathfrak{L}_{\alpha,-\alpha,\gamma}$ is of the form

$$\mathfrak{L}_{\alpha,-\alpha,\gamma}(z) = \frac{2(1-\gamma)z}{1-\alpha^2 z^2}, \tag{1.11}$$

and the equation (1.5) or (1.6) reduces to $(u^2+v^2)^2 = c^2u^2 + d^2v^2$, with $c = 2(1-\gamma)/(1-\alpha^2)$, $d = 2(1-\gamma)/(1+\alpha^2)$, that is

$$\partial\mathfrak{D}(\alpha,-\alpha,\gamma) = \left\{u+iv\colon \frac{4(1-\gamma)^2}{(1-\alpha^2)^2}u^2 + \frac{4(1-\gamma)^2}{(1+\alpha^2)^2}v^2 = \left(u^2+v^2\right)^2\right\}. \tag{1.12}$$



(A) $\alpha = -\sqrt{2/3}, \gamma = 0.5$

Fig. 6 The image of $\mathbb{D}$ under $\mathfrak{L}_{\alpha,-\alpha,\gamma}(z)$

We remind that the hippopede is the bicircular rational algebraic curve of degree 4, symmetric with respect to both axes. Any hippopede is the intersection of a torus with one of its tangent planes that is parallel to its axis of rotational symmetry. When $c > d > 0$ (that is $\alpha \neq 0$) such a curve is known as an oval or leminiscate of Booth, see Fig. 6. Since the case $d = -c$ does not hold, the leminiscate (1.12) do not reduce to the leminiscate of Bernoulli.

We note that for $c/\sqrt{2} < d < c\sqrt{2}$, the domain bounded by the hippopede is convex, that is for $-\sqrt{3-2\sqrt{2}} < \alpha < 0$, and the curve is called Booth's oval. For $d = c/\sqrt{2}$, that is for $\alpha = -\sqrt{3-2\sqrt{2}}$ the hippopede has a flattened segment of the boundary. Summarizing, the domain bounded by the hippopede is bounded and convex for $\alpha \in \left[-\sqrt{3-2\sqrt{2}}, 0\right)$, and concave for $\alpha \in \left(-1, -\sqrt{3-2\sqrt{2}}\right)$.

1.6 Remaining cases

In the remaining range of parameters, i.e. $-1 < \alpha < \beta < 1$, not considered in previous subsections, the curve $\mathfrak{L}_{\alpha,\beta,\gamma}(e^{it})$ is the generalized Pascal snail, that has the form

$$(u^2 + v^2 - au)^2 = c^2u^2 + d^2v^2 \tag{1.13}$$

with

$$a = \frac{2(1-\gamma)(\alpha+\beta)}{(1-\alpha^2)(1-\beta^2)}, \; c = \frac{2(1-\gamma)(1+\alpha\beta)}{(1-\alpha^2)(1-\beta^2)},$$
$$d = \frac{2(1-\gamma)(1+\alpha\beta)}{(1-\alpha\beta)\sqrt{(1-\alpha^2)(1-\beta^2)}}. \tag{1.14}$$

We note, that the curve given by (1.13) has similar properties to the Pascal snail, but symmetric only with respect to real axis. It has either horizontal eight-like shape,

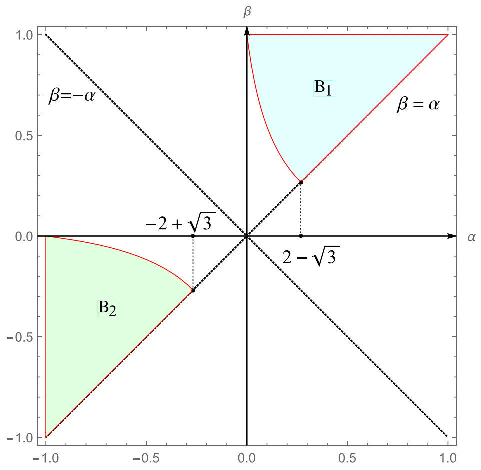


Fig. 7 The range of the parameters α, β

bean-shape, pear-shape or is convex. From this reason the region bounded by (1.13) is convex, or concave. As we can see in the Theorem 1.1 the minimum and maximum of real part are not always achieved on the real axis. Taking into account the geometrical properties of set $\mathfrak{L}(\mathbb{D})$, we get the following.

Theorem 1.1 [5] *Let* $-1 \le \alpha \le \beta \le 1$ *and* $\alpha\beta \ne \pm 1$. *Then*

$$\max_{0\le\theta<2\pi} \Re\,\mathfrak{L}(e^{i\theta}) = \begin{cases} \frac{(1+\alpha\beta)^2}{2(1-\alpha\beta)[2\sqrt{\alpha\beta(1-\alpha^2)(1-\beta^2)}-(\alpha+\beta)(1-\alpha\beta)]} & \text{for } (\alpha,\beta)\in B_2, \\ \frac{1}{(1-\alpha)(1-\beta)} & \text{otherwise}, \end{cases}$$

$$\min_{0\le\theta<2\pi} \Re\,\mathfrak{L}(e^{i\theta}) = \begin{cases} \frac{-(1+\alpha\beta)^2}{2(1-\alpha\beta)[2\sqrt{\alpha\beta(1-\alpha^2)(1-\beta^2)}+(\alpha+\beta)(1-\alpha\beta)]} & \text{for } (\alpha,\beta)\in B_1, \\ \frac{-1}{(1+\alpha)(1+\beta)} & \text{otherwise}, \end{cases}$$

where

$$B_1 = \{0 < \alpha < 1,\ \beta_1(\alpha) < \beta < 1\}, \quad B_2 = \{-1 < \alpha < 0,\ \alpha < \beta < \beta_2(\alpha)\},$$

with

$$\beta_1(\alpha) = \frac{(1+\alpha)\sqrt{\alpha^2+14\alpha+1}-(\alpha^2+6\alpha+1)}{2\alpha(1-\alpha)},$$

$$\beta_2(\alpha) = \frac{(1-\alpha)\sqrt{\alpha^2-14\alpha+1}-\alpha^2+6\alpha-1}{2\alpha(1+\alpha)}.$$

The sets B_1, B_2 *are represented on a Fig.* 7.

In the special cases we have

$$\max_{0\le\theta<2\pi} \Re\,\mathfrak{L}(e^{i\theta}) = \begin{cases} -\frac{1}{8\alpha}\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^2 & \textit{for } -1<\alpha\le\sqrt{3}-2,\ \beta=\alpha,\\ \frac{1}{(1-\alpha)^2} & \textit{for } \sqrt{3}-2\le\alpha<1,\ \beta=\alpha,\\ -\frac{1+\alpha}{2(1-\alpha)^2} & \textit{for } -1<\alpha\le 0,\ \beta=1,\\ -\frac{1}{2(1+\alpha)} & \textit{for } 0\le\alpha<1,\ \beta=1,\\ \frac{1}{1-\alpha^2} & \textit{for } -1<\alpha<0,\ \beta=-\alpha. \end{cases}$$

$$\min_{0\le\theta<2\pi} \Re\,\mathfrak{L}(e^{i\theta}) = \begin{cases} -\frac{1}{8\alpha}\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^2 & \textit{for } 2-\sqrt{3}\le\alpha<1,\ \beta=\alpha,\\ -\frac{1}{(1+\alpha)^2} & \textit{for } -1<\alpha\le\sqrt{3}-2,\ \beta=\alpha,\\ -\frac{1}{2(1+\alpha)} & \textit{for } -1<\alpha\le 0,\ \beta=1,\\ -\frac{1-\alpha}{2(1+\alpha)^2} & \textit{for } 0\le\alpha<1,\ \beta=1,\\ -\frac{1}{1-\alpha^2} & \textit{for } -1<\alpha<0,\ \beta=-\alpha. \end{cases}$$

1.7 Conclusions

In general $\mathfrak{L}(\mathbb{D})$ is a domain symmetric about the real axis and starlike with respect to origin and such that $\mathfrak{L}(0)=0$, $\mathfrak{L}'(0)=1>0$. The geometrical properties of the regions $\mathfrak{L}(\mathbb{D})$ provides a natural bridge between the convex and concave domains. We also note that such domains were discussed in relation of generalized typically-real functions and generalized Chebyshev polynomials of the second kind [4,5].

From Theorem 1.1 we conclude the following Corollary.

Corollary 1.2 *Let $-1\le\alpha\le\beta\le 1$, $\alpha\beta\ne\pm1$ and $0\le\gamma<1$, and let $\mathfrak{L}_{\alpha,\beta,\gamma}$ be the function defined by (1.1). Then, for $z\in\mathbb{D}$ we have*

$$\Re\left\{\mathfrak{L}_{\alpha,\beta,\gamma}(z)\right\} > \mathfrak{L}_0(\alpha,\beta,\gamma) = \begin{cases} \frac{-(1+\alpha\beta)^2(1-\gamma)}{(1-\alpha\beta)[2\sqrt{\alpha\beta(1-\alpha^2)(1-\beta^2)}+(\alpha+\beta)(1-\alpha\beta)]} & \textit{for } (\alpha,\beta)\in B_1,\\ \frac{-2(1-\gamma)}{(1+\alpha)(1+\beta)} & \textit{otherwise}, \end{cases}$$

and

$$\Re\left\{\mathfrak{L}_{\alpha,\beta,\gamma}(z)\right\} < \mathfrak{M}_0(\alpha,\beta,\gamma) = \begin{cases} \frac{(1+\alpha\beta)^2(1-\gamma)}{(1-\alpha\beta)[2\sqrt{\alpha\beta(1-\alpha^2)(1-\beta^2)}-(\alpha+\beta)(1-\alpha\beta)]} & \textit{for } (\alpha,\beta)\in B_2,\\ \frac{2(1-\gamma)}{(1-\alpha)(1-\beta)} & \textit{otherwise}, \end{cases}$$

Also, if $\alpha = 0$*, then*

$$\left| \mathfrak{L}_{0,\beta,\gamma}(z) - \frac{2(1-\gamma)\beta}{1-\beta^2} \right| \leq \frac{2(1-\gamma)}{1-\beta^2}.$$

In the sequel we will use the following lemma.

Lemma 1.3 [1] *Let z is a complex number with positive real part. Then for any real number t such that $t \in [0, 1]$, we have* $\Re \left\{ z^t \right\} \geq (\Re z)^t$.

2 Subclass of the Caratheodory class related to the generalized Pascal snail

Denote by $\mathcal{P}$ the Carathèodory class of functions i.e. $\mathcal{P} = \{p : p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \ \Re p(z) > 0 \ (z \in \mathbb{D})\}$. The fundamental importance of $\mathcal{P}$ in geometric functions theory relies on the construction of several related families of analytic functions and is well known. Hence, various subclasses of $\mathcal{P}$ were defined and studied. Classical cases are related to the halfplane and angular domain i.e. $\mathcal{P}(\alpha)$ that denotes a subclass of $\mathcal{P}$ consisting of functions with real part greater than α $(0 \leq \alpha < 1)$, and $\mathcal{P}_\gamma$ the class with argument between $-\gamma\pi/2$ and $\gamma\pi/2$ $(0 < \gamma \leq 1)$. Also, several subfamilies of $\mathcal{P}$ were determined by the fact that some functionals are contained in convex subdomains of right halfplane. Therefore any subfamily of halfplane domains were considered in the context to a subfamily of $\mathcal{P}$. Hence a definition of the domains related to the Pascal snail was a motivation to the definition of some subclass of $\mathcal{P}$ associated with such domains. To do this we first translate a domain $\mathfrak{L}_{\alpha,\beta,\gamma}(\mathbb{D})$ with a vector $(1, 0)$ in order to obtain a domain $\mathcal{D}_{\alpha,\beta,\gamma}$ contained in a right halfplane such that $1 \in \mathcal{D}_{\alpha,\beta,\gamma}$. The boundary of the domain $\mathcal{D}_{\alpha,\beta,\gamma}$ is then described as follows:

$$\partial\mathcal{D}_{\alpha,\beta,\gamma} = \left\{ u + iv : \frac{\big((2-2\gamma)(u-1) + (\alpha+\beta)((u-1)^2+v^2)\big)^2}{(1+\alpha\beta)^2} + \frac{4(1-\gamma)^2 v^2}{(1-\alpha\beta)^2} - ((u-1)^2+v^2)^2 = 0 \right\}.$$

We note that $\mathcal{D}_{\alpha,\beta,\gamma}$ is contained in a halfplane $\Re w > 1 + \mathfrak{L}_0$, where $\mathfrak{L}_0$ is given in Corollary 1.2. Anyway, there is substantial difference between $\mathcal{D}_{\alpha,\beta,\gamma}$ and a halfplane because $\mathcal{D}_{\alpha,\beta,\gamma}$ is not always a convex domain. However, when $\alpha = 0$ and $\beta \to 1^-$ then $\mathcal{D}_{\alpha,\beta,\gamma}$ tends to a halfplane $\Re w > \gamma - 1$. Thus $\mathcal{D}_{\alpha,\beta,\gamma}$ provides a natural bridge between the convex and the concave domains.

Now, we define a function $\mathcal{T}_{\alpha,\beta,\gamma}$ as

$$\mathcal{T}_{\alpha,\beta,\gamma}(z) = 1 + \mathfrak{L}_{\alpha,\beta,\gamma}(z), \tag{2.1}$$

that map $\mathbb{D}$ univalently onto a domain $\mathcal{D}_{\alpha,\beta,\gamma}$. Rewriting Corollary 1.2 for the function $\mathcal{T}_{\alpha,\beta,\gamma}$ we conclude the following theorem.

Theorem 2.1 *Let* $-1 < \alpha \le \beta < 1$, $0 \le \gamma < 1$, *and let* $\mathcal{T}_{\alpha,\beta,\gamma}(\cdot)$ *be defined by* (2.1). *Then*

$$\Re\left\{\mathcal{T}_{\alpha,\beta,\gamma}(z)\right\} > 1 + \mathfrak{L}_0(\alpha,\beta,\gamma)$$

and

$$\Re\left\{\mathcal{T}_{\alpha,\beta,\gamma}(z)\right\} < 1 + \mathfrak{M}_0(\alpha,\beta,\gamma),$$

where $\mathfrak{L}_0(\alpha,\beta,\gamma)$ *and* $\mathfrak{M}_0(\alpha,\beta,\gamma)$ *are given in Corollary* 1.2.

Now, we are ready to construct a class $\mathcal{P}_{snail}(\alpha,\beta,\gamma)$ as follows

$$\mathcal{P}_{snail}(\alpha,\beta,\gamma) = \{p \in \mathcal{P}:\ p(\mathbb{D}) \subset \mathcal{D}_{\alpha,\beta,\gamma}\} = \{p \in \mathcal{P}:\ p \prec \mathcal{T}_{\alpha,\beta,\gamma}\}.$$

3 The classes $\mathcal{ST}_{snail}(\alpha,\beta,\gamma)$, $\mathcal{CV}_{snail}(\alpha,\beta,\gamma)$ and their properties

In this Section we give a concise presentation of some families of analytic functions related to the generalized Pascal snail $\mathcal{T}_{\alpha,\beta,\gamma}$. We will study some subclasses of $\mathcal{S}$ with functions analytic and *univalent* in $\mathbb{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{3.1}$$

We also recall a class $\mathcal{ST}(\beta) \subset \mathcal{S}$, called *starlike functions of order* $0 \le \beta < 1$, that consist of functions f satisfying a condition

$$\Re\left\{zf'(z)/f(z)\right\} > \beta \quad (z \in \mathbb{D})$$

and a class $\mathcal{CV}(\beta)$, called *convex functions of order* $0 \le \beta < 1$, with analytic condition

$$\Re\left\{1 + zf''(z)/f'(z)\right\} > \beta \quad (z \in \mathbb{D}).$$

Let f and g be analytic in $\mathbb{D}$. Then the function f is said to *subordinate* to g in $\mathbb{D}$ written by $f(z) \prec g(z)$, if there exists a self-map of the unit disk ω, analytic in $\mathbb{D}$ with $\omega(0) = 0$ and such that $f(z) = g(\omega(z))$. If g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Also, let $\mathcal{ST}[\beta]$ be the subclass of $\mathcal{ST}$ defined by

$$\mathcal{ST}[\beta] := \left\{ f \in \mathcal{A}:\ \frac{zf'(z)}{f(z)} \prec \frac{1}{1-\beta z} \right\}$$

where $-1 \le \beta \le 1$, $\beta \ne 0$. Notice that for $\beta = \pm 1$ the function $w = 1/(1-\beta z)$ maps the unit disc $\mathbb{D}$ onto the half-plane $\Re w > 1/2$, and for $-1 < \beta < 1$ the function $w = 1/(1-\beta z)$ maps the unit disc $\mathbb{D}$ onto the disc $D(C(\beta), R(\beta))$ with the center $C(\beta) = 1/(1-\beta^2)$ and the radius $R(\beta) = |\beta|/(1-\beta^2)$.

Lemma 3.1 *Let* $-1 < \alpha \le \beta < 1$, $0 \le \gamma < 1$, *and* $\mathfrak{L}_{\alpha,\beta,\gamma}$ *be defined by* (1.1). *Then* $\mathfrak{L}_{\alpha,\beta,\gamma}$ *is starlike in* $\mathbb{D}$, *moreover*

$$\frac{\mathfrak{L}_{\alpha,\beta,\gamma}(z)}{2-2\gamma} \in \mathcal{ST}\left(\frac{1-|\alpha\beta|}{(1+|\alpha|)(1+|\beta|)}\right) \quad \textit{and} \quad \frac{\mathfrak{L}_{\alpha,\beta,\gamma}(z)}{2-2\gamma} \in \mathcal{CV}\left(t_0(\alpha,\beta)\right),$$

where

$$0 \le t_0(\alpha,\beta) = \begin{cases} \dfrac{1-|\alpha|}{1+|\alpha|} + \dfrac{1-|\beta|}{1+|\beta|} - \dfrac{1+\alpha\beta}{1-\alpha\beta} & \textit{for} \quad \alpha\beta \ge 0, \\ \dfrac{1-\alpha^2}{1+\alpha^2} + \dfrac{1-\beta^2}{1+\beta^2} - \dfrac{1-\alpha\beta}{1+\alpha\beta} & \textit{for} \quad \alpha\beta < 0. \end{cases} \tag{3.2}$$

Also, if $|z| = r < 1$, *then (see Figs. 1, 2 and 6)*

$$\max_{|z|=r} \left|\mathfrak{L}_{\alpha,\beta,\gamma}(z)\right| = \begin{cases} \mathfrak{L}_{\alpha,\beta,\gamma}(r) & \textit{for } \alpha\beta > 0 \textit{ with } \alpha+\beta > 0 \textit{ or } \alpha = 0, \\ -\mathfrak{L}_{\alpha,\beta,\gamma}(-r) & \textit{for } \alpha\beta > 0 \textit{ with } \alpha+\beta < 0 \textit{ or } \beta = 0, \\ \mathfrak{L}_{\alpha,\beta,\gamma}(r) & \textit{for } \alpha\beta < 0 \textit{ with } \alpha+\beta > 0, \\ -\mathfrak{L}_{\alpha,\beta,\gamma}(-r) & \textit{for } \alpha\beta < 0 \textit{ with } \alpha+\beta < 0, \\ \left|\mathfrak{L}_{\alpha,-\alpha,\gamma}(\pm r)\right| & \textit{for } \alpha+\beta = 0, \end{cases}$$

$$\min_{|z|=r} \left|\mathfrak{L}_{\alpha,\beta,\gamma}(z)\right| = \begin{cases} -\mathfrak{L}_{\alpha,\beta,\gamma}(-r) & \textit{for } \alpha\beta > 0 \textit{ with } \alpha+\beta > 0 \textit{ or } \alpha = 0, \\ \mathfrak{L}_{\alpha,\beta,\gamma}(r) & \textit{for } \alpha\beta > 0 \textit{ with } \alpha+\beta < 0 \textit{ or } \beta = 0, \\ \frac{4(1-\gamma)r\sqrt{|\alpha\beta|}}{(\beta-\alpha)(1-\alpha\beta r^2)} & \textit{for } \alpha\beta < 0 \textit{ with } \alpha+\beta > 0, \\ \frac{4(1-\gamma)r\sqrt{|\alpha\beta|}}{(\beta-\alpha)(1-\alpha\beta r^2)} & \textit{for } \alpha\beta < 0 \textit{ with } \alpha+\beta < 0, \\ \left|\mathfrak{L}_{\alpha,-\alpha,\gamma}(\pm ir)\right| & \textit{for } \alpha+\beta = 0. \end{cases}$$

Proof A straightforward calculation shows that $G := \mathfrak{L}_{\alpha,\beta,\gamma}$ satisfy

$$\Re\left\{\frac{zG'(z)}{G(z)}\right\} = 1 + \Re\left\{\frac{\alpha z}{1-\alpha z}\right\} + \Re\left\{\frac{\beta z}{1-\beta z}\right\} > 1 - \frac{|\alpha|}{1+|\alpha|} - \frac{|\beta|}{1+|\beta|},$$

from which the result concerning starlikeness follows.

In addition, we have

$$1 + \frac{zG''(z)}{G'(z)} = \frac{1+\alpha z}{1-\alpha z} + \frac{1+\beta z}{1-\beta z} - \frac{1+\alpha\beta z^2}{1-\alpha\beta z^2} \quad (z \in \mathbb{D}).$$

Thus for $\theta \in [0, 2\pi)$

$$\Re\left\{1 + \frac{e^{i\theta}G''(e^{i\theta})}{G'(e^{i\theta})}\right\} = \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos\theta} + \frac{1-\beta^2}{1+\beta^2-2\beta\cos\theta} - \frac{1-\alpha^2\beta^2}{1+\alpha^2\beta^2-2\alpha\beta\cos 2\theta}.$$

A convexity result yield from the estimating the value of the function $g(t)$ of the variable $t := \cos\theta$ of the form

$$g(t) := \frac{1-\alpha^2}{1+\alpha^2-2\alpha t} + \frac{1-\beta^2}{1+\beta^2-2\beta t} - \frac{1-\alpha^2\beta^2}{(1+\alpha\beta)^2-4\alpha\beta t^2},$$

where $-1 \le t \le 1$.

In order to prove the second part of lemma, define for $\theta \in [0, 2\pi)$ the function

$$\begin{aligned} Q(\theta) &:= \left|\mathfrak{L}_{\alpha,\beta,\gamma}\left(re^{i\theta}\right)\right|^2 \\ &= \frac{4(1-\gamma)^2r^2}{\left(1+\alpha^2r^2-2\alpha r\cos\theta\right)\left(1+\beta^2r^2-2\beta r\cos\theta\right)} \quad (0<r<1). \end{aligned}$$

We see that min or max of $Q(\theta)$ are attained at the critical points of the above function, equivalently

$$8(1-\gamma)^2r^3\sin\theta\left(4\alpha\beta r\cos\theta-(\alpha+\beta)(1+\alpha\beta r^2)\right)=0.$$

For $\alpha\beta = 0$ and $\alpha+\beta \neq 0$ the only ones critical points are $\theta = 0, \theta = \pi$. Next, let $\alpha\beta > 0$. Then, similarly, $Q'(\theta) = 0$ for $\theta = 0$ and $\theta = \pi$ since $|(\alpha+\beta)(1+\alpha\beta r^2)/4\alpha\beta| \le 1$ does not hold. If $\alpha+\beta > 0$, then for such θ we have

$$-\mathfrak{L}_{\alpha,\beta,\gamma}(-r) \le \left|\mathfrak{L}_{\alpha,\beta,\gamma}\left(re^{i\theta}\right)\right| \le \mathfrak{L}_{\alpha,\beta,\gamma}(r).$$

And, if $\alpha+\beta < 0$, then we obtain

$$\mathfrak{L}_{\alpha,\beta,\gamma}(r) \le \left|\mathfrak{L}_{\alpha,\beta,\gamma}\left(re^{i\theta}\right)\right| \le -\mathfrak{L}_{\alpha,\beta,\gamma}(-r).$$

For the case $\alpha\beta < 0$, the critical points are $\theta = 0, \theta = \pi$, and the solutions of the equation

$$4\alpha\beta r\cos\theta-(\alpha+\beta)(1+\alpha\beta r^2)=0. \tag{3.3}$$

We consider three separate cases, the first is $\alpha+\beta > 0$. Then for critical points $\theta = 0$ and the solutions of the equation (3.3) we have

$$\frac{4(1-\gamma)r\sqrt{|\alpha\beta|}}{(\beta-\alpha)(1-\alpha\beta r^2)} \le \left|\mathfrak{L}_{\alpha,\beta,\gamma}\left(re^{i\theta}\right)\right| \le \mathfrak{L}_{\alpha,\beta,\gamma}(r).$$

The second case is $\alpha+\beta < 0$. Then for the critical points $\theta = \pi$ and the solutions of the equation (3.3), we obtain

$$\frac{4(1-\gamma)r\sqrt{|\alpha\beta|}}{(\beta-\alpha)(1-\alpha\beta r^2)} \le \left|\mathfrak{L}_{\alpha,\beta,\gamma}\left(re^{i\theta}\right)\right| \le -\mathfrak{L}_{\alpha,\beta,\gamma}(-r).$$

Finally for the case $\alpha+\beta=0$, the critical points are $\theta=0, \theta=\pi/2, \theta=\pi, \theta=3\pi/2$ and $\theta=2\pi$. For such θ we conclude

$$\frac{2(1-\gamma)r}{1+\alpha^2r^2}=\left|\mathfrak{L}_{\alpha,-\alpha,\gamma}(\pm ir)\right|\leq\left|\mathfrak{L}_{\alpha,-\alpha,\gamma}\left(re^{i\theta}\right)\right|\leq\left|\mathfrak{L}_{\alpha,-\alpha,\gamma}(\pm r)\right|=\frac{2(1-\gamma)r}{1-\alpha^2r^2}.$$

□

Let $\alpha=\pm\beta$. From (3.2), the function $\mathfrak{L}_{\alpha,\alpha,\gamma}(z)/(2-2\gamma)$ is univalent in $\mathbb{D}$ if $t_0(\alpha,\alpha)=\frac{1+\alpha^2-4|\alpha|}{1-\alpha^2}\geq 0$ and this is equivalent to the range $-2+\sqrt{3}\leq\alpha\leq 2-\sqrt{3}$.

Now, we define a family of functions related to the Pascal snail $\mathcal{T}_{\alpha,\beta,\gamma}$ and present various relations of that family with the previously known classes.

Definition 3.2 For $-1<\alpha\leq\beta<1$, and $0\leq\gamma<1$ with $\gamma\geq\mathcal{T}_0(\alpha,\beta)$ and $\mathcal{T}_0(\alpha,\beta)$ defined by

$$\mathcal{T}_0(\alpha,\beta)=\begin{cases}1-\frac{(1-\alpha\beta)[2\sqrt{\alpha\beta(1-\alpha^2)(1-\beta^2)}+(\alpha+\beta)(1-\alpha\beta)]}{(1+\alpha\beta)^2} & for\ (\alpha,\beta)\in B_1,\\ 1-\frac{(1+\alpha)(1+\beta)}{2} & otherwise,\end{cases}\tag{3.4}$$

let $\mathcal{ST}_{snail}(\alpha,\beta,\gamma)$ denote the subfamily of $\mathcal{S}$ consisting of the functions f, satisfying the condition

$$zf'(z)/f(z)\prec\mathcal{T}_{\alpha,\beta,\gamma}(z)\quad(z\in\mathbb{D}),\tag{3.5}$$

and let $\mathcal{CV}_{snail}(\alpha,\beta,\gamma)$ be a class of analytic functions f such that

$$1+zf''(z)/f'(z)\prec\mathcal{T}_{\alpha,\beta,\gamma}(z)\quad(z\in\mathbb{D}),\tag{3.6}$$

where $\mathcal{T}_{\alpha,\beta,\gamma}$ is given by (2.1). Geometrically, the condition (3.5) and (3.6) means that the expression $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ lies in a domain bounded by the generalized Pascal snail $\mathcal{T}_{\alpha,\beta,\gamma}$ (Fig. 8) given by

$$\left[(u-1)^2+v^2-a(u-1)\right]^2=c^2(u-1)^2+d^2v^2,$$

where a, c and d given by (1.14).

By the properties of $\mathcal{T}_{\alpha,\beta,\gamma}$, given in Theorem 2.1, we have

$$\Re\left\{zf'(z)/f(z)\right\}>1+\mathfrak{L}_0\quad(z\in\mathbb{D}),\tag{3.7}$$

for $f\in\mathcal{ST}_{snail}(\alpha,\beta,\gamma)$, and for $f\in\mathcal{CV}_{snail}(\alpha,\beta,\gamma)$

$$\Re\left\{1+zf''(z)/f'(z)\right\}>1+\mathfrak{L}_0\quad(z\in\mathbb{D}),\tag{3.8}$$

where $\mathfrak{L}_0=\mathfrak{L}_0(\alpha,\beta,\gamma)$ is given in Corollary 1.2.

Additionally $\mathcal{CV}_{snail}(\alpha, \beta, \gamma) \subset \mathcal{G}$ for γ satisfying

$$\gamma \geq \gamma_0(\alpha, \beta) = \begin{cases} 1 - \frac{3(1-\alpha\beta)[2\sqrt{\alpha\beta(1-\alpha^2)(1-\beta^2)}+(\alpha+\beta)(1-\alpha\beta)]}{2(1+\alpha\beta)^2} & for\ (\alpha, \beta) \in B_1, \\ 1 - \frac{3(1+\alpha)(1+\beta)}{4} & otherwise, \end{cases} \tag{3.9}$$

where $\mathcal{G}$ is the family of function univalent, convex in one direction, and satisfying $\Re\{1 + zf''(z)/f'(z)\} > -1/2$, see [12].

Taking into account (3.2) the function $\mathfrak{L}_{\alpha,\alpha,\gamma}(z)/(2-2\gamma) \in \mathcal{G}$ for $|\alpha| \leq 4 - \sqrt{13}$, and $\mathfrak{L}_{\alpha,-\alpha,\gamma}(z)/(2-2\gamma) \in \mathcal{G}$ for $-\sqrt{6-\sqrt{33}} \leq \alpha < 0$, [8].

Summarizing, $\mathcal{T}_{\alpha,\beta,\gamma}$ is a analytic univalent function with positive real part in $\mathbb{D}$, $\mathcal{T}_{\alpha,\beta,\gamma}(\mathbb{D})$ is symmetric with respect to the real axis, starlike with respect to $\mathcal{T}_{\alpha,\beta,\gamma}(0) = 1$ and convex in one direction under some conditions on α and β. Moreover $\mathcal{T}'_{\alpha,\beta,\gamma}(0) = 2(1-\gamma) > 0$ hence $\mathcal{T}_{\alpha,\beta,\gamma}(\mathbb{D})$ satisfies Ma and Minda condition [9]. We refer to [2,3,6,11] for a detailed discussion about similar subclasses of related to functions mapping the unit disk onto domains contained in a right halfplane and starlike with respect to 1.

For $\beta = \alpha$ with $-1 < \alpha < 1$ and $\beta = -\alpha$ with $-1 < \alpha < 0$, the quantities $\mathcal{T}_0(\alpha, \beta)$ and $\gamma_0(\alpha, \beta)$ are the following

$$\mathcal{T}_0(\alpha, \alpha) = \begin{cases} 1 - 4\alpha\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^2 & for\ \ 2-\sqrt{3} \leq \alpha < 1, \\ \frac{1-2\alpha-\alpha^2}{2} & for\ \ -1 < \alpha \leq 2-\sqrt{3}, \end{cases} \qquad \mathcal{T}_0(\alpha, -\alpha) = \frac{1+\alpha^2}{2}$$

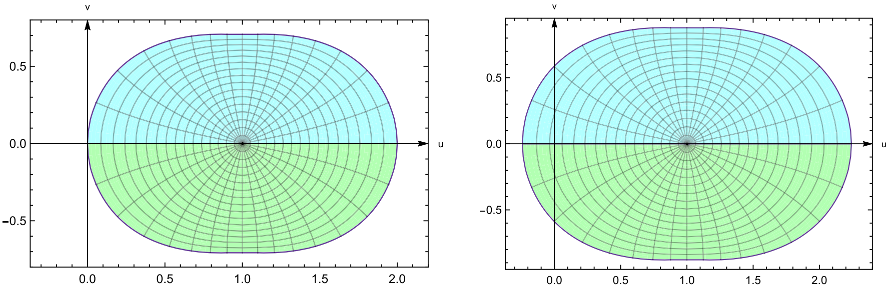
and

$$\gamma_0(\alpha, \alpha) = \begin{cases} 1 - 6\alpha\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^2 & for\ \ 2-\sqrt{3} \leq \alpha < 1, \\ \frac{1-6\alpha-3\alpha^2}{4} & for\ \ -1 < \alpha \leq 2-\sqrt{3}, \end{cases} \qquad \gamma_0(\alpha, -\alpha) = \frac{1+3\alpha^2}{4}.$$

For $\gamma = 1/2$, classes $\mathcal{ST}_{snail}(\alpha, \alpha, 1/2)$ and $\mathcal{CV}_{snail}(\alpha, \alpha, 1/2)$ are defined under the condition $0 \leq \alpha \leq \alpha_0$, where $\alpha_0 = 0.615331\ldots$ is a root of equation $8\alpha\left(1-\alpha^2\right)^2 = \left(1+\alpha^2\right)^2$. We also note that $\mathcal{ST}_{snail}(\alpha, -\alpha, 1/2)$ and $\mathcal{CV}_{snail}(\alpha, -\alpha, 1/2)$ of starlike and convex functions of Ma-Minda type [9], can not be defined, because it should satisfy $\gamma \geq \frac{1+\alpha^2}{2}$ that is $\alpha^2 \leq 0$ which is impossible.

Further properties of $\mathcal{T}_{\alpha,\beta,\gamma}$ yield:

$$f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma) \Longrightarrow \phi(z) := \int_0^z \left(\frac{f(t)}{t}\right)^{-\frac{1}{\mathfrak{L}_0}} dt \in \mathcal{CV}.$$



(A) $\alpha = 1-\sqrt{2}, \gamma = 2-\sqrt{2}$ $(\gamma \geq \mathcal{T}_0(\alpha, -\alpha))$

(B) $\alpha = 1-\sqrt{2}, \gamma = 1.9-\sqrt{2}$ $(\gamma < \mathcal{T}_0(\alpha, -\alpha))$

Fig. 8 Image of $\mathbb{D}$ under $\mathcal{T}_{\alpha,-\alpha,\gamma}(z)$

Indeed, by logarithmic differentiation of $\phi'(z) = \left(\frac{f(z)}{z}\right)^{-\frac{1}{\mathfrak{L}_0}}$ we obtain

$$1 + \frac{z\phi''(z)}{\phi'(z)} = 1 - \frac{1}{\mathfrak{L}_0}\left(\frac{zf'(z)}{f(z)} - 1\right) = 1 + \frac{1}{\mathfrak{L}_0} - \frac{1}{\mathfrak{L}_0}\frac{zf'(z)}{f(z)} \quad (z \in \mathbb{D}).$$

Since $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$, we conclude that

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} = 1 + \frac{1}{\mathfrak{L}_0} - \frac{1}{\mathfrak{L}_0}\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{D}).$$

The equivalence $g \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ if and only if $zg'(z)/g(z) \prec \mathcal{T}_{\alpha,\beta,\gamma}(z)$ allows to determine the structural formula for functions in $\mathcal{ST}_{snail}(\alpha, \beta, \gamma)$. A function g is in the class $\mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ if and only if there exists an analytic function $p \prec \mathcal{T}_{\alpha,\beta,\gamma}$, such that

$$g(z) = z\exp\left(\int_0^z \frac{p(t) - 1}{t}\,dt\right). \tag{3.10}$$

The above integral representation provides many examples of functions of the class $\mathcal{ST}_{snail}(\alpha, \beta, \gamma)$. Let $p(z) = \mathcal{T}_{\alpha,\beta,\gamma}(z^n) \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ for $n = 1, 2, \ldots$. Then, for $\alpha \neq \beta,\ n \geq 1$, the function

$$\begin{aligned}\Psi_{\alpha,\beta,\gamma,n}(z) &= z\exp\left(\int_0^z \frac{2(1-\gamma)t^{n-1}}{(1-\alpha t^n)(1-\beta t^n)}\,dt\right) = z\left(\frac{1-\beta z^n}{1-\alpha z^n}\right)^{\frac{2(1-\gamma)}{n(\alpha-\beta)}}\\ &= z + \frac{2(1-\gamma)}{n}z^{n+1} + \frac{(1-\gamma)[2(1-\gamma) + n(\alpha+\beta)]}{n^2}z^{2n+1} + \cdots,\end{aligned} \tag{3.11}$$

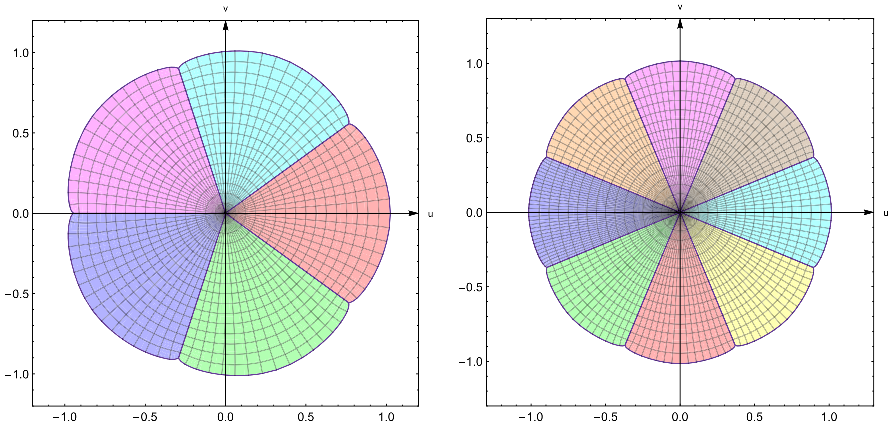


Fig. 9 $\Psi_{\alpha,\beta,\gamma,n}(\mathbb{D})$ for $\alpha = -0.9$, $\beta = 0.4$, $\gamma = 0.93$, $n = 5, 8$

is extremal for several problems in the class $\mathcal{ST}_{snail}(\alpha, \beta, \gamma)$. For $n = 1$ we have

$$\Psi_{\alpha,\beta,\gamma}(z) := \Psi_{\alpha,\beta,\gamma,1}(z) = z\left(\frac{1-\beta z}{1-\alpha z}\right)^{\frac{2(1-\gamma)}{\alpha-\beta}}, \tag{3.12}$$

and for $\alpha = \beta$

$$\Psi_{\alpha,\alpha,\gamma,n}(z) = z\exp\left(\frac{2(1-\gamma)z^n}{n(1-\alpha z^n)}\right) = z + \frac{2(1-\gamma)z^{n+1}}{n(1-\alpha z^n)} + \cdots \tag{3.13}$$

and

$$\Psi_{\alpha,\alpha,\gamma}(z) := \Psi_{\alpha,\alpha,\gamma,1}(z) = z\exp\left(\frac{2(1-\gamma)z}{1-\alpha z}\right). \tag{3.14}$$

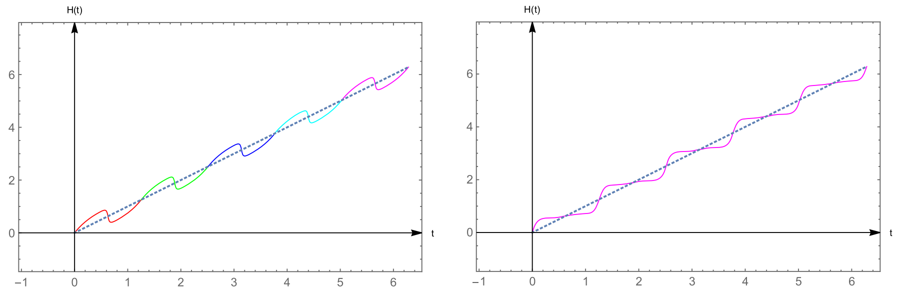
We note that $\Psi_{\alpha,\beta,\gamma,n}(\mathbb{D})$ is sunflower's domain (Fig. 9).

Indeed, for $\alpha \neq \beta$ let

$$G(t) = \left|\Psi_{\alpha,\beta,\gamma,n}(e^{it})\right| = \left(\frac{1+\beta^2-2\beta\cos nt}{1+\alpha^2-2\alpha\cos nt}\right)^p,$$

where $p = \frac{1-\gamma}{n(\alpha-\beta)}$ and $n \geq 2$. Since

$$G'(t) = \left(\frac{1+\beta^2-2\beta\cos nt}{1+\alpha^2-2\alpha\cos nt}\right)^{p-1} \frac{pn(\beta-\alpha)(1-\alpha\beta)\sin nt}{(1+\alpha^2-2\alpha\cos nt)^2},$$



(A) $\alpha = -0.9, \beta = 0.4, \gamma = 0.2, n = 5$ with $(\gamma < \mathcal{T}_0(\alpha, \beta))$

(B) $\alpha = 0.4, \beta = 0.5, \gamma = 0.3, n = 5$ with $(\gamma \geq \mathcal{T}_0(\alpha, \beta))$

Fig. 10 Graph of function $H(t)$

we see that the points of extreme of modulus occur at $t = \frac{k\pi}{n}$, where $k = 0, 1, 2, ..., 2(n-1)$. At these points $G(t)$ alternately attains its maximum and minimum, equal $\left(\frac{1+\beta}{1+\alpha}\right)^{2p}$ and $\left(\frac{1-\beta}{1-\alpha}\right)^{2p}$, respectively.

Additionally, the argument of Ψ i.e.

$$H(t) = \text{Arg}\, \Psi_{\alpha,\beta,\gamma,n}(e^{it}) = t - 2p\, \tan^{-1} \frac{(\beta-\alpha)\sin nt}{1+\alpha\beta-(\alpha+\beta)\cos nt},$$

is also alternately increasing and decreasing (see Fig. 10) as $t \in [0, 2\pi)$ and $\gamma > \mathcal{T}_0(\alpha, \beta)$, where $\mathcal{T}_0(\alpha, \beta)$ is defined by (3.4).

In the case $\alpha = \beta$ the function $G(t)$ has the form

$$G(t) = \exp \Re \left(\frac{2(1-\gamma)z^n}{n(1-\alpha z^n)}\right) = \exp\left(\frac{2(1-\gamma)(\cos nt - \alpha)}{n(1+\alpha^2 - 2\alpha \cos nt)}\right),$$

whose behavior is similar to the behavior of $G(t)$ for $\alpha \neq \beta$. The same situation holds for $H(t), \alpha = \beta$.

If $\gamma < \mathcal{T}_0(\alpha, \beta)$, the function $\Psi_{\alpha,\beta,\gamma,n}$ is not starlike in a whole unit disk as well as not univalent there (Fig. 11).

From Lemma 3.1 it can be seen that the smallest disk with center $(1, 0)$ that contains $\mathcal{T}_{\alpha,\beta,\gamma}(\mathbb{D})$ and the largest disk with center at $(1, 0)$ contained in $\mathcal{T}_{\alpha,\beta,\gamma}(\mathbb{D})$ (see Fig 12) are, below.

Proposition 3.3 *Let* $-1 < \alpha \leq \beta < 1,\ \alpha\beta \neq \pm 1$. *Then*

$$\mathcal{T}_{\alpha,\beta,\gamma}(\mathbb{D}) \supset \begin{cases} \left\{w \in \mathbb{C}\colon |w-1| < \frac{2(1-\gamma)}{(1+\alpha)(1+\beta)}\right\} & \textit{for } \ \alpha\beta > 0 \textit{ with } \alpha+\beta < 0 \textit{ or } \beta = 0, \\ \left\{w \in \mathbb{C}\colon |w-1| < \frac{2(1-\gamma)}{(1-\alpha)(1-\beta)}\right\} & \textit{for } \ \alpha\beta > 0 \textit{ with } \alpha+\beta > 0 \textit{ or } \alpha = 0, \\ \left\{w \in \mathbb{C}\colon |w-1| < \frac{4(1-\gamma)\sqrt{|\alpha\beta|}}{(\beta-\alpha)(1-\alpha\beta)}\right\} & \textit{for } \ \alpha\beta < 0 \textit{ with } \alpha+\beta \neq 0, \\ \left\{w \in \mathbb{C}\colon |w-1| < \frac{2(1-\gamma)}{1+\alpha^2}\right\} & \textit{for } \ \alpha+\beta = 0, \end{cases} \tag{3.15}$$

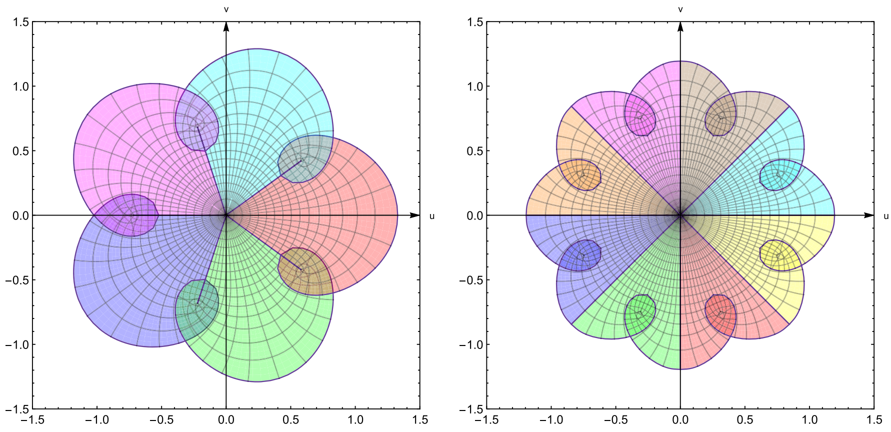
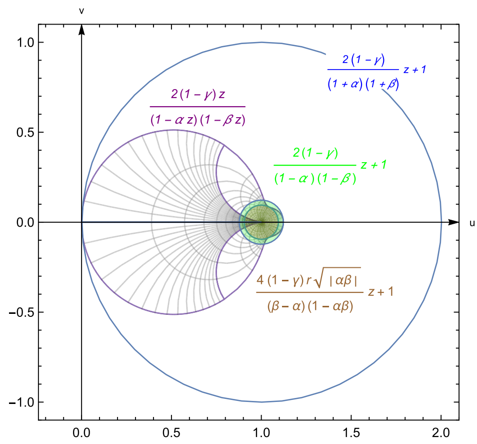


Fig. 11 $\Psi_{\alpha,\beta,\gamma,n}(\mathbb{D})$ for $\alpha = -0.9$, $\beta = 0.4$, $\gamma = 0.2$, $n = 5, 8$ with $\gamma < \mathcal{T}_0(\alpha, \beta)$

Fig. 12 The range of the functions $\mathfrak{L}_{\alpha,\beta,\gamma}$, $\frac{2(1-\gamma)}{(1+\alpha)(1+\beta)}z + 1$, $\frac{2(1-\gamma)}{(1-\alpha)(1-\beta)}z + 1$ and $\frac{4(1-\gamma)\sqrt{|\alpha\beta|}}{(\beta-\alpha)(1-\alpha\beta)}z + 1$ for $\alpha = -0.9$, $\gamma = 0.93$ and $\beta = 0.4$



$$\mathcal{T}_{\alpha,\beta,\gamma}(\mathbb{D}) \subset \begin{cases} \left\{w \in \mathbb{C}\colon |w-1| < \frac{2(1-\gamma)}{(1-\alpha)(1-\beta)}\right\} & \textit{for} \ \ \alpha\beta \neq 0 \textit{ with } \alpha+\beta > 0 \textit{ or } \alpha = 0, \\ \left\{w \in \mathbb{C}\colon |w-1| < \frac{2(1-\gamma)}{(1+\alpha)(1+\beta)}\right\} & \textit{for} \ \ \alpha\beta \neq 0 \textit{ with } \alpha+\beta < 0 \textit{ or } \beta = 0 \\ \left\{w \in \mathbb{C}\colon |w-1| < \frac{2(1-\gamma)}{1-\alpha^2}\right\} & \textit{for} \ \ \alpha+\beta = 0. \end{cases} \tag{3.16}$$

The function $\Psi_{\alpha,\beta,\gamma}$ given by (3.11)*, and* (3.14) *shows that the bounds are the best possible.*

Theorem 3.4 *Let $-1 < \alpha \leq \beta < 1$, and let f be analytic in $\mathbb{D}$. If $P_f = f/f' \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$, then*

$$\frac{zf'(z)}{f(z)} \prec \frac{z}{\Psi_{\alpha,\beta,\gamma}(z)} \quad (z \in \mathbb{D}).$$

Proof Let $p(z) = zf'(z)/f(z)$. Then $P_f(z) = z/p(z)$ and $zP'_f/P_f = 1 - zp'/p$. Since $P_f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$, we have

$$-\frac{zp'(z)}{p(z)} \prec \mathcal{T}_{\alpha,\beta,\gamma}(z) - 1 = \mathfrak{L}_{\alpha,\beta,\gamma}(z) \quad (z \in \mathbb{D}).$$

The function F defined by

$$F(z) = \int_0^z \frac{\mathfrak{L}_{\alpha,\beta,\gamma}(t)}{t}\, dt = \log\left(\frac{\Psi_{\alpha,\beta,\gamma}(z)}{z}\right)$$

where $\Psi_{\alpha,\beta,\gamma}$ given by (3.12), is analytic in $\mathbb{D}$, $F(0) = F'(0) - 1 = 0$ and

$$1 + \frac{zF''(z)}{F'(z)} = \frac{z\mathfrak{L}'_{\alpha,\beta,\gamma}(t)}{\mathfrak{L}_{\alpha,\beta,\gamma}(t)} \quad (z \in \mathbb{D}).$$

Taking into account Lemma 3.1, we deduce that the function F is convex in $\mathbb{D}$. Applying [14], we conclude that

$$-\log p(z) \prec \log\left(\frac{\Psi_{\alpha,\beta,\gamma}(z)}{z}\right) \quad or \quad \log p(z) \prec \log\left(\frac{z}{\Psi_{\alpha,\beta,\gamma}(z)}\right),$$

and by (3.12), the required result follows. □

Since for $z \in \mathbb{D}$ and $\alpha \neq \beta$

$$\Re\left\{\frac{z}{\Psi_{\alpha,\beta,\gamma}(z)}\right\} = \left|\frac{1-\alpha z}{1-\beta z}\right|^{-\frac{2(1-\gamma)}{\beta-\alpha}} \cos\left(\frac{2(1-\gamma)}{\beta-\alpha} \arg\frac{1-\alpha z}{1-\beta z}\right)$$

and $\arg\frac{1-\alpha z}{1-\beta z} \in (-\pi/2, \pi/2)$ the above and Theorem 3.4 leads to the following conclusion.

Corollary 3.5 *Let $f \in \mathcal{A}$ be a locally univalent function. If $P_f = f/f' \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ with $\alpha \neq \beta$ and $\gamma \geq 1 - \frac{\beta-\alpha}{2}$, then $f \in \mathcal{ST}$.*

Now we get a representation of functions in class $\mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ with the help of the class $\mathcal{ST}[\beta]$.

Lemma 3.6 *Let $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ with $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. Then there exists $h \in \mathcal{ST}[\beta]$, and $g \in \mathcal{ST}[\alpha]$ such that*

$$f(z) = z\left(\frac{h(z)}{g(z)}\right)^{\frac{2(1-\gamma)}{\beta-\alpha}} \quad (z \in \mathbb{D}).$$

Proof Let $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$. Then, by (3.10), there exists a self-map ω, which is analytic in $\mathbb{D}$, $\omega(0) = 0,\ |\omega(z)| < 1$, and such that

$$\begin{aligned} f(z) &= z \exp\left(\int_0^z \frac{\mathcal{T}_{\alpha,\beta,\gamma}(\omega(t)) - 1}{t}\, dt\right) \\ &= z \exp \int_0^z q \left[\frac{\beta\omega(t)}{t(1-\beta\omega(t))} - \frac{\alpha\omega(t)}{t(1-\alpha\omega(t))}\right] dt \\ &= z \left(\frac{z \exp \int_0^z \frac{\beta\omega(t)}{t[1-\beta\omega(t)]}\, dt}{z \exp \int_0^z \frac{\alpha\omega(t)}{t[1-\alpha\omega(t)]}\, dt}\right)^q \\ &= z \left(\frac{z \exp \int_0^z \frac{\frac{1}{1-\beta\omega(t)} - 1}{t}\, dt}{z \exp \int_0^z \frac{\frac{1}{1-\alpha\omega(t)} - 1}{t}\, dt}\right)^q = z\left(\frac{h(z)}{g(z)}\right)^q, \end{aligned}$$

where $q = \frac{2(1-\gamma)}{\beta-\alpha}$. The assertion now follows. □

From the relation $h \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ if and only if $1 + zh''(z)/h'(z) \prec \mathcal{T}_{\alpha,\beta,\gamma}(z)$ we obtain the structural formula for functions in $\mathcal{CV}_{snail}(\alpha, \beta, \gamma)$. A function h is in the class $\mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ if and only if there exists an analytic function p with $p \prec \mathcal{T}_{\alpha,\beta,\gamma}$, such that

$$h(z) = \int_0^z \exp\left(\int_0^w \frac{p(t) - 1}{t}\, dt\right) dw. \tag{3.17}$$

The above representation supply many examples of functions in class $\mathcal{CV}_{snail}(\alpha, \beta, \gamma)$. Let $p(z) = \mathcal{T}_{\alpha,\beta,\gamma}(z) \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$, then for some $n \geq 1$ and $\alpha \neq \beta$, the functions

$$\begin{aligned} K_{\alpha,\beta,\gamma,n}(z) &= \int_0^z \exp\left(\int_0^w \frac{2(1-\gamma)t^{n-1}}{(1-\alpha t^n)(1-\beta t^n)}\, dt\right) dw \\ &= \int_0^z \left(\frac{1-\alpha t^n}{1-\beta t^n}\right)^{\frac{2(1-\gamma)}{n(\alpha-\beta)}} dt, \end{aligned} \tag{3.18}$$

are extremal functions for several problems in the class $\mathcal{CV}_{snail}(\alpha, \beta, \gamma)$. For $n = 1$ we have

$$K_{\alpha,\beta,\gamma}(z) := K_{\alpha,\beta,\gamma,1}(z) = \int_0^z \left(\frac{1-\alpha t}{1-\beta t}\right)^{\frac{2(1-\gamma)}{\alpha-\beta}} dt. \tag{3.19}$$

and for $\alpha = \beta$

$$K_{\alpha,\alpha,\gamma,n}(z) = \int_0^z \exp\left(\frac{2(1-\gamma)t^n}{n(1-\alpha t^n)}\right) dt \quad \textit{and} \quad K_{\alpha,\alpha,\gamma}(z) := K_{\alpha,\alpha,\gamma,1}(z). \tag{3.20}$$

Now we get a representation of functions in class $\mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ with the help of class $\mathcal{ST}[\beta]$. From Lemma 3.6, we conclude the following Corollary.

Corollary 3.7 *Let $f \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ with $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. Then there exists $h \in \mathcal{ST}[\beta]$ and $g \in \mathcal{ST}[\alpha]$ such that*

$$f'(z) = \left(\frac{h(z)}{g(z)}\right)^{\frac{2(1-\gamma)}{\beta-\alpha}} \qquad (z \in \mathbb{D}).$$

From (3.15), we conclude that $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < L = \begin{cases} \frac{2(1-\gamma)}{(1+\alpha)(1+\beta)} & \textit{for} \quad \alpha\beta > 0 \textit{ with } \alpha+\beta < 0 \textit{ or } \beta = 0, \\ \frac{2(1-\gamma)}{(1-\alpha)(1-\beta)} & \textit{for} \quad \alpha\beta > 0 \textit{ with } \alpha+\beta > 0 \textit{ or } \alpha = 0, \\ \frac{4(1-\gamma)\sqrt{|\alpha\beta|}}{(\beta-\alpha)(1-\alpha\beta)} & \textit{for} \quad \alpha\beta < 0 \textit{ with } \alpha+\beta \neq 0, \\ \frac{2(1-\gamma)}{1+\alpha^2} & \textit{for} \quad \alpha+\beta = 0 \end{cases}$$

and the fact that $f \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ if and only if $zf'(z) \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$, we get the following conclusions.

Proposition 3.8 *Let $-1 < \alpha \leq \beta < 1$. The classes $\mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ and $\mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ are nonempty. The following functions are the examples of their members.*

(1) Let $a_n \in \mathbb{C}$ with $n = 2, 3, \ldots$. Then $f(z) = z + a_n z^n \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ $\iff |a_n| \leq \frac{L}{n-1+L}$.
(2) Let $a_n \in \mathbb{C}$ with $n = 2, 3, \ldots$. Then $f(z) = z + a_n z^n \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ $\iff n\,|a_n| \leq \frac{L}{n-1+L}$.
(3) Let $A \in \mathbb{C}$. Then $z/(1-Az)^2 \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma) \iff |A| \leq \frac{2}{2+L}$.
(4) Let $A \in \mathbb{C}$. Then $z/(1-Az) \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma) \iff |A| \leq \frac{2}{2+L}$.
(5) Let $A \in \mathbb{C}$. Then $z\exp(Az) \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma) \iff |A| \leq L$.
(6) Let $A \in \mathbb{C}$. Then $\frac{\exp(Az)-1}{A} \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma) \iff 0 < |A| \leq L$,
where L is given in the Corollary 3.7.

The following corollary is the consequence of Lemma 3.1, and Theorems in [9].

Corollary 3.9 *For $-1 < \alpha \leq \beta < 1$, $|z| = r < 1$, and $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$, it holds*

$$-\Psi_{\alpha,\beta,\gamma}(-r) \leq |f(z)| \leq \Psi_{\alpha,\beta,\gamma}(r),$$
$$\Psi'_{\alpha,\beta,\gamma}(-r) \leq \left|f'(z)\right| \leq \Psi'_{\alpha,\beta,\gamma}(r) \quad \textit{for} \quad \alpha\beta > 0 \textit{ with } \alpha+\beta > 0 \textit{ or } \alpha = 0,$$

$$\Psi'_{\alpha,\beta,\gamma}(r) \le \left|f'(z)\right| \le \Psi'_{\alpha,\beta,\gamma}(-r) \quad \textit{for} \quad \alpha\beta > 0 \textit{ with } \alpha+\beta < 0 \textit{ or } \beta = 0,$$
$$|\mathrm{Arg}\,\{f(z)/z\}| \le \max_{|z|=r} \mathrm{Arg}\left\{\Psi_{\alpha,\beta,\gamma}(z)/z\right\}.$$

Equalities in the above inequalities hold at a given point other than origin for the functions

$$\psi_{\alpha,\gamma,\mu}(z) = \overline{\mu}\Psi_{\alpha,\beta,\gamma}(\mu z) \quad (|\mu| = 1)\,. \tag{3.21}$$

Moreover

$$\frac{f(z)}{z} \prec \frac{\Psi_{\alpha,\beta,\gamma}(z)}{z} \quad (z \in \mathbb{D}). \tag{3.22}$$

If $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$*, then either* f *is a rotation of* $\Psi_{\alpha,\beta,\gamma}$ *given by* (3.12) *and* (3.14) *or*

$$\left\{w \in \mathbb{C}\colon |w| \le -\Psi_{\alpha,\beta,\gamma}(-1)\right\} \subset f(\mathbb{D}),$$

where $-\Psi_{\alpha,\beta,\gamma}(-1) = \lim_{r\to 1^-}[-\Psi_{\alpha,\beta,\gamma}(-r)]$.

Corollary 3.10 *Let* $-1 < \alpha \le \beta < 1$*. If* $f \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ *and* $|z| = r < 1$*, then*

$$-K_{\alpha,\beta,\gamma}(-r) \le |f(z)| \le K_{\alpha,\beta,\gamma}(r),$$
$$K'_{\alpha,\beta,\gamma}(-r) \le |f'(z)| \le K'_{\alpha,\beta,\gamma}(r),$$
$$|\mathrm{Arg}\left\{f'(z)\right\}| \le \max_{|z|=r} \mathrm{Arg}\left\{K'_{\alpha,\beta,\gamma}(z)\right\}.$$

Equalities in the above inequalities hold at a given point other than 0 *for functions* $\overline{\mu}K_{\alpha,\beta,\gamma}(\mu z)$ *with* $(|\mu| = 1)$*. Moreover*

$$f'(z) \prec K'_{\alpha,\beta,\gamma}(z) \quad (z \in \mathbb{D}).$$

If $f \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$*, then either* f *is a rotation of* $K_{\alpha,\beta,\gamma}$ *given by* (3.19) *and* (3.20) *or*

$$\{w \in \mathbb{C}\colon |w| \le -K_{\alpha,\beta,\gamma}(-1)\} \subset f(\mathbb{D}),$$

where $-K_{\alpha,\beta,\gamma}(-1) = \lim_{r\to 1^-}[-K_{\alpha,\beta,\gamma}(-r)]$.

Theorem 3.11 *Let* $-1 < \alpha < \beta < 1$*. If* $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$*, then*

(1) $\Re\left\{\frac{f(z)}{z}\right\} > \left(\frac{1+\alpha}{1+\beta}\right)^{\frac{2-2\gamma}{\beta-\alpha}}$ *for* $\mathcal{T}_0(\alpha, \beta) \le 1 - \frac{\beta-\alpha}{2} \le \gamma$ $(z \in \mathbb{D})$,

(2) $\Re\left\{\frac{f(z)}{z}\right\}^{\frac{\beta-\alpha}{2-2\gamma}} > \frac{1+\alpha}{1+\beta}$ $(z \in \mathbb{D})$,

(3) $\left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| \leq \frac{2(1-\gamma)}{\beta-\alpha} \sin^{-1}\left(\frac{|z|(\beta-\alpha)}{1-|z|^2\alpha\beta}\right) \quad (z \in \mathbb{D}).$

Proof Let $q := \frac{2(1-\gamma)}{\beta-\alpha}$.

Case 1. From $1 - \frac{\beta-\alpha}{2} \leq \gamma$ it follows that $0 < (2-2\gamma)/(\beta-\alpha) \leq 1$, and from $f \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$ it follows that $\mathcal{T}_0 \leq 1 - \frac{\beta-\alpha}{2}$. Then, making use Corollary 3.9 and Lemma 1.3, we conclude that

$$\begin{aligned} \Re\left\{\frac{f(z)}{z}\right\} &> \Re\left\{\frac{\Psi_{\alpha,\beta,\gamma}(z)}{z}\right\} = \Re\left\{\left(\frac{1-\alpha z}{1-\beta z}\right)^q\right\} \\ &\geq \left\{\Re\left(\frac{1-\alpha z}{1-\beta z}\right)\right\}^q > \left(\frac{1+\alpha}{1+\beta}\right)^q. \end{aligned}$$

The function $\psi_{\alpha,\gamma,\mu}$ given by (3.21), shows that the bound is the best possible.

Case 2. From Corollary 3.9 we have

$$\left[\frac{f(z)}{z}\right]^{1/q} \prec \left[\frac{\Psi_{\alpha,\beta,\gamma}(z)}{z}\right]^{1/q}.$$

Thus

$$\Re\left\{\frac{f(z)}{z}\right\}^{1/q} > \Re\left\{\frac{1-\alpha z}{1-\beta z}\right\} > \frac{1+\alpha}{1+\beta}.$$

Case 3. By Corollary 3.9 it is enough to consider $\operatorname{Arg}\left\{\Psi_{\alpha,\beta,\gamma}(z)/z\right\}$. Since the image of the disk $\{z \in \mathbb{C} \colon |z| \leq r\}$ by the function $w = \Psi_{\alpha,\beta,\gamma}(z)/z$ or $w^{1/q} = (1-\alpha z)/(1-\beta z)$ is contained in closed disc with center $\left(1-\alpha\beta r^2\right)/\left(1-\beta^2 r^2\right)$ and radius $(r(\beta-\alpha))/\left(1-\beta^2 r^2\right)$, then

$$\left|w^{1/q} - \frac{1-\alpha\beta r^2}{1-\beta^2 r^2}\right| \leq \frac{r(\beta-\alpha)}{1-\beta^2 r^2} \quad \textit{and} \quad \left|\operatorname{Arg} w^{1/q}\right| < \frac{\pi}{2}.$$

Thus

$$\left|\operatorname{Arg} w^{1/q}\right| \leq \sin^{-1}\left(\frac{r(\beta-\alpha)}{1-r^2\alpha\beta}\right).$$

The proof is now complete. □

It is clear that $f(z) \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$ if and only if $zf'(z) \in \mathcal{ST}_{snail}(\alpha, \beta, \gamma)$. Using the same notation and the same reasoning as in the proof of Theorem 3.11 we have the following Corollary.

Corollary 3.12 *Let* $-1 < \alpha < \beta < 1$. *If* $f \in \mathcal{CV}_{snail}(\alpha, \beta, \gamma)$, *then*

(1) $\Re\left\{f'(z)\right\} > \left(\frac{1+\alpha}{1+\beta}\right)^{\frac{2-2\gamma}{\beta-\alpha}}$ *for* $\mathcal{T}_0(\alpha, \beta) \leq 1 - \frac{\beta-\alpha}{2} \leq \gamma \ (z \in \mathbb{D}),$

(2) $\Re\left\{f'(z)\right\}^{\frac{\beta-\alpha}{2-2\gamma}} > \dfrac{1+\alpha}{1+\beta} \quad (z \in \mathbb{D}),$

(3) $\left|\mathrm{Arg}\left\{f'(z)\right\}\right| \le \dfrac{2(1-\gamma)}{\beta-\alpha}\sin^{-1}\left(\dfrac{|z|(\beta-\alpha)}{1-|z|^2\alpha\beta}\right) \quad (z \in \mathbb{D}).$

The conditions $\mathcal{T}_0(\alpha,\beta) \le 1-\dfrac{\beta-\alpha}{2} \le \gamma$ in Theorem 3.11 for requirement $\beta = -\alpha$ are equivalent to conditions $1-\sqrt{2} \le \alpha < 0, \gamma \ge 1+\alpha$, and so we have the following.

Corollary 3.13 *For* $1-\sqrt{2} \le \alpha < 0$ *and* $\gamma \ge 1+\alpha$*, we have:*

$$f \in \mathcal{ST}_{snail}(\alpha,-\alpha,\gamma) \Longrightarrow \Re\left\{\frac{f(z)}{z}\right\} > \left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1-\gamma}{\alpha}} \quad (z \in \mathbb{D}),$$

and

$$f \in \mathcal{CV}_{snail}(\alpha,-\alpha,\gamma) \Longrightarrow \Re\left\{f'(z)\right\} > \left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1-\gamma}{\alpha}} \quad (z \in \mathbb{D}).$$

Acknowledgements The authors thank the editor and the anonymous referees for constructive and pertinent suggestions.

Author contributions Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

Availability of supporting data Not applicable.

Declarations

Conflict of interest The authors declare that they have no competing interests.

References

1. Babalola, K.O.: On λ-pseudo-starlike functions. J. Class. Anal. **3**(2), 137–147 (2013)
2. Kanas, S., Masih, V.S., Ebadian, A.: Relations of a planar domain bounded by hyperbola with family of holomorphic functions. J. Inequal. Appl., 246 (2019), 14pp
3. Kanas, S., Sugawa, T.: On conformal representations of the interior of an ellipse. Annales. Acad. Sci. Fennicae **31**, 329–348 (2006)
4. Kanas, S., Tatarczak, A.: Constrained coefficients problem for generalized typically real functions. Complex Var. Elliptic Equ. **61**(8), 1052–1063 (2016)

5. Kanas, S., Tatarczak, A.: Generalized typically real functions. Filomat **30**(7), 1697–1710 (2016). https://doi.org/10.2298/FIL1607697K
6. Kanas, S., Masih, V.S., Ebadian, A.: Coefficients problems for families of holomorphic functions related to hyperbola. Math. Slovaca **70**(3), 605–616 (2020)
7. Lentini, M., Cardona, L., Paluszny, M.: Grid generation using leminiscates with two foci. Math. Comput. Model. **57**, 2186–2195 (2013)
8. Ma, W., Minda, D.: A unified treatment of some special classes of univalent functions. In: Proceedings of Conference on Complex Analysis, Tianjin, 1992, Conference Proceedings and Lecture Notes in Analysis, Vol. 1 (International Press, Cambridge, MA, 1994), pp. 157–169
9. Masih, V.S., Ebadian, A., Yalçin, S.: Some properties associated to a certain class of starlike functions. Math. Slovaca **69**(6), 1329–1340 (2019)
10. Mendiratta, R., Sumit, N., Ravichandran, V.: On a subclass of strongly starlike functions associated with exponential function. Bull. Malays. Math. Sci. Soc. **38**(1), 365–386 (2015)
11. Ozaki, S.: On the theory of multivalent functions. II. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 45–87 (1941)
12. Piejko, K., Sokół, J.: On booth leminiscate and Hadamard product of analytic functions. Math. Slovaca **65**(6), 1337–1344 (2015)
13. Rogosinski, W.: On the coefficients of subordinate functions. Proc. Lond. Math. Soc. **2**(1), 48–82 (1945)
14. Suffridge, T.J.: Some remarks on convex maps of the unit disk. Duke Math. J. **37**, 775–777 (1970)