ON THE BEHAVIOUR OF SOLUTIONS TO THE DIRICHLET PROBLEM FOR THE $p(x)$–LAPLACIAN WHEN $p(x)$ GOES TO 1 IN A SUBDOMAIN

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ABSTRACT. In this paper we prove a stability result for some classes of elliptic problems involving variable exponents. More precisely, we consider the Dirichlet problem for an elliptic equation in a domain Ω, which is the p–Laplacian equation, $-\text{div}(|\nabla u|^{p-2}\nabla u) = f$, in a subdomain Ω_1 of Ω and the Laplace equation, $-\Delta u = f$, in its complementary (that is, our equation involves a $p(x)$–Laplacian with a discontinuous exponent $p(x)$). We assume that the right-hand side f belongs to $L^\infty(\Omega)$. For this problem, we study the behaviour of the solutions as p goes to 1, showing that they converge to a function u, which is almost everywhere finite when the size of the datum f is small enough. Moreover, we prove that this u is a solution of a limit problem involving the 1–Laplacian operator in Ω_1. We also discuss uniqueness under a favorable geometry.

1. INTRODUCTION.

Let us consider a bounded domain (open and connected) Ω of $\mathbb{R}^N$ and Ω_1, Ω_2 a partition of it into two subdomains, that is, $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, with $\Gamma = \partial\Omega_1 \cap \Omega = \partial\Omega_2 \cap \Omega$; three cases for the relative locations of the domains are possible, namely: $\partial\Omega_1 \cap \partial\Omega = \emptyset$ and $\Gamma = \partial\Omega_1$ (that is, $\overline{\Omega_1}$ lies in the interior of Ω); $\partial\Omega_2 \cap \partial\Omega = \emptyset$ and $\Gamma = \partial\Omega_2$ ($\overline{\Omega_2}$ lies in the interior of Ω); $\partial\Omega_1 \cap \partial\Omega \neq \emptyset \neq \partial\Omega_2 \cap \partial\Omega$ and $\partial\Omega_1 \neq \Gamma \neq \partial\Omega_2$ (the closure of both subdomains do not lie in the interior of Ω). We will assume that all the involved domains are Lipschitz continuous and that the interface Γ is smooth and it satisfies $\mathcal{H}^{N-1}(\Gamma) < \infty$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure.

Our aim is to study the Dirichlet problem for an elliptic equation which is the 1–Laplacian equation in Ω_1 and the usual Laplace equation in Ω_2. To this end we introduce the variable exponent $l(x)$ defined by

$$l(x) = \begin{cases} 1, & \text{if } x \in \Omega_1; \\ 2, & \text{if } x \in \Omega_2; \end{cases} \tag{1.1}$$

and we consider the Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{l(x)-2}\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.2}$$

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where the right-hand side f belongs to $L^\infty(\Omega)$. Observe that the problem involves the $p(x)$–Laplacian, that is, the operator $\Delta_{l(x)}u = \operatorname{div}(|\nabla u|^{l(x)-2}\nabla u)$, with a discontinuous exponent $l(x)$. Common hypotheses in the literature of $p(x)$–Laplacian type equations are that $p(x)$ is a continuous function (with a modulus of continuity) and, moreover, the variable exponent $p(x)$ is usually assumed to be bounded away from 1. To the best of our knowledge, this paper is the first attempt to analyze a problem where the exponent $p(x)$ is discontinuous and equal to 1 in some part of the domain.

Note that the solution must solve $-\Delta_1 u = -\operatorname{div}(\frac{Du}{|Du|}) = f$ in Ω_1 and $-\Delta u = f$ in Ω_2. Hence, to handle this equation, (1.2), we have to give a notion of solution and then consider a suitable functional framework. Adapting the definition of solution for the 1–Laplacian equation given in [4], we look for a solution to problem (1.2) in the set of measurable functions in Ω which belongs to $BV(\Omega)$ and to $H^1(\Omega_2)$. Obviously our notion of solution needs to give sense to the quotient $\frac{Du}{|Du|}$ in Ω_1, where, in general, Du is not a function but a Radon measure, while it is a solution of the Laplace equation with datum f in Ω_2. To this aim our definition of solution (see Section 4) is based on a vector field $z \in L^\infty(\Omega_1;\mathbb{R}^N)$ satisfying $\|z\|_\infty \le 1$, $-\operatorname{div} z = f$ in $\mathcal{D}'(\Omega_1)$ and $(z, Du) = |Du|$. Observe that, formally, $\|z\|_\infty \le 1$ and $(z, Du) = |Du|$ imply $z = \frac{Du}{|Du|}$ in Ω_1. To give sense to (z, Du) we use the theory of L^∞–divergence–measure vector fields due to Anzellotti [7] and Chen–Frid [8, 9, 10].

In this paper, for small right-hand side f, we prove the existence and uniqueness of such a solution to problem (1.2), as well as we present explicit examples. Our main result reads as follows:

Theorem 1.1. *There exists a constant C depending on the domains, Ω_1, Ω_2, such that, if $\|f\|_{L^\infty(\Omega)} < C$, then there is, at least, a solution to problem* (1.2).

Moreover, if $\mathcal{H}^{N-1}(\partial\Omega_2 \cap \partial\Omega) \neq 0$, then the problem has a unique solution in the sense that two solutions u_1, u_2 must agree in Ω_2.

Remark 1.2. The fact that $l(x) = 2$ in Ω_2 is used only to simplify the presentation of our results. From the proofs it is clear that we can consider more general exponents of the form

$$l(x) = \begin{cases} 1\,, & \text{if } x \in \Omega_1\,; \\ q(x)\,, & \text{if } x \in \Omega_2\,; \end{cases}$$

as long as $q(x)$ is greater than 1.

Our strategy to find a solution to problem (1.2) is to consider a limit of a "sequence" of approximating problems involving the so-called $p(x)-$laplacian operator. This means that we consider the variable exponent $p : \Omega \to [p, 2]$ which is the measurable function defined by

$$p(x) = \begin{cases} p\,, & \text{if } x \in \Omega_1 : \\ 2\,, & \text{if } x \in \Omega_2\,; \end{cases} \tag{1.3}$$

with $1 < p < 2$, and we study the behaviour, as p goes to 1, of the solutions u_p to the Dirichlet problem

$$\begin{cases} -\operatorname{div}\,(|\nabla u_p|^{p(x)-2}\nabla u_p) = f(x)\,, & \text{in } \Omega\,; \\ u_p = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

It is well–known (see [13, 17]) that a weak solution u_p to the Dirichlet problem (1.4) is a function belonging to the so-called generalized Sobolev space $W_0^{1,p(x)}(\Omega)$

such that the following identity holds true

$$\int_\Omega |\nabla u_p|^{p(x)-2}\nabla u_p \cdot \nabla v = \int_\Omega f v\,, \qquad v \in W_0^{1,p(x)}(\Omega). \tag{1.5}$$

We prove that the approximate solutions u_p converge to a BV–function u that turns out to be a solution to equation (1.2) according to our definition (given in Section 4).

