REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 6, número 1; 1993.
http://dx.doi.org/10.5209/rev_REMA.1993.v6.n1.17860

# *On the Behaviour of Generalized Solutions to Genuinly Nonlinear First Order Equations for Small and Large Values of Time*

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**ABSTRACT.** The paper deals with the asymptotic behaviour of generalized solutions to nonlinear first order equations. With the aid of explicit variational representation one studies the decrease of solutions for a large time. And for the small time an asymptotics of the perturbation's front is calculated.

## 1. INTRODUCTION

This paper investigates generalized solutions (briefly:g.s.) of the equation

$$Lu \equiv u_t + f(t,x,u_x) + g(t,x,u) = H(t,x), \ (t,x) \in \mathbb{R}_+ \times \mathbb{R} \qquad (1.1)$$

with initial data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R} \qquad (1.2)$$

Here $f \in C^3$, $g \in C$, $g(t,x,u)$ satisfies local Hölder condition in $u \in \mathbb{R}_+$

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1991 Mathematics Subject Classification: 35L60.
Editorial Complutense. Madrid, 1993.

with exponent $n > 0$ and has three continuous derivatives in $u \in \mathbb{R}_+$, $f(t,x,0)=g(t,x,0)=0$, $g(t,x,u)$ monotonically nondecreases in $u$, $f_{pp}(t,x,p)\geq\mu>0$, $H(t,x)$ is a measurable function bounded for bounded $t$; $u_0(x) \in W^1_\infty(\mathbb{R})$, $u_0(x)\geq 0$.

The Cauchy problem for (1.1) was considered in [2], [8]-[10]. The papers [11], [3]-[7] are devoted to the case of $f$ nonconvex with respect to $u_x$. Further references may be found in the cited articles.

In §2 we deal with the decreasing of g.s. to (1.1), (1.2) as $t$ converges to $+\infty$ in the case

$$f(t,x,u_x)=f(u_x),\ \ g(t,x,u)\equiv 0,\ \ \lim_{|x|\to+\infty} u_0(x)=0,\ \ H(t,x)\equiv 0.$$

In [8] for this case stabilization to inf $u_0(x)$ was proved and the rate of the stabilization was estimated. Similar results but in the case of nonconvex $f$ are presented in [1]. Our results are based on explicit formula for the g.s. of the problem (1.1), (1.2) obtained in [9]. Some function $v(t,x)$ satisfying (1.1) in the generalized sense will be found such that for every $u_0(x)$ with compact support and nonsmooth at the points of the support boundary the corresponding g.s. of (1.1), (1.2) identically equals $v(t,x)$ beginning from some fixed time. If $u_0(x)$ is smooth then the g.s. tends to $v(t,x)$ as $t$ converges to $+\infty$ and the rate of convergence will be estimated.

In case $u_0(x)$ does not possess compact support but tends to zero as $|x| \to +\infty$ the rate of g.s. decreasing also will be estimated.

§3 is devoted to the support behaviour of g.s. to (1.1), (1.2) for small values of $t$ provided $u_0(x)$ has compact support. For example, consider the equation

$$u_t+f(u_x)+Nu_x+K|u|^{n-1}u,\quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

where $f(0)=f'(0)=0, f''>0, N\geq 0, K\geq 0, 0<n<1$.

Suppose $u_0(x)=P(1-x)^\alpha$ for $1-\varepsilon \leq x \leq 1$, $u_0(x)\equiv 0$ for $x > 1$, where $\alpha\geq 1$, $P>0$, $\varepsilon>0$. For this particular case as a corollary of the results contained in §3 one obtains necessary and sufficient conditions for the support of the g.s. of (1.1), (1.2) to expand in the left or the right direction.

**Definition 1.1.** *A function $u(t,x)$ Lipschitz continuous in $[0,T]\times\mathbb{R}$, $T>0$, is called a g.s. of the problem (1.1), (1.2) if:*

*1) $u(t,x)$ satisfies (1.1) almost everywhere in $\mathbb{R}_+\times\mathbb{R}$ and takes initial values (1.2);*

*2) for every $l\neq 0$ in each rectangle $(0,T]\times[-R,R]$, $T>0$, $R>0$, the following inequality is fulfilled*

$$u(t,x+l)-2u(t,x)+u(t,x-l)\leq\lambda(t,R)l^2, \tag{1.3}$$

*where $\lambda(t,R)\geq 0$ is defined for $t\in(0,T]$ and $\lambda(t,R)<\lambda_\delta<+\infty$ for $0<\delta\leq t\leq T$.*

The existence theorem for the problem (1.1), (1.2) one can prove by analogy with [8].

**Theorem 1.1.** *Suppose $v(t,x)\in W^1_\infty$, $w(t,x)\in W^1_\infty$ in each strip $[0,T]\times\mathbb{R}$, $T>0$. Suppose $v(t,x)$ is a g.s. of the equation $Lv=h_1(t,x)$ with data $v(0,x)=v_0(x)\in W^1_\infty(\mathbb{R})$, and $w(t,x)$ is a g.s. of the equation $Lw=h_2(t,x)$ with data $w(0,x)=w_0(x)\in W^1_\infty(\mathbb{R})$, where $h_1(t,x),h_2(t,x)$ are measurable functions bounded for bounded $t$. Suppose $h_1(t,x)\leq h_2(t,x)$ in $\mathbb{R}_+\times\mathbb{R}$ and $v_0(x)\leq w_0(x)$ in $\mathbb{R}$. Then $v(t,x)\leq w(t,x)$ in $\mathbb{R}_+\times\mathbb{R}$.*

Due to monotonicity of $g(t,x,u)$ in $u$ the proof of this theorem is analogous to [8]. It follows from the Theorem 1.1 that the g.s. of (1.1), (1.2) is unique.

Below $u(t,x)$ denotes the g.s. of the problem (1.1), (1.2) with $H(t,x)\equiv 0$.

## 2. THE BEHAVIOUR OF GENERALIZED SOLUTIONS AS t CONVERGES TO $+\infty$

Suppose $f(t,x,u_x)\equiv f(u_x)$, $g(t,x,u)\equiv 0$, $f'(0)=0$, $f\in C^4(\mathbb{R})$. Under these conditions (see [9]) the g.s. of the problem (1.1), (1.2) with $H(t,x)\equiv 0$ is given by the formula

$$u(t,x)=\inf_{q\in\mathbb{R}} G(t,x,q,u_0(q)),\ G(t,x,q,u_0(q))\equiv u_0(q)+t\cdot\Phi((x-q)/t), \tag{2.1}$$

where $\Phi(q)$ is Legendre transformation of the function $f(q)$, $\Phi(q)=f'^{-1}(q)q - f\circ f'^{-1}(q)\geq 0$, because of the convexity of $f(q)$. Notice, that $\Phi'(q)=f'^{-1}(q)$.

Let us fulfil some auxiliary research and study the roots of the equation (with respect to $q$)

$$G_q=u_0'(q)-f'^{-1}((x-q)/t)=0,\quad q\in[a,b], \tag{2.2}$$

where $t>0$, $x\in\mathbb{R}$ are fixed.

**Lemma 2.1.** *Suppose $u_0(t,x)$ is monotone in $[a,b]$, $u_0\in C([a,b])$. Then if $[a,b]$ does not contain zero roots for $u'_0(q)=0$ there exists such $t_0$ that for $t>t_0$ the segment $[a,b]$ does not contain roots of (2.2); if $q_*\in[a,b]$, $u'_0(q_*)=0$ and $q_*$ is the unique root for $u'_0(q)=0$ then there exists such $t_1$ that for $t>t_1$ the equation (2.2) has only one root tending to $q_*$ as $t\to+\infty$.*

**Proof.** The equation (2.2) is equivalent to the following

$$f'\circ u_0'(q)=(x-q)/t \tag{2.3}$$

The left hand side of (2.3) equals zero if and only if $u'_0(q)=0$. The right hand side of (2.3) for fixed $x$ tends to zero as $t\to+\infty$ uniformly with respect to $q\in[a,b]$. Now, the statements of Lemma 2.1 easily follow.

**Lemma 2.2.** *Suppose supp $u_0(x) \subset [a,b]$. Then supp $u(t,x) \subset \mathbb{R}_+ \times [a,b]$.*

**Proof.** Suppose $x \notin [a,b]$, $t \in \mathbb{R}_+$. Let us take $q=x$, then $G(t,x,x,u_0(x)) = u_0(x) = 0$. Our result follows from (2.1).

Let us introduce the notation

$$v(t,x)=\min[G(t,x,0,u_0(0)),G(t,x,1,u_0(1))].$$

**Theorem 2.1.** *Suppose the following conditions hold:*

*1) supp $u_0(x) = [0,1]$;*
*2) $u_0(x) \geq l_1(x)$, $l_1(x) \in W^1_\infty(\mathbb{R})$ and $l_1(x) > 0$, $l'_1(x\pm 0) \neq 0$ in $(0,1)$;*
*3) $u_0(x)=l_1(x)$ for $|x|<\varepsilon$ and for $|x-1|<\varepsilon$ for some $\varepsilon > 0$;*
*4) $l_1(x) \sim ax^\alpha$, $l'_1(x) \sim \alpha a x^{\alpha-1}$ as $x \to +0$;*
*5) $l_1(x) \sim a(1-x)^\alpha$, $l'_1(x) \sim -\alpha a(1-x)^{\alpha-1}$ as $x \to 1-0$;*

*here $a$=const>0, $\alpha$=const>1.*

*Then $t^{\alpha/(\alpha-1)}[v(t,x)-u(t,x)] \to B(x)$ as $t \to +\infty$ for every $x \in \mathbb{R}$, where $B(x)=0$ for $x \notin (0,1)$ and*

$$B(x)=(\alpha-1)a^{-1/(\alpha-1)}[\alpha f''(0)]^{-\alpha/(\alpha-1)}\min\{x^{\alpha/(\alpha-1)},(1-x)^{\alpha/(\alpha-1)}\}$$

*for $x \in (0,1)$.*

**Proof.** It follows from the conditions of Theorem 2.1 that $G(t,x,q,u_0(q)) \geq G(t,x,q,l_1(q))$. By virtue of Lemma 2.1 $G(t,x,q,l_1(q))$ as a function of $q$ can have a local minimum at $\varepsilon$-neighbourhoods of the points $q$=0 and $q$=1 provided $t$ is sufficiently large. But there $u_0(q)=l_1(q)$. It is easy to see that $G(t,x,q,u_0(q))>0$, $q \in (0,1)$ because of $u_0(q)>0$ for $q \in (0,1)$. So global minimum with respect to $q$ of the function $G(t,x,q,u_0(q))$ coincides with the smallest of the values $G(t,x,0,0)$, $G(t,x,1,0)$, $G(t,x\ q_0,u_0(q_0))$, $G(t,x,q_1,u_0(q_1))$, where $q_0=q_0(t,x)$, $q_1=q_1(t,x)$ are the roots of (2.3) tending to 0 and 1 respectively as $t \to +\infty$.

Notice, that $G(t,x,0,0)=t\Phi(x/t)$, $G(t,x,1,0)=t\Phi((x-1)/t)$. It is easy to see that for sufficiently large $t$ the following inequalities are true:

$$G(t,x,0,0) \le G(t,x,1,0) \quad \text{for } 0 \le x \le 1/2 \qquad \text{and}$$
$$G(t,x,0,0) \ge G(t,x,1,0) \quad \text{for } 1/2 \le x \le 1.$$

Hence, provided $t$ is large enough we obtain

$$G(t,x,q_0,u_0(q_0)) \le G(t,x,q_1,u_0(q_1)) \quad \text{for } 0 \le x < 1/2 \qquad \text{and}$$
$$G(t,x,q_0,u_0(q_0)) \ge G(t,x,q_1,u_0(q_1)) \quad \text{for } 1/2 < x \le 1.$$

Let us seek $q_0(t,x)$ in the form $q_0(t,x)=b_0(x)t^{-\beta}+o(t^{-\beta})$, where $b_0 \in C(\mathbb{R})$, $\beta > 0$. Substituting $q_0(t,x)$ into (2.2) one gets $\alpha a q_0(t,x)^{\alpha-1} + o(q_0(t,x)^{\alpha-1}) = \Phi'((x-q_0(t,x))/t)$, $\alpha a b_0(x)^{\alpha-1}t^{-\beta(\alpha-1)} + o(t^{-\beta(\alpha-1)}) = \Phi''(0)(x-q_0(t,x))/t + o(1/t)$. Hence $\beta=1/(\alpha-1)$ and equating coefficients at $t^{-1}$ one finds $a\alpha b_0^{\alpha-1}(x) = x\Phi''(0)$. Further, $G(t,x,0,0) - G(t,x,q_0(t,x),u_0(q_0(t,x))) = t\Phi(x/t)-t\Phi((x-q_0(t,x))/t)-u_0(q_0(t,x)) = t[\Phi'((x-q_0(t,x))/t)q_0(t,x)/t + \int_{(x-q_0(t,x))/t}^{x/t}\Phi''(\xi)(x/t-\xi)d\xi] - aq_0(t,x)^{\alpha} + o(t^{-\alpha/(\alpha-1)})$. One estimates $t\left|\int_{(x-q_0(t,x))/t}^{x/t}\Phi''(\xi)(x/t-\xi)d\xi\right| \le \text{const}\; t\int_{(x-q_0(t,x))/t}^{x/t}|x/t-\xi|\,d\xi \le \text{const}\; q_0(t,x)^2/t = o(t^{-\beta-1}) = o(t^{-\alpha/(\alpha-1)})$. Besides, $\Phi'((x-q_0(t,x))/t) = \Phi''(0)(x-q_0(t,x))/t + o(1/t)$.

Thus, $G(t,x,0,0) - G(t,x,q_0(t,x),u_0(q_0(t,x))) = \Phi''(0)xq_0(t,x)/t - ab_0(x)^{\alpha,-\alpha/(\alpha-1)} + o(t^{-\alpha/(\alpha-1)}) = [\Phi''(0)xb_0(x) - ab_0(x)^{\alpha}]t^{-\alpha/(\alpha-1)} + o(t^{-\alpha/(\alpha-1)})$. Now, taking into account that $\Phi''(0)=1/f''(f'^{-1}(0))=1/f''(0)$, one finds $G(t,x,0,0) - G(t,x,q_0(t,x),u_0(q_0(t,x))) = b_0(x)x(1-1/\alpha)t^{-\alpha/(\alpha-1)}/f''(0) + o(t^{-\alpha/(\alpha-1)})$. The first term of the right hand side is positive for $x > 0$.

By analogy one can prove the equality $G(t,x,1,0) - G(t,x,q_1(t,x), u_0(q_1(t,x))) = t^{-\alpha/(\alpha-1)}b_0(1-x)(1-x)(1-1/\alpha)/f''(0) + o(t^{-\alpha/(\alpha-1)})$. Here the first term of the right hand side is positive for $x < 1$.

Comparing the last two formulas one proves the statement of Theorem 2.1 for $x \in (0,1)$. For other $x$ $B(x)=0$ by virtue of Lemma 2.2.

**Theorem 2.2.** *Suppose the following conditions hold:*

*1) supp* $u_0(x) = [0,1]$;

*2)* $u_0(x) \geq l_2(x)$ *in* $(0,1)$, $l_2(x)=Mx/\varepsilon$ *for* $0 < x \leq \varepsilon$, $l_2(x)=M(1-x)/(1-\varepsilon)$ *for* $\varepsilon \leq x < 1$, $M>0$, $1>\varepsilon>0$;

*3)* $\Phi(s) \leq Ms/\varepsilon$ *for* $0\leq s\leq s_0(M,\varepsilon)$, $\Phi(-s) \leq Ms/(1-\varepsilon)$ *for* $0\leq s\leq s_1(M,\varepsilon)$, *where* $\Phi$ *is the Legendre transformation of* $f$.

*Then* $u(t,x) = v(t,x)$ *for* $t \geq T = \max\,[-1/f'(-M/(1-\varepsilon)),\ 1/f'(M/\varepsilon),\ s_0(M,\varepsilon)^{-1},\ s_1(M,\varepsilon)^{-1}]$.

**Proof.** Let us investigate infimum (with respect to $q$) of the function $G(t,x,q,l_2(q))$. Because of Lemma 2.2 one may consider only $0 < x < 1$. We are interested in the values of the function $G$ at the points $q$, where either $G_q=0$ or $G$ is not smooth. $G$ is not smooth at three points: 0, $\varepsilon$, 1.

If $q < 0$ then $G_q=0$ only at $q=x$; if $0<q<\varepsilon$ then $G_q=0$ at $q=x-tf'(M/\varepsilon)$; if $\varepsilon<q<1$ then $G_q=0$ at $q=x-tf'(-M/(1-\varepsilon))$; if $q>1$ then $G_q=0$ only at $q=x$.

Thus, the global minimum on the funcion $G(t,x,q,l_2(q))$ with respect to $q$ coincides with the smallest of the values:

$\omega_1 \equiv t\Phi(x/t)$, $\omega_2 \equiv t\Phi((x-1)/t)$, $\omega_3 \equiv M+t\Phi((x-\varepsilon)/t)$, $\omega_4 \equiv l_2(x)$, $\omega_5 \equiv l_2(x-tf'(M/\varepsilon)) + t\Phi\circ f'(M/\varepsilon)$, $\omega_6 \equiv l_2(x-tf'(-M/(1-\varepsilon))) + t\Phi\circ f'(-M/(1-\varepsilon))$.

For $0<x<1$, $x<tf'(M/\varepsilon)$, $x > 1 + tf'(-M/(1-\varepsilon))$ the following inequalities are true: $\omega_1 \leq \omega_5$, $\omega_2 \leq \omega_6$. Because of assumption 3) $\min(\omega_1,\omega_2) \leq \omega_4$ for $t\geq T$. Further $\omega_3 > M$ in consequence to $\Phi > 0$. Thus, it sufficies to find $\min(\omega_1,\omega_2)$ for $t \geq T$. But $G(t,x,q,u_0(q)) \geq G(t,x,q,l_2(q))$ for every $q$. Hence

$$\inf_{q\in\mathbf{R}} G(t,x,q,u_0(q))=\inf_{q\in\mathbf{R}} G(t,x,q,l_2(q))=\min\{t\Phi(x/t),t\Phi((x-1)/t)\}\equiv v(t,x) \text{ for } t\geq T.$$

**Remark 2.1.** It follows from the theorems 2.1, 2.2 that $u(t,x)$ tends to zero as $t \to +\infty$ in case $u_0(x)$ is not smooth at the points 0 and 1 slower than in case $u_0(x)$ is smooth there. But this difference presents only in the second term of the asymptotics as $t \to +\infty$.

**Theorem 2.3.** *Suppose the following conditions hold:*

1) $u_0(x) > 0,\ x \in \mathbb{R}$;
2) $u_0(x) \geq l_3(x),\ l_3 \in W^1_\infty(\mathbb{R}),\ l_3(x) > 0,\ l_3'(x\pm 0)\neq 0$ *for* $x \in \mathbb{R}$;
3) $u_0(x) = l_3(x)$ *for* $|x|$ *sufficiently large;*
4) $l_3(x) \sim C_1|x|^{-\beta} + C_2|x|^{-\beta-1}$,
$l_3'(x) \sim -\beta C_1|x|^{-\beta-1}\mathrm{sign}\, x - (\beta+1)C_2|x|^{-\beta-2}\mathrm{sign}\, x$,
$|x| \to +\infty,\ C_1>0,\ C_2>0,\ \beta>0$.

*Then* $t^{(\beta+1)/(\beta+2)}u(t,x) - C_1^{2/(\beta+2)}(\beta+2)2^{-1}[\Phi''(0)/\beta]^{\beta/(\beta+2)}t^{1/(\beta+2)} \to \phi''(0)^{(\beta+1)/(\beta+2)}(\beta C_1)^{1/(\beta+2)}[C_2/(\beta C_1) - |x|]$ *as* $t \to +\infty$ *for every* $x \in \mathbb{R}$.

**Proof.** By analogy with the proof of Theorem 2.1 it sufficies to find the minimum with respect to $q$ of the function $G(t,x,q,l_3(q))$. The equation (2.3) with respect to $q$ has two roots $q_+(t,x)$ and $q_-(t,x)$ tending to $+\infty$ and $-\infty$ respectively as $t \to +\infty$. Let us seek these roots in the form

$$q_\pm(t,x)=\pm a_1 t^{\gamma}(1+a_2(x)t^{-\delta}+o(t^{-\delta})),$$

where $a_1 = \mathrm{const} > 0,\ a_2 \in C(\mathbb{R}),\ \gamma > 0,\ \delta > 0$.

Let us consider $q_+$. Applying Taylor's formula with the remainder term in Peano's form one gets

$$l_3'(q_+(t,x)) = -\beta C_1 a_1^{-(\beta+1)}t^{-\gamma(\beta+1)}(1+a_2(x)t^{-\delta} + o(t^{-\delta}))^{-\beta-1} -$$
$$(\beta+1)C_2 a_1^{-(\beta+2)}t^{-\gamma(\beta+2)}(1+a_2(x)t^{-\delta} + o(t^{-\delta}))^{-\beta-2} + o(t^{-\gamma(\beta+2)});$$
$$\Phi'((x-q_+(t,x))/t) = \Phi''(0)[x/t-a_1t^{\gamma-1}(1+a_2(x)t^{-\delta} + o(t^{-\delta}))] +$$
$$\Phi'''(0)2^{-1}[x/t-a_1t^{\gamma-1}(1+a_2(x)t^{-\delta} + o(t^{-\delta}))]^2 + o(t^{-2(1-\gamma)}).$$

Substituting these expresions into (2.3) and comparing the terms with the lowest absolute value of powers of $t$ one obtains

$$\gamma=1/(\beta+2),\ a_1^{\beta+2}=\beta C_1/\Phi''(0).$$

If $\delta > \gamma$ then $-\gamma(\beta+1)-\delta < -1$ and the supplementary condition on $a_1$ will arise: $(\beta+1)C_2a_1^{-(\beta+2)} = x\Phi''(0)$ which is in the contrary with the former.

If $\delta < \gamma$ then $-\gamma(\beta+1)-\delta > -1$ and $-\gamma(\beta+1)-\delta = -1+\gamma-\delta$; besides $-a_1a_2(x)\Phi''(0) = \beta(\beta+1)C_1a_1^{-(\beta+1)}a_2(x)$. It is impossible because the left hand the right hand quantities are of opposite signs.

Hence $\delta=\gamma$ and $a_1^{-(\beta+2)}[C_1\beta(\beta+1)a_1a_2(x)-(\beta+1)C_2] = \Phi''(0)(x-a_1a_2(x))$, that is

$$a_1^{(\beta+2)}=\beta C_1/\Phi''(0),\ a_2(x)=[\beta C_1x+C_2(\beta+1)]/[\beta(\beta+2)C_1a_1] \quad (2.4)$$

Using formulas (2.4) one gets

$$G(t,x,q_+(t,x),\ l_3(q_+(t,x))) = C_1q_+(t,x)^{-\beta} + C_2q_+(t,x)^{-\beta-1} + o(t^{-(\beta+1)/(\beta+2)}) + t[2^{-1}\Phi''(0)((x-q_+(t,x))/t)^2 + 6^{-1}\phi'''(0) \times ((x-q_+(t,x))/t)^3 + o(t^{-3(\beta+1)/(\beta+2)})] = [C_1a_1^{-\beta} + 2^{-1}\phi''(0)a_1^2] \times t^{-\beta/(\beta+2)} + [\Phi''(0)a_1^2a_2(x) - \beta C_1a_1^{-\beta}a_2(x) + C_2a_1^{-\beta-1} - \Phi''(0)a_1x] \times t^{-(\beta+1)/(\beta+2)} + o(t^{-(\beta+1)/(\beta+2)}) = a_1^2(1/\beta+1/2)\Phi''(0)t^{-\beta/(\beta+2)} + a_1\Phi''(0)[C_2/(\beta C_1)-x]t^{-(\beta+1)/(\beta+2)} + o(t^{-(\beta+1)/(\beta+2)}).$$

By analogy one finds

$$G(t,x,q_-(t,x),\ l_3(q_-(t,x))) = a_1^2(1/\beta+1/2)\Phi''(0)t^{-\beta/(\beta+2)} + a_1\Phi''(0)[C_2/(\beta C_1)+x]t^{-(\beta+1)/(\beta+2)} + o(t^{-(\beta+1)/(\beta+2)}).$$

It is easy to see that

$$u(t,x) = \min\ [G(t,x,q_+(t,x),l_3(q_+(t,x))),G(t,x,q_-(t,x),l_3(q_-(t,x)))]$$

and get the statement of Theorem 2.3.

**Theorem 2.4.** *Suppose assumptions 1), 2) of the Theorem 2.3 hold and assumptions 3), 4) hold only for $x < 0$; besides $f'(0) > 0$. Then (independently of the behaviour of $u_0(x)$ as $x \to +\infty$) the following relation is valid*

$$u(t,x)t^{\beta+1}-C_1(f'(0))^{-\beta}t \to (f'(0))^{-\beta-1}(C_2+\beta C_1x)$$

*as $t \to +\infty$, for every $x \in \mathbb{R}$.*

**Proof.** In this case for large $t$ equation (2.3) has only one unbounded root $q_-(t,x) \to -\infty$ as $t \to +\infty$. Other roots, if they exist, tend to finite limits. It sufficies to establish asymptotics only for the function $p(t,x) \equiv G(t,x,q_-(t,x),l_3(q_-(t,x)))$ as $t \to +\infty$ because $G(t,x,q_-(t,x),u_0(q_-(t,x)))$ tends to zero as $t \to +\infty$.

Let us seek $q_-(t,x)$ in the form

$$q_-(t,x) = -a_1 t^{\gamma}(1+a_2(x)t^{-\delta}+a_3 t^{-\Delta}+o(t^{-\Delta})),$$

where $a_1$ = const > 0, $a_2 \in C(\mathbb{R})$, $a_3$ = const, $\gamma>0$, $\Delta>\delta>0$. Then $l_3'(q_-(t,x)) = \beta C_1 a_1^{-\beta-1} t^{-\gamma(\beta+1)}[1+a_2(x)t^{-\delta}+a_3 t^{-\Delta}+o(t^{-\Delta})]^{-\beta-1}+C_2(\beta+1)a_1^{-\beta-2}t^{-\gamma(\beta+2)}[1+a_2(x)t^{-\delta}+a_3 t^{-\Delta}+o(t^{-\Delta})]^{-\beta-2} + o(t^{-\gamma(\beta+2)})$. Now we have $u'_0(q_-(t,x)) \to 0$ as $t \to +\infty$. It follows from (2.3) that $(x-q_-(t,x))/t$ tends to $b$ = const as $t \to +\infty$ and $\Phi'(b)=0$ so $\Phi(b)=0$. However, $(x-q_-(t,x))/t = x/t+a_1 t^{\gamma-1}(1+a_2(x)t^{-\delta}+a_3 t^{-\Delta}+o(t^{-\Delta}))$. So $\gamma=1$, $a_1=b$. Further,
$\Phi'((x-q_-(t,x))/t) = \Phi''(a_1)[x/t+a_1a_2(x)t^{-\delta}+a_1a_3t^{-\Delta}+o(t^{-\Delta})] + 2^{-1}\Phi'''(a_1)[x/t+a_1a_2(x)t^{-\delta}+a_1a_3t^{-\Delta}+o(t^{-\Delta})]^2 + o(t^{-2\min(1,\Delta)})$.

Substituting $\Phi'$ and $l_3'$ into (2.3) one obtains $\delta=1$, $a_2(x)=-x/a_1$; further, $\Delta=\beta+1$, $\beta C_1 a_1^{-\beta-1}=a_1a_3\Phi''(a_1)$. Thus,

$p(t,x) = l_3(q_-(t,x)) + t\Phi((x-q_-(t,x))/t) = C_1a_1^{-\beta}t^{-\beta}(1+a_2(x)t^{-1}+ a_3t^{-\beta-1}+o(t^{-\beta-1}))^{-\beta} + C_2a_1^{-\beta-1}t^{-\beta-1}(1+a_2(x)t^{-1}+a_3t^{-\beta-1}+o(t^{-\beta-1}))^{-\beta-1} + o(t^{-\beta-1}) + t[2^{-1}\Phi''(a_1)(a_1a_3t^{-\beta-1}+o(t^{-\beta-1}))^2] = C_1a_1^{-\beta}t^{-\beta} + (C_2a_1^{-\beta-1} - C_1a_1^{-\beta}\beta a_2(x))t^{-\beta-1} + o(t^{-\beta-1})$.

Substituting $a_2(x)$ one gets the required asymptotics.

## 3. THE BEHAVIOUR OF GENERALIZED SOLUTIONS FOR SMALL t

In this paragraph we consider the equation (1.1) in the form

$$L_1u \equiv u_t+f_1(t,x,u_x)+Bu_x+g(t,x,u)=0, \tag{3.1}$$

where $f_1 \geq 0$, $B$ = const$\geq 0$, $f_1'(t,x,0)=0$.

**Theorem 3.1.** *Suppose the following conditions hold:*

*1) supp* $u_0(x)=[0,1]$; $u_0(x)\leq P(1-x)^\alpha$ *for* $1-\varepsilon \leq x \leq 1$, $\alpha=const\geq 1$, $P=const>0$, $\varepsilon=const>0$;

*2)* $g(t,x,u)\geq g_1(u)$, $g_1\geq 0$, $g_1\in C$, $\int_0^1 d\xi/g_1(\xi)<+\infty$;

*3)* $A_1(t)\equiv P^{-1/\alpha}(H^{-1}(t))^{1/\alpha}-Bt \geq 0$ *for small t, where* $H(s)\equiv\int_0^s d\xi/g_1(\xi)$.

*Then there exists such* $t_0>0$ *that* $u(t,x)=0$ *for* $1-A_1(t)\leq x\leq 1$, $0\leq t\leq t_0$.

**Proof.** Let us construct such a function $v_0(x)$ that $v_0(x)=P(1-x)^\alpha$ for $1-\varepsilon \leq x \leq 1$ and $v_0(x) \geq u_0(x)$ for other $x$. Let us consider a comparison function $w_1(t,x)$, defined by the relation

$$\int_{w_1}^{v_0(x-Bt)} d\xi/g_1(\xi)=t \text{ for } \int_0^{v_0(x-Bt)} d\xi/g_1(\xi)\geq t \text{ and}$$

equals to zero for other $t,x$. It is easy to see that $w_1(0,x)\geq u_0(x)$, $L_1w_1\geq 0$ at the points where $w_1(t,x)$ is smooth and (1.3) is valid with $w_1$ instead of $u$. It follows with the aid of Theorem 1.1 that $u(t,x)\leq w_1(t,x)$. From the definition of $w_1(t,x)$ one has $w_1(t,x)=0$ for $H(v_0(x-Bt))\leq t$, in particular for $x-Bt \geq 1-\varepsilon$ and $1-x+Bt\leq[H^{-1}(t)/P]^{1/\alpha}$. To finish the proof it remains to choose such $t_0$ that two previous inequalities are valid for $0\leq t\leq t_0$.

**Remark 3.1.** If $A_1(t)=0$ for small $t$ then $u(t,x)\equiv 0$ for $x\geq 1$ and small $t$: it follows from the proof of Theorem 3.1.

**Theorem 3.2.** *Suppose the following conditions hold:*

*1)* $g(t,x,s) \leq Ks^n$ *for small* $s\geq 0$, $0 < n < 1$, $K = const > 0$;

*2) supp* $u_0(x)=[0,1]$, $u_0\geq P(1-x)^\alpha$, $1-\varepsilon\leq x\leq 1$, $\varepsilon>0$, $\alpha\geq 1$, $P = const > 0$;

*3)* $A_2(t)\equiv Bt-P^{-1/\alpha}[t(K(1-n)+\delta)]^{1/(\alpha(1-n))} \geq 0$ *for small t and some* $\delta>0$.

*Then there exists such* $t_0>0$ *that* $u(t,x)>0$ *for* $1+A_2(t)\geq x\geq 1$, $t\leq t_0$.

**Proof.** Let us choose such a continuous in $\mathbb{R}$ and smooth for $x \neq 1$ function $v_0(x)$ that $v_0(x)=[P(1-x)^\alpha]^{1-n}$ for $1-\varepsilon \leq x \leq 1$, $0 \leq v_0(x) \leq u_0(x)^{1-n}$ for other $x$. Let us consider the function

$$w_2(t,x)=\{[v_0(x-Bt)-(K(1-n)+\delta)t]_+\}^{1/(1-n)} \equiv \lambda^{1/(1-n)}.$$

It is easy to see that (1.3) is valid for $w_2(t,x)$. Further, at the points where $w_2(t,x)$ is smooth one obtains $L_1 w_2 = -(1-n)^{-1}\lambda^{n/(1-n)}[v'_0 B + (K(1-n) + \delta] + f_1(t,x,(w_2)_x) + B(1-n)^{-1}\lambda^{n/(1-n)}v'_0 + g(t,x,\lambda^{1/(1-n)}) \leq (1-n)^{-1}\lambda^{n/(1-n)} \times [f_1(t,x,(w_2)_x)/(w_2)_x - \delta] \leq (1-n)^{-1}\lambda^{n/(1-n)}[f'_1(t,x,(w_2)_x) - \delta] \leq 0$, if $\varepsilon$ is small and $v_0(x)$ is suitably chosen.

The function $w_2(t,x)$ is positive if $v_0(x-Bt) > (K(1-n) + \delta)t$, in particular for $1-\varepsilon+Bt \leq x \leq 1+Bt$ and for $x < 1 + Bt - P^{-1/\alpha}(K(1-n) + \delta)^{1/[\alpha(1-n)]}t^{1/[\alpha(1-n)]}$. Applying Theorem 1.1 on gets the required result.

**Remark 3.2** If $A_2(t)=0$ for small $t$ then $u(t,x)>0$ for $1-\varepsilon \leq x \leq 1$, $\varepsilon>0$ and small $t$: it follows from the proof of Theorem 3.2.

**Remark 3.3.** Suppose $g(t,x,s) \equiv K|s|^{n-1}s$, $0 < n < 1$, $K = \text{const} > 0$, $u_0(x)=P(1-x)^\alpha$ for $1-\varepsilon \leq x \leq 1$, $\alpha \geq 1$, $P = \text{const} > 0$, $\varepsilon>0$. then the following statements follow from Theorems 3.1, 3.2:

1) If $\alpha(1-n) > 1$ then $\sup_x \operatorname{supp} u(t,x) < 1$ for small $t > 0$.
2) If $\alpha(1-n) < 1$ then $\sup_x \operatorname{supp} u(t,x) > 1$ for small $t > 0$.
3) Suppose $\alpha(1-n) = 1$; then:

    a) If $B < KP^{-1/\alpha}(1-n)$ then $\sup_x \operatorname{supp} u(t,x) < 1$ for small $t > 0$;
    b) If $B > KP^{-1/\alpha}(1-n)$ then $\sup_x \operatorname{supp} u(t,x) > 1$ for small $t > 0$;

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Recibido: 9 de julio de 1992