A similar approach has been used to study the problem (1.4) (and its limit as $p \to 1$), when the variable exponent $p(x)$ is constant in the whole Ω, i.e. $p(x) = p$. In such a case there is no stability result for solutions to p–Laplacian equation as p goes to 1, in the sense that solutions of the p–Laplacian equation converge to a function that can be infinity on a set of positive measure when the datum f is large enough (see [15, 16] for particular data and [19, 20] for more general data, see also [12]). Furthermore, there is no uniqueness of the solution of the limit problem. On the contrary, in [18] we consider an anisotropic operator that is as the q–Laplacian ($q > 1$) in just one direction to find a unique solution to the limit equation for each datum f in a suitable Lebesgue space (regardless of the size of f). Moreover, this unique solution is the limit, as p goes to 1, of solutions to anisotropic equations which are equal to the p–Laplacian in some directions and as the q–Laplacian in the others. In view of these results one may wonder if the fact that the equation in (1.2) contains a part in which the usual Laplacian occurs, plays a role in the existence of solutions or in their uniqueness. As we can see from our main result, Theorem 1.1, existence is also restricted to small data f (as in the case that $p(x) = p$ is constant in Ω), but uniqueness holds (in a weak sense) under a favorable position of Ω_2 in Ω (as in the anisotropic case).

Elliptic problems involving variable exponents became popular a few years ago in relation to applications to elasticity and electrorheological fluids. Meanwhile, the underlying functional analytical tools have been extensively developed and new applications, for example to image processing, have kept the subject as the focus of an intensive research activity. For general references on the $p(x)$-Laplacian we refer to [11] and [14] that include a large bibliography, and [17]. The delicate regularity properties of $p(x)$-harmonic functions have been established in [1] and [2].

The rest of this paper is organized as follows: in Section 2 we study our functional setting and we recall a few properties of the generalized Sobolev spaces; in Section 3, we begin by studying the asymptotic behaviour of the sequence u_p of approximate solutions to problem (1.4), as $p \to 1$, we get a limit function u and a vector field z which is the weak limit of $|\nabla u_p|^{p-2}\nabla u_p$ in Ω_1; in Section 4 we introduce our notion of solution and we prove the existence result stated in Theorem 1.1, which consists in proving that the limit function u above is a solution to (1.2), finally we prove the uniqueness result and compute explicit examples.

2. Preliminary results.

Throughout this paper, we will denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}^N$ and by $\mathcal{H}^{N-1}(E)$ its $N-1$–dimensional Hausdorff measure. We denote by Ω an open bounded and connected subset of $\mathbb{R}^N$ and Ω_1, Ω_2 are subdomains of Ω satisfying $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, with $\Gamma = \partial\Omega_1 \cap \Omega = \partial\Omega_2 \cap \Omega$. We assume that Ω, Ω_1 and Ω_2 are Lipschitz continuous and we denote by ν_i the unit outward normal to Ω_i, $i = 1, 2$. We also assume that the interface Γ is smooth and satisfies $\mathcal{H}^{N-1}(\Gamma) < \infty$.

Some remarks on our functional setting are in order. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. It is well–known that the functional

$$u \mapsto \int_{\Omega_1} |Du| + \int_{\partial\Omega_1 \cap \partial\Omega} |u| \, d\mathcal{H}^{N-1} + \int_\Gamma |u - g| \, d\mathcal{H}^{N-1}, \quad g \in L^1(\Gamma), \tag{2.6}$$

is lower semi–continuous in $BV(\Omega_1)$ with respect to the $L^1(\Omega_1)$–convergence.

Next we will recall some well-known facts about the generalized Lebesgue and Sobolev spaces (see for example [13, 17]). Let us consider a bounded domain Ω of $\mathbb{R}^N$ and let $p : \Omega \to (1, +\infty)$ be a measurable function. The so-called "generalized Lebesgue space" $L^{p(x)}(\Omega)$ is the class of all measurable functions u on Ω such that

$$\int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} < +\infty,$$

for some $\lambda > 0$. If the variable exponent $p(x)$ satisfies the condition

$$1 < p^- = \text{ess inf}_{x\in\Omega} \, p(x) \le \text{ess sup}_{x\in\Omega} \, p(x) = p^+ < +\infty, \tag{2.7}$$

then the space $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_p = \inf \left\{ \lambda > 0 \ : \ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \le 1 \right\}. \tag{2.8}$$

Since $p(x) > 1$, the space $L^{p(x)}(\Omega)$ is reflexive and its dual space is $L^{p'(x)}(\Omega)$, where

$$\frac{1}{p'(x)} = 1 - \frac{1}{p(x)}, \quad \text{a.e. } x \in \Omega.$$

If $p_1(x), p_2(x)$ are measurable functions which satisfy condition (2.7), then

$$L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega), \tag{2.9}$$

if, and only if, $p_1(x) \le p_2(x)$ a.e. in Ω; in this case the embedding is continuous.

The generalized Hölder inequality holds

$$\int_\Omega |f(x)g(x)| \le 2\|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^{p'(x)}(\Omega)}, \tag{2.10}$$

for any $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$. The generalized Sobolev space $W^{1,p(x)}(\Omega)$ with $p(x)$ satisfying (2.7) is the set of measurable functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$, i.e. $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \ : \ |\nabla u| \in L^{p(x)}(\Omega)\}$. It is a reflexive and separable Banach space equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

As a consequence of the embedding (2.9), if $p_1(x), p_2(x)$ are measurable functions satisfying (2.7) and $p_1(x) \le p_2(x)$ a.e. in Ω, then $W^{1,p_2(x)}(\Omega) \subset W^{1,p_1(x)}(\Omega)$, continuously. Finally $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Sobolev's embedding theorems are well-known (cf. [14]) when $p(x)$ is a continuous function. Nevertheless, in our framework, these results do not apply and we have to prove the corresponding Sobolev embedding theorem.

Proposition 2.1. *Let $1 < p < 2$ and let $p(x)$ be the following step function*

$$p(x) = \begin{cases} p & \text{if } x \in \Omega_1, \\ 2 & \text{if } x \in \Omega_2, \end{cases} \tag{2.11}$$

then there exist a positive constant c_1 such that

$$\frac{1}{2}\left[\left(\int_{\Omega_1}|v|^p\right)^{1/p} + \left(\int_{\Omega_2}|v|^2\right)^{1/2}\right] \le ||v||_{L^{p(x)}(\Omega)}$$
$$\le c_1\left[\left(\int_{\Omega_1}|v|^p\right)^{1/p} + \left(\int_{\Omega_2}|v|^2\right)^{1/2}\right]$$

for every $v \in L^{p(x)}(\Omega)$.

Remark 2.2. Note that one can choose c_1 independent of p as long as p remains bounded. For our purposes we let $p \in (1, 2]$.

Proof. Let v be a function belonging to $L^{p(x)}(\Omega)$ and let

$$\mu = \left(\int_{\Omega_1}|v|^p\right)^{1/p} + \left(\int_{\Omega_2}|v|^2\right)^{1/2}.$$

We have

$$\mu^p \ge \int_{\Omega_1}|v|^p, \quad \mu^2 \ge \int_{\Omega_2}|v|^2$$

and then, for every $\alpha > 0$,

$$\int_{\Omega_1}\frac{|v|^p}{(\alpha\mu)^p} + \int_{\Omega_2}\frac{|v|^2}{(\alpha\mu)^2} \le \frac{1}{\alpha^p} + \frac{1}{\alpha^2}.$$

Given $\bar{\alpha} > 0$ such that $\frac{1}{\bar{\alpha}^p} + \frac{1}{\bar{\alpha}^2} = 1$, by definition of norm we obtain

$$||v||_{L^{p(x)}} \le \bar{\alpha}\mu = \bar{\alpha}\left[\left(\int_{\Omega_1}|v|^p\right)^{1/p} + \left(\int_{\Omega_2}|v|^2\right)^{1/2}\right]. \tag{2.12}$$

On the other hand, for every $\beta > 0$,

$$\int_{\Omega_1}\frac{|v|^p}{\beta^p} \le \int_{\Omega_1}\frac{|v|^p}{\beta^p} + \int_{\Omega_2}\frac{|v|^2}{\beta^2} \tag{2.13}$$

which implies

$$\left(\int_{\Omega_1}|v|^p\right)^{1/p} \le ||v||_{L^{p(x)}(\Omega)}.$$

Analogously,

$$\left(\int_{\Omega_2}|v|^2\right)^{1/2} \le ||v||_{L^{p(x)}(\Omega)},$$

and then adding the two inequalities we get

$$\left(\int_{\Omega_1}|v|^p\right)^{1/p} + \left(\int_{\Omega_2}|v|^2\right)^{1/2} \le 2||v||_{L^{p(x)}(\Omega)}. \tag{2.14}$$

■

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ smooth enough. If $\Gamma \subset \partial\Omega$ satisfies $\mathcal{H}^{N-1}(\Gamma) \neq 0$, then for each $p \in (1, +\infty)$ there exists a constant $C_p > 0$ such that*

$$\left(\int_\Omega |v|^p\right)^{1/p} + \left(\int_\Omega |\nabla v|^p\right)^{1/p} \le C_p \left[\int_\Gamma |v| + \left(\int_\Omega |\nabla v|^p\right)^{1/p}\right] \tag{2.15}$$

for all $v \in W^{1,p}(\Omega)$. Moreover, these constants do not degenerate as p goes to 1, and so a constant can be chosen independently on p (when p is close to 1).

Proof. The existence, for each $p \in (1, +\infty)$, of a constant C_p satisfying (2.15) is well–known (see for instance [21], pp 247–248).

Let us prove the second assertion. Assume to get a contradiction that constants C_p degenerate as p goes to 1. Then there exist $v_p \in W^{1,p}(\Omega)$ satisfying

(1) $\displaystyle\left(\int_\Omega |v_p|^p\right)^{1/p} + \left(\int_\Omega |\nabla v_p|^p\right)^{1/p} = 1.$

(2) $\displaystyle\lim_{p\to 1} \int_\Gamma |v_p| + \left(\int_\Omega |\nabla v_p|^p\right)^{1/p} = 0.$

Observe that it follows that

$$\lim_{p\to 1} \int_\Omega |v_p|^p = 1\,. \tag{2.16}$$

Since, by Hölder's inequality,

$$\int_\Omega |v_p| + \int_\Omega |\nabla v_p| \le |\Omega|^{1-(1/p)} \left(\int_\Omega |v_p|^p\right)^{1/p} + |\Omega|^{1-(1/p)} \left(\int_\Omega |\nabla v_p|^p\right)^{1/p} \le 1 + |\Omega|\,,$$

we may find $v \in BV(\Omega)$ such that, up to "subsequences",

$$v_p \rightharpoonup v\,, \quad \text{weakly* in } BV(\Omega)\,.$$

As a consequence, we have

$$v_p \to v\,, \quad \text{strongly in } L^\alpha(\Omega), \text{ for every } 1 < \alpha < \frac{N}{N-1} \tag{2.17}$$

and

$$\int_\Omega |Dv| \le \liminf_{p\to 1} \int_\Omega |\nabla v_p| \le \liminf_{p\to 1} |\Omega|^{1-(1/p)} \left(\int_\Omega |\nabla v_p|^p\right)^{1/p} = 0\,.$$

Hence, $Dv = 0$ in Ω and so v is constant in Ω. Since

$$\int_\Gamma |v| \le \liminf_{p\to 1} \int_\Omega |v_p| \le \liminf_{p\to 1} |\Gamma|^{1-(1/p)} \left(\int_\Gamma |v_p|^p\right)^{1/p} = 0\,,$$

it follows that this constant must be 0. Therefore, $v = 0$. So that (2.17) becomes

$$v_p \to 0\,, \quad \text{strongly in } L^{\frac{N}{N-1}}(\Omega)\,.$$

It implies, for p close to 1, that

$$\int_\Omega |v_p|^p \le |\Omega|^{1-\frac{(N-1)p}{N}} \left(\int_\Omega |v_p|^{\frac{N}{N-1}}\right)^{\frac{(N-1)p}{N}} \le |\Omega| \left(\int_\Omega |v_p|^{\frac{N}{N-1}}\right)^{\frac{(N-1)p}{N}}\,.$$

Thus, $\displaystyle\lim_{p\to 1} \int_\Omega |v_p|^p = 0$ and it contradicts (2.16). ∎

Proposition 2.4. *Let $p(x)$ be the function defined in Proposition 2.1. Then there exists a positive constant A_p such that*

$$||v||_{L^{p(x)}(\Omega)} \leq A_p ||\nabla v||_{L^{p(x)}(\Omega)} \tag{2.18}$$

for every $v \in W_0^{1,p(x)}(\Omega)$.

REMARK 2.5. The constant A_p appearing in (2.18) can be chosen independent of p as long as p is bounded, say $p \in (1, 2]$. Indeed, one has to observe that in the following proof the constant depends essentially on the constants appearing in the previous proposition and on the best constant for the Sobolev trace embedding $W^{1,p}(D) \hookrightarrow L^p(\partial D)$. We explicitely remark that these constants do not degenerate as $p \to 1$ (see [6] for the trace embedding). In the sequel, we will denote

$$A = \limsup_{p\to 1} A_p \,.$$

Proof. Let us distinguish three cases according to the relative location of the subdomains Ω_1 and Ω_2.

FIRST CASE: $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, and $\Gamma = \partial\Omega_1$. In this case $\partial\Omega_1 \cap \partial\Omega = \emptyset$, that is $\overline{\Omega}_1$ lies in the interior of Ω. Let $v \in W_0^{1,p(x)}(\Omega)$, then

$$\left(\int_{\Omega_2} |v|^2\right)^{1/2} \leq C \left(\int_{\Omega_2} |\nabla v|^2\right)^{1/2}, \quad \int_\Gamma |v|^2 \leq C \int_{\Omega_2} |\nabla v|^2. \tag{2.19}$$

Moreover

$$\left(\int_\Gamma |v|^p\right)^{1/p} \leq |\Gamma|^{(1/p)-(1/2)} \left(\int_\Gamma |v|^2\right)^{1/2} \leq C \left(\int_{\Omega_2} |\nabla v|^2\right)^{1/2},$$

and

$$\begin{aligned} \left(\int_{\Omega_1} |v|^p\right)^{1/p} &\leq C \left(\int_\Gamma |v|^p\right)^{1/p} + C \left(\int_{\Omega_1} |\nabla v|^p\right)^{1/p} \\ &\leq C \left(\int_{\Omega_2} |\nabla v|^2\right)^{1/2} + C \left(\int_{\Omega_1} |\nabla v|^p\right)^{1/p}. \end{aligned} \tag{2.20}$$

Adding (2.19) and (2.20) we obtain

$$\left(\int_{\Omega_1} |v|^p\right)^{1/p} + \left(\int_{\Omega_2} |v|^2\right)^{1/2} \leq C \left(\int_{\Omega_1} |\nabla v|^p\right)^{1/p} + C \left(\int_{\Omega_2} |\nabla v|^2\right)^{1/2}$$

and the claim is proved.

SECOND CASE: $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, and $\Gamma = \partial\Omega_2$. In this case $\overline{\Omega}_2$ lies in the interior of Ω. We have

$$\left(\int_{\Omega_1} |v|^p\right)^{1/p} \leq C \left(\int_{\Omega_1} |\nabla v|^p\right)^{1/p}, \quad \int_\Gamma |v|^p \leq C \int_{\Omega_1} |\nabla v|^p, \tag{2.21}$$

and

$$\left(\int_{\Omega_2} |v|^2\right)^{1/2} \leq C \left(\int_\Gamma |v|\right) + C \left(\int_{\Omega_2} |\nabla v|^2\right)^{1/2}.$$

Since

$$\int_\Gamma |v| \leq |\Gamma|^{1-(1/p)} \left(\int_\Gamma |v|^p\right)^{1/p} \leq C \left(\int_{\Omega_1} |\nabla v|^p\right)^{1/p},$$

then

$$\left(\int_{\Omega_2} |v|^2\right)^{1/2} \le C\left(\int_{\Omega_1} |\nabla v|^p\right)^{1/p} + C\left(\int_{\Omega_2} |\nabla v|^2\right)^{1/2}, \tag{2.22}$$

and the claim follows adding (2.21) and (2.22).

THIRD CASE: $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, and $\partial\Omega_1 \cap \partial\Omega \neq \emptyset \neq \partial\Omega_2 \cap \partial\Omega$. In this last case the claim follows by adding the following two inequalities

$$\left(\int_{\Omega_2} |v|^2\right)^{1/2} \le C\left(\int_{\Omega_2} |\nabla v|^2\right)^{1/2}, \quad \left(\int_{\Omega_1} |v|^p\right)^{1/p} \le C\left(\int_{\Omega_1} |\nabla v|^p\right)^{1/p}.$$

This ends the proof. ■

We conclude this section with the Green formula for a vector field $F \in L^2(\Omega; \mathbb{R}^N)$ such that $\operatorname{div} F \in L^{2N/(N+2)}(\Omega)$. This vector field defines a functional

$$[F, \nu] \,:\, W^{1/2,2}(\partial\Omega) \to \mathbb{R}$$

by the expression

$$[F, \nu](u) = \int_\Omega w \, \operatorname{div} F + \int_\Omega \nabla w \cdot F\,, \tag{2.23}$$

where $w \in H^1(\Omega)$ verifies $w\big|_{\partial\Omega} = u$. This definition does not depend on the function w we have chosen. Indeed, suppose that $w_1, w_2 \in H^1(\Omega)$ satisfy $w_i\big|_{\partial\Omega} = u$, $i = 1, 2$. Then $w_1 - w_2 \in H_0^1(\Omega)$ and so

$$\int_\Omega (w_1 - w_2) \, \operatorname{div} F + \int_\Omega \nabla(w_1 - w_2) \cdot F = 0\,.$$

We point out that the functional defined by (2.23) is continuous. To see it consider the function $w \in H^1(\Omega)$ whose trace is u and minimizes the norm $\|w\|_{H^1(\Omega)}$. Now the desired continuity is as a consequence of the following computations

$$\begin{aligned}
\big|[F, \nu](u)\big| &\le \int_\Omega |w|\,|\operatorname{div} F| + \int_\Omega |\nabla w|\,|F| \\
&\le \Big[\Big(\int_\Omega |\operatorname{div} F|^{2N/(N+2)}\Big)^{(N+2)/2N} + \Big(\int_\Omega |F|^2\Big)^{1/2}\Big] \|w\|_{H^1(\Omega)} \\
&\le C(F, N, \Omega)\, \|u\|_{W^{1/2,2}(\partial\Omega)}\,.
\end{aligned}$$

Hence, $[F, \nu] \in (W^{1/2,2}(\partial\Omega))^*$ and it is an extension of a distribution in $\mathbb{R}^N$, namely the one given by

$$[F, \nu](\varphi) = \int_\Omega \varphi \, \operatorname{div} F + \int_\Omega \nabla\varphi \cdot F \quad \varphi \in C_0^\infty(\mathbb{R}^N)\,.$$

The support of this distribution lies on $\partial\Omega$. From now on, we will write

$$\int_{\partial\Omega} w\,[F, \nu]\, d\mathcal{H}^{N-1} = [F, \nu](w\big|_{\partial\Omega})\,, \quad w \in H^1(\Omega)\,.$$

With this notation, (2.23) becomes

$$\int_{\partial\Omega} w\,[F, \nu]\, d\mathcal{H}^{N-1} = \int_\Omega w \, \operatorname{div} F + \int_\Omega \nabla w \cdot F\,, \quad w \in H^1(\Omega)\,,$$

and it is in this sense that Green's formula holds true. When the trace $w\big|_{\partial\Omega}$ vanishes on $\partial\Omega\backslash\Gamma$, we will write

$$\int_\Gamma w\,[F,\nu]\,d\mathcal{H}^{N-1}\,,$$

instead of

$$\int_{\partial\Omega} w\,[F,\nu]\,d\mathcal{H}^{N-1}.$$

3. Convergence of u_p as p goes to 1.

In this section we want to study the behaviour as p goes to 1 of the solution u_p to problem (1.4). Our first step is to prove that such a problem has a unique weak solution.

Proposition 3.1. *For every $f \in L^\infty(\Omega)$, problem* (1.4) *has a unique weak solution.*

Proof. Consider the functional in $W_0^{1,p(x)}(\Omega)$ defined by

$$I[u] = \frac{1}{p}\int_{\Omega_1} |\nabla u|^p + \frac{1}{2}\int_{\Omega_2} |\nabla u|^2 - \int_\Omega fu\,.$$

It is straightforward that this functional is strictly convex and lower semicontinuous with respect to the weak convergence. Moreover, it is bounded from below, since it is bigger (up to a constant) than the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega fu.$$

Therefore I has a unique minimum in $W_0^{1,p(x)}(\Omega)$, which is the unique weak solution to (1.4). ■

In what follows, with abuse of notation, we will say that u_p is a sequence and we will consider subsequences of it, as p goes to 1.

Proposition 3.2. *If $f \in L^\infty(\Omega)$ and*

$$2A\max\{\sqrt{|\Omega_2|},1\}\|f\|_{L^\infty(\Omega)} < 1\,, \tag{3.24}$$

then there exists a function $u \in BV(\Omega)\cap H^1(\Omega_2)$ and a vector field $z \in L^\infty(\Omega_1;\mathbb{R}^N)$ such that, up to subsequences,

$$u_p \to u \quad \textit{strongly in } L^1(\Omega) \textit{and a.e. in } \Omega, \tag{3.25}$$

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \textit{weakly in } L^q(\Omega_1;\mathbb{R}^N) \textit{ for all } 1 \le q < +\infty\,, \tag{3.26}$$

with

$$\|z\|_{L^\infty(\Omega_1;\mathbb{R}^N)} \le 1. \tag{3.27}$$

Proof. Step 1. A priori estimates. Let u_p be the weak solution to problem (1.4). We want to prove an a priori estimate for the gradients of u_p in the space $L^{p(x)}(\Omega;\mathbb{R}^N)$. Let us denote λ_p the norm of ∇u_p in such a space. Then it holds

$$\int_\Omega \left|\frac{\nabla u_p}{\lambda_p}\right|^{p(x)} = 1\,. \tag{3.28}$$

Choosing u_p/λ_p as test function in (1.5), we get

$$\frac{1}{\lambda_p}\int_\Omega |\nabla u_p|^{p(x)} = \frac{1}{\lambda_p}\int_\Omega f u_p. \tag{3.29}$$

By generalized Holder's inequality (2.10) and inequality (2.18), we get

$$\begin{aligned}\frac{1}{\lambda_p}\int_\Omega |\nabla u_p|^{p(x)} &\le \frac{1}{\lambda_p}\|f\|_{L^\infty(\Omega)}\int_\Omega |u_p| \\ &\le \frac{2}{\lambda_p}\|f\|_{L^\infty(\Omega)}\|1\|_{L^{p'(x)}(\Omega)}\|u_p\|_{L^{p(x)}(\Omega)} \\ &\le \frac{2A_p}{\lambda_p}\|f\|_{L^\infty(\Omega)}\|1\|_{L^{p'(x)}(\Omega)}\|\nabla u_p\|_{L^{p(x)}(\Omega;\mathbb{R}^N)} \\ &\le 2A_p\|f\|_{L^\infty(\Omega)}\|1\|_{L^{p'(x)}(\Omega)}\,.\end{aligned}$$

On the other hand, by (3.28), we can estimate the left-hand side of (3.29) in the following way

$$\frac{1}{\lambda_p}\int_\Omega |\nabla u_p|^{p(x)} = \lambda_p^{p-1}\int_{\Omega_1}\left|\frac{\nabla u_p}{\lambda_p}\right|^p + \lambda_p\int_{\Omega_2}\left|\frac{\nabla u_p}{\lambda_p}\right|^2 \ge \min\{\lambda_p^{p-1},\lambda_p\}. \tag{3.30}$$

Combining (3.29) and (3.30), we obtain

$$\min\{\lambda_p^{p-1},\lambda_p\} \le 2A_p\|f\|_{L^\infty(\Omega)}\|1\|_{L^{p'(x)}(\Omega)}\,. \tag{3.31}$$

We claim that

$$\limsup_{p\to 1}\|1\|_{L^{p'(x)}(\Omega)} = \max\{\sqrt{|\Omega_2|},1\}\,. \tag{3.32}$$

Indeed, denoted $\tau_p = \|1\|_{L^{p'(x)}(\Omega)}$, by definition of norm (2.8), we get

$$\frac{|\Omega_1|}{\tau_p^{p'}} + \frac{|\Omega_2|}{\tau_p^{2'}} = 1\,.$$

Let τ a limit point of τ_p as p goes to 1. The inequality

$$\frac{|\Omega_1|}{\tau_p^{p'}} \le 1$$

implies that $1 \le \tau < +\infty$. Moreover if $\tau > 1$, we deduce that $\lim_{p\to 1}\tau_p = \sqrt{|\Omega_2|}$. This yields (3.32).

Now, since (3.24) holds, then $\min\{\lambda_p^{p-1},\lambda_p\} < 1$, for $p \le p_0$, and hence, from the definition of λ_p, we obtain the following a priori estimate

$$\|\nabla u_p\|_{L^{p(x)}(\Omega;\mathbb{R}^N)} < 1. \tag{3.33}$$

Step 2. Convergence of u_p and of the gradients. By classical Holder's inequality, Proposition 2.1 and the a priori estimate (3.33), we deduce

$$\begin{aligned}\int_\Omega |\nabla u_p| &\le |\Omega_1|^{1-\frac{1}{p}}\left(\int_{\Omega_1}|\nabla u_p|^p\right)^{\frac{1}{p}} + |\Omega_2|^{\frac{1}{2}}\left(\int_{\Omega_2}|\nabla u_p|^2\right)^{\frac{1}{2}} \\ &\le C\max\{|\Omega_1|^{1-\frac{1}{p}},|\Omega_2|^{\frac{1}{2}}\}\|\nabla u_p\|_{L^{p(x)}(\Omega;\mathbb{R}^N)} \\ &\le C(1+1+|\Omega_2|^{\frac{1}{2}})\end{aligned} \tag{3.34}$$

for any $p \le p_0$. In an analogous way, we obtain

$$\left(\int_\Omega |\nabla u_p|^2\right)^{1/2} \le c_p\|\nabla u_p\|_{L^{p(x)}(\Omega;\mathbb{R}^N)} \le c_1 + 1\,. \tag{3.35}$$

This implies that there exist a subsequence still denoted by u_p and a function $u \in BV(\Omega) \cap H^1(\Omega_2)$ such that

$$\begin{cases} \nabla u_p \rightharpoonup Du & \text{weakly* in the sense of measures in } \Omega_1\,, \\ \nabla u_p \rightharpoonup \nabla u & \text{weakly in } L^2(\Omega_2;\mathbb{R}^N)\,, \\ u_p \to u & \text{strongly in } L^1(\Omega)\,, \\ u_p \to u & \text{a.e. in } \Omega\,. \end{cases}$$

Now we prove (3.26). For every q such that $1 \le q < p'$, by Proposition 2.1 and our a priori estimate (3.33), we have

$$\begin{aligned} \int_{\Omega_1} |\nabla u_p|^{(p-1)q} &\le \frac{1}{c_2}|\Omega_1|^{1-\frac{(p-1)q}{p}} \|\nabla u\|_{L^{p(x)}(\Omega;\mathbb{R}^N)} \\ &\le \frac{1}{c_2}|\Omega_1|^{1-\frac{(p-1)q}{p}} \le \frac{1}{c_2}(|\Omega_1|+1)\,, \end{aligned} \tag{3.36}$$

for any $p \le p_1$.

It yields that, for any q fixed, the sequence $|\nabla u_p|^{p-2}\nabla u_p$ is bounded in the space $L^q(\Omega_1;\mathbb{R}^N)$ and then there exists $z_q \in L^q(\Omega_1;\mathbb{R}^N)$ such that, up to subsequences,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z_q \quad \text{in } L^q(\Omega_1;\mathbb{R}^N) \quad \text{for all } 1 \le q < +\infty\,.$$

Moreover, by a diagonal argument we can find a limit z that does not depend on q, that is,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \text{in } L^q(\Omega_1;\mathbb{R}^N) \quad \text{for all } 1 \le q < +\infty\,. \tag{3.37}$$

Now by (3.36) we deduce

$$\||\nabla u_p|^{p-2}\nabla u_p\|_{L^q(\Omega_1;\mathbb{R}^N)} \le \left(\frac{1}{c_2}(|\Omega_1|+1)\right)^{1/q}.$$

Therefore, by the lower semicontinuity of the norm, we have

$$\|z\|_{L^q(\Omega_1;\mathbb{R}^N)} \le \left(\frac{1}{c_2}(|\Omega_1|+1)\right)^{1/q} \quad \text{for all } 1 \le q < +\infty\,. \tag{3.38}$$

Hence, we obtain that z belongs to $L^\infty(\Omega;\mathbb{R}^N)$ and

$$\|z\|_{L^\infty(\Omega_1;\mathbb{R}^N)} = \lim_{q\to+\infty} \|z\|_{L^q(\Omega_1;\mathbb{R}^N)} \le 1\,,$$

that is, (3.27) is proved. ■

4. Main results.

In this Section, we will study the limit problem. For the sake of simplicity, in what follows we will deal with the case $\partial\Omega_1 \cap \partial\Omega \ne \emptyset \ne \partial\Omega_2 \cap \partial\Omega$, the existence of solutions in the other cases can be handled analogously using Proposition 2.4. Note that for our uniqueness result we need a condition on the location of the subdomains, namely, $\mathcal{H}^{N-1}(\partial\Omega_2 \cap \partial\Omega) \ne 0$.

We begin by introducing the definition of solution to the limit problem

$$\begin{cases} -\Delta_{l(x)} u = f\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases} \tag{4.39}$$

where

$$l(x) = \begin{cases} 1\,, & \text{if } x \in \Omega_1\,; \\ 2\,, & \text{if } x \in \Omega_2\,. \end{cases} \tag{4.40}$$

Let u be a function belonging to $BV(\Omega)$ and let z be a vector field belonging to $L^\infty(\Omega_1;\mathbb{R}^N)$ such that $\operatorname{div} z$, in the sense of distributions, belongs to $L^\infty(\Omega_1)$, i.e.

$$< \operatorname{div} z, \varphi >= -\int_{\Omega_1} z\cdot\nabla\varphi,$$

for all $\varphi \in C_0^\infty(\Omega_1)$. Then we define the distribution $(z, Du) : C_0^\infty(\Omega_1) \to \mathbb{R}$ by

$$< (z, Du), \varphi >= -\int_{\Omega_1} u\, z\cdot\nabla\varphi - \int_{\Omega_1} u\,\varphi \operatorname{div} z\,, \tag{4.41}$$

for every $\varphi \in C_0^\infty(\Omega_1)$. Since $u \in BV(\Omega) \subset L^{N',1}(\Omega_1)$, $\varphi \in C_0^\infty(\Omega_1)$, $z \in L^\infty(\Omega_1;\mathbb{R}^N)$ and $\operatorname{div} z \in L^\infty(\Omega_1)$, all terms in (4.41) make sense.

Moreover, as in [7] (see also [4] pp. 126–127) we may define the weak trace of the exterior normal component of z on $\partial\Omega_1$, which will be denoted by $[z, \nu_1]$. Then a generalized Green's formula holds, relating the function $[z, \nu_1]$ and the measure (z, Du):

$$\int_{\Omega_1} u \operatorname{div} z + \int_{\Omega_1} (z, Du) = \int_{\partial\Omega_1} [z, \nu_1]u \; d\mathcal{H}^{N-1}. \tag{4.42}$$

We point out that $u \in BV(\Omega)$ and so jump discontinuities are allowed. Thus, function u may have a jump discontinuity on Γ and so different traces on Γ. To distinguish these different values, we will write the restriction of u to Ω_i as $u_{|\Omega_i}$, $i = 1, 2$.

Definition 4.1. *We say that a function $u \in BV(\Omega) \cap H^1(\Omega_2)$ is a solution to* (4.39) *if the following conditions hold:*

There exists $z \in L^\infty(\Omega_1;\mathbb{R}^N)$ satisfying

$$||z||_{L^\infty(\Omega_1;\mathbb{R}^N)} \le 1\,; \tag{4.43}$$

$$-\operatorname{div} z = f \quad \text{in } \mathcal{D}'(\Omega_1), \qquad -\Delta u = f \quad \text{in } \mathcal{D}'(\Omega_2); \tag{4.44}$$

$$\int_{\Omega_1} z\cdot\nabla\varphi + \int_{\Omega_2} \nabla u\cdot\nabla\varphi = \int_\Omega f\varphi,\ \forall\varphi \in C_0^\infty(\Omega); \tag{4.45}$$

$$(z, Du) = |Du| \quad \text{as measures in } \Omega_1; \tag{4.46}$$

$$[z, \nu_1] = [\nabla u_{|\Omega_2}, \nu_1] \quad \mathcal{H}^{N-1}\text{–}a.e.\ on\ \Gamma; \tag{4.47}$$

$$u_{|\Omega_2} = 0 \quad \mathcal{H}^{N-1}\text{–}a.e.\ on\ \partial\Omega_2 \cap \partial\Omega; \tag{4.48}$$

$$[z, \nu_1] \in \operatorname{sign}(-u_{|\Omega_1}) \quad \mathcal{H}^{N-1}\text{–}a.e.\ on\ \partial\Omega_1 \cap \partial\Omega; \tag{4.49}$$

$$[z, \nu_1] \in \operatorname{sign}(u_{|\Omega_2} - u_{|\Omega_1}) \quad \mathcal{H}^{N-1}\text{–}a.e.\ on\ \Gamma. \tag{4.50}$$

Remark 4.2. The following weak formulation holds: if $u \in BV(\Omega) \cap H^1(\Omega_2)$ is a solution to (4.39) in the sense of Definition 4.1, then

$$\begin{aligned} &\int_{\Omega_1} |Du| - \int_{\Omega_1} (z, Dw) + \int_{\partial\Omega_1\cap\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \\ &\int_{\partial\Omega_1\cap\partial\Omega} w[z, \nu_1]\, d\mathcal{H}^{N-1} + \int_{\Omega_2} \nabla u\cdot\nabla(u - w) = \int_\Omega f(u - w) \end{aligned} \tag{4.51}$$

for every $w \in BV(\Omega) \cap H^1(\Omega_2)$ with $w = 0$ on $\partial\Omega \cap \partial\Omega_2$.

Now we are ready to prove the following existence result.

Theorem 4.3. *If $f \in L^\infty(\Omega)$ verifies* (3.24)*, then there exists, at least, a solution to problem* (4.39).

Proof. **Proof of** (4.44)**-**(4.45)**.** Let z be the vector field given in (3.26), and let u_p be the solution to (1.4), then

$$\int_{\Omega_1} |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi + \int_{\Omega_2} \nabla u_p \cdot \nabla\varphi = \int_\Omega f\varphi, \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{4.52}$$

and also

$$\int_{\Omega_1} |\nabla u_p|^{p-2}\nabla u_p \nabla\varphi = \int_{\Omega_1} f\varphi, \quad \forall\, \varphi \in C_0^\infty(\Omega_1), \tag{4.53}$$

and

$$\int_{\Omega_2} \nabla u_p \nabla\varphi = \int_{\Omega_2} f\varphi, \quad \forall\, \varphi \in C_0^\infty(\Omega_2). \tag{4.54}$$

Letting $p \to 1$ in all the above equations, we obtain (4.45)-(4.44).

Proof of (4.46)**.** By Anzellotti's theory,

$$|(z, Du)| \le \|z\|_\infty |Du|$$

and, since $\|z\|_\infty \le 1$, we have

$$(z, Du) \le |Du|\,, \quad \text{as measures in } \Omega_1\,. \tag{4.55}$$

Now we prove that in fact equality holds in (4.55). Consider $u_p\phi$, with $\phi \ge 0$, $\phi \in C_0^\infty(\Omega_1)$ as test function. We obtain

$$\int_{\Omega_1} \phi|\nabla u_p|^p + \int_{\Omega_1} u_p|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\phi = \int_{\Omega_1} f u_p \phi\,. \tag{4.56}$$

On the other hand, by Young inequality, we have

$$\int_{\Omega_1} \phi|\nabla u_p| \le \frac{1}{p}\int_{\Omega_1} \phi|\nabla u_p|^p + \frac{p-1}{p}\int_{\Omega_1} \phi. \tag{4.57}$$

Combining (4.56)-(4.57) and letting p go to 1, we deduce

$$\int_{\Omega_1} \phi|Du| + \int_{\Omega_1} uz \cdot \nabla\phi \le \int_{\Omega_1} f u\phi\,. \tag{4.58}$$

On the other hand, we have

$$\int_{\Omega_1} f u\phi = \int_{\Omega_1} \phi(z, Du) + \int_{\Omega_1} uz \cdot \nabla\phi\,. \tag{4.59}$$

Combining (4.58) and (4.59), we deduce

$$\int_{\Omega_1} \phi|Du| \le \int_{\Omega_1} \phi(z, Du)$$

This proves the claim.

Proof of (4.47)**.** From (4.45) and the generalized Green formula we have that for every $\varphi \in C_0^\infty(\Omega)$,

$$-\int_{\Omega_1} \varphi \operatorname{div} z + \int_{\partial\Omega_1} \varphi[z, \nu_1]\, d\mathcal{H}^{N-1} - \int_{\Omega_2} \varphi \operatorname{div} \nabla u + \int_{\partial\Omega_2} \varphi[\nabla u, \nu_2]\, d\mathcal{H}^{N-1} = \int_\Omega f\varphi.$$

Then taking into account (4.44) and the fact that $\varphi \in C_0^\infty(\Omega)$ we obtain

$$\int_\Gamma \varphi[z, \nu_1] + \int_\Gamma \varphi[\nabla u_{|\Omega_2}, \nu_2] = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. Having in mind that $\nu_1 = -\nu_2$ on Γ, it gives (4.47).

Proof of (4.48)**.** This is a straightforward consequence of the following facts: $u_p = 0$ on $\partial\Omega_2 \cap \partial\Omega$ for all p; $u_p \rightharpoonup u$ weakly in $H^1(\Omega_2)$ and the embedding $H^1(\Omega_2) \hookrightarrow L^2(\partial\Omega_2 \cap \partial\Omega)$ is continuous.

Proof of (4.49) **and** (4.50)**.** Denote by g_p the trace of u_p on Γ, and by g the trace of $u_{|\Omega_2}$ on Γ. It follows from $u_p \rightharpoonup u_{|\Omega_2}$ weakly in $H^1(\Omega_2)$ that $g_p \to g$ strongly in $L^2(\Gamma)$. Taking u_p as test function in (1.5) and using Young's inequality we get

$$\begin{aligned} &\int_{\Omega_1} |\nabla u_p| + \frac{1}{p}\int_{\Omega_2} |\nabla u_p|^2 \\ &\leq \frac{1}{p}\int_{\Omega_1} |\nabla u_p|^p + \frac{p-1}{p}|\Omega_1|^{1-1/p} + \frac{1}{p}\int_{\Omega_2} |\nabla u_p|^2 \\ &= \frac{1}{p}\int_{\Omega} f u_p + \frac{p-1}{p}|\Omega_1|^{1-1/p} \end{aligned} \tag{4.60}$$

In other words, we have

$$\begin{aligned} &\int_{\Omega_1} |\nabla u_p| + \int_{\partial\Omega_1\cap\partial\Omega} |u_p|\, d\mathcal{H}^{N-1} + \int_\Gamma |u_p - g|\, d\mathcal{H}^{N-1} + \frac{1}{p}\int_{\Omega_2} |\nabla u_p|^2 \\ &\leq \frac{1}{p}\int_\Omega f u_p + \frac{p-1}{p}|\Omega_1|^{1-1/p} + \int_\Gamma |g_p - g|\, d\mathcal{H}^{N-1}\,. \end{aligned} \tag{4.61}$$

Having in mind the lower semi–continuity of the functional (2.6), we may let p go to 1, obtaining

$$\int_{\Omega_1} |Du| + \int_{\partial\Omega_1\cap\partial\Omega} |u_{|\Omega_1}|\, d\mathcal{H}^{N-1} + \int_\Gamma |u_{|\Omega_1} - g|\, d\mathcal{H}^{N-1} + \int_{\Omega_2} |\nabla u|^2 \leq \int_\Omega f u\,.$$

Taking into account that $u_{|\Omega_2} = g$ on Γ on the left-hand side and applying the generalized Green formula on the right-hand, we get

$$\begin{aligned} &|Du|(\Omega_1) + \int_{\partial\Omega_1\cap\partial\Omega} |u^{\Omega_1}|\, d\mathcal{H}^{N-1} + \int_\Gamma |u_{|\Omega_1} - u_{|\Omega_2}|\, d\mathcal{H}^{N-1} + \int_{\Omega_2} |\nabla u|^2 \\ &\leq \int_{\Omega_1} f u + \int_{\Omega_2} f u \\ &= \int_{\Omega_1} (z, Du) - \int_{\partial\Omega_1} u_{|\Omega_1}[z,\nu_1]\, d\mathcal{H}^{N-1} + \int_{\Omega_2} |\nabla u|^2 - \int_\Gamma u_{|\Omega_2}[\nabla u_{|\Omega_2}, \nu_2]\, d\mathcal{H}^{N-1}. \end{aligned} \tag{4.62}$$

Since $|Du|(\Omega_1) = \int_{\Omega_1}(z, Du)$ and $[z,\nu_1] = -[\nabla u_{|\Omega_2}, \nu_2]$, we can simplify the remainders terms, so (4.62) becomes

$$\begin{aligned} &\int_{\partial\Omega_1\cap\partial\Omega} |u_{|\Omega_1}| + u_{|\Omega_1}[z,\nu_1]\, d\mathcal{H}^{N-1} + \\ &\int_\Gamma |u_{|\Omega_1} - u_{|\Omega_2}| + (u_{|\Omega_1} - u_{|\Omega_2})[z,\nu_1]\, d\mathcal{H}^{N-1} \leq 0\,, \end{aligned}$$

which prove (4.49) and (4.50). ■

Now we prove the uniqueness of the solution to the limit problem.

Theorem 4.4. *If $\mathcal{H}^{N-1}(\partial\Omega_2 \cap \partial\Omega) \neq 0$, then problem* (4.39) *has a unique solution in the sense that two solutions u_1, u_2 must agree in Ω_2.*

Proof. Let us assume that u_1, u_2 are two solutions to problem (4.39). By Remark 4.2, we can use u_2 as test function in the weak formulation written for u_1 and u_1 as test function in the weak formulation written for u_2. Adding the two equations, we have

$$\int_{\Omega_1} |Du_1| + \int_{\Omega_1} |Du_2| - \int_{\Omega_1} (z_1, Du_2) - \int_{\Omega_1} (z_2, Du_1) + \int_{\partial\Omega\cap\partial\Omega_1} |u_1|\, d\mathcal{H}^{N-1}$$
$$+ \int_{\partial\Omega\cap\partial\Omega_1} |u_2|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega\cap\partial\Omega_1} u_2[z_1;\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega\cap\partial\Omega_1} u_1[z_2;\nu]\, d\mathcal{H}^{N-1}$$
$$+ \int_{\Omega_2} |\nabla(u_1 - u_2)|^2 \le 0.$$

Since, we have

$$\int_{\Omega_1} (z_1, Du_2) \le \int_{\Omega_1} |Du_2|, \quad \text{and} \quad \int_{\Omega_1} (z_2, Du_1) \le \int_{\Omega_1} |Du_2|,$$

and moreover

$$-u_1[z_2;\nu] \le |u_1|, \quad \text{and} \quad -u_2[z_1;\nu] \le |u_2|,$$

we deduce

$$\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^2 \le 0,$$

which implies, since $u_1 = u_2 = 0$ on $\partial\Omega_2 \cap \partial\Omega$, that $u_1 = u_2$ in Ω_2. ■

Remark 4.5. As in [19] we can prove that, if the norm of f is large enough, there is not any $u \in BV(\Omega_1)$ which is the weak* limit of u_p. Precisely if $\|f\|_{W^{-1,\infty}(\Omega_1)} > 1$, then

$$\lim_{p\to 1} \int_{\Omega_1} |\nabla u_p| = +\infty\,.$$

Indeed, since $\|f\|_{W^{-1,\infty}(\Omega_1)} = \lim_{p\to1} \|f\|_{W^{-1,p'}(\Omega_1)}$, we may take $\epsilon > 0$ and $p_0 > 1$ such that $\|f\|_{W^{-1,p'}(\Omega_1)} > 1+\epsilon$, for all $p \le p_0$. On the other hand, if $\varphi \in W_0^{1,p}(\Omega_1)$ with $\|\nabla\varphi\|_{L^p(\Omega_1;\mathbb{R}^N)} \le 1$, then

$$< f, \varphi >_{W^{-1,p'}(\Omega_1), W_0^{1,p}(\Omega_1)} = \int_{\Omega_1} |\nabla u_p|^{p-1} \nabla u_p \cdot \nabla\varphi \le \left(\int_{\Omega_1} |\nabla u_p|^p \right)^{\frac{p-1}{p}}.$$

This implies

$$\|f\|_{W^{-1,p'}(\Omega_1)}^{\frac{p}{p-1}} \le \int_{\Omega_1} |\nabla u_p|^p.$$

Therefore,

$$(1+\epsilon)^{\frac{p}{p-1}} \le \int_{\Omega_1} |\nabla u_p|^p$$

for $p \le p_0$ and then

$$\lim_{p\to1} \int_{\Omega_1} |\nabla u_p|^p = +\infty.$$

Since for a suitable $g \in L^\infty(\Omega_1)$, we have $f = \operatorname{div} g$, the conclusion follows from

$$\int_{\Omega_1} |\nabla u_p|^p = < f, u_p >_{W^{-1,\infty}(\Omega_1), W_0^{1,1}(\Omega_1)} = \int_{\Omega_1} g \cdot \nabla u_p \le \|g\|_\infty \int_{\Omega_1} |\nabla u_p|\,.$$

Finally, to illustrate our results, we compute explicit solutions in the 1−dimensional case.

EXAMPLE 4.6. Assume that $\Omega =]-1,1[$, with $\Omega_1 =]-1,0[$ and $\Omega_2 =]0,1[$. Consider $\lambda, \Lambda > 0$ and take as right-hand side the function f defined by

$$f(x) = \begin{cases} \lambda\,, & \text{if } x < 0\,, \\ \Lambda\,, & \text{if } x > 0\,. \end{cases}$$

The solution at level p can explicitly be computed, namely,

$$u_p(x) = \begin{cases} \dfrac{p-1}{\lambda p}\left[(\lambda + k_p)^{p/(p-1)} - (-\lambda x + k_p)^{p/(p-1)}\right], & \text{if } x < 0\,, \\ -\dfrac{\Lambda}{2}x^2 + k_p x + \dfrac{\Lambda}{2} - k_p\,, & \text{if } x > 0\,, \end{cases}$$

where $k_p \geq 0$ satisfies

$$k_p + \frac{p-1}{\lambda p}\left[(\lambda + k_p)^{p/(p-1)} - (k_p)^{p/(p-1)}\right] = \frac{\Lambda}{2}.$$

Now, assume that there is a limit function $u(x) = \lim_{p\to 1} u_p(x)$. If this limit is finite, we get that there exists $k = \lim_{p\to 1} k_p$. Then we deduce that this limit must satisfy $\lambda + k \leq 1$ and $k = \frac{\Lambda}{2}$. Thus, $\lambda + \frac{\Lambda}{2} \leq 1$ and consequently $\Lambda \leq 2 - 2\lambda$.

In this case, the limit function is given by

$$u(x) = \begin{cases} 0\,, & \text{if } x < 0\,, \\ -\dfrac{\Lambda}{2}x^2 + \dfrac{\Lambda}{2}x\,, & \text{if } x > 0\,, \end{cases}$$

and so

$$\begin{aligned} z(x) &= \lim_{p\to 1} |u_p'(x)|^{p-2}u_p'(x) = -\lambda x + \frac{\Lambda}{2}\,, && \text{if } x < 0, \\ u'(x) &= -\Lambda x + \frac{\Lambda}{2}\,, && \text{if } x > 0\,. \end{aligned}$$

REFERENCES

[1] E. ACERBI AND G. MINGIONE: "Regularity results for stationary electro–rheological fluids", *Arch. Ration. Mech. Anal.*, **164**, (2002), 213–259.

[2] E. ACERBI AND G. MINGIONE: "Gradient estimates for the $p(x)$-Laplacian system", *J. Reine Angew. Math.*, **584**, (2005), 117–148.

[3] L. AMBROSIO, N. FUSCO AND D. PALLARA: *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 2000.

[4] F. ANDREU, V. CASELLES AND J.M. MAZON: *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals* Birkhäuser, Basel-Boston-Berlin, 2004.

[5] F. ANDREU, C. BALLESTER, V, CASELLES AND J.M. MAZON: "The Dirichlet problem for the total variation flow", *J. Func. Anal.* **180** (2001), 347–403.

[6] F. ANDREU, J.M. MAZON AND J.D. ROSSI: "The best constant for the Sobolev trace embedding from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$", *Nonlinear Anal. TM&A.* **59(7)**, (2004), 1125–1145.

[7] G. ANZELLOTTI: "Pairings between measures and bounded functions and compensated compactness" *Ann. Mat. Pura Appl.* **135** (1983), 293–318

[8] G.–Q. CHEN AND H. FRID: "Divergence-measure fields and hyperbolic conservation laws." *Arch. Ration. Mech. Anal.* **147** (1999), no. 2, 89–118.

[9] G.–Q. CHEN AND H. FRID: "On the theory of divergence-measure fields and its applications." Dedicated to Constantine Dafermos on his 60th birthday. *Bol. Soc. Brasil. Mat. (N.S.)* **32** (2001), no. 3, 401–433.

[10] G.–Q. CHEN AND H. FRID: "Extended divergence-measure fields and the Euler equations for gas dynamics." *Comm. Math. Phys.* **236** (2003), no. 2, 251–280.

[11] L. DIENING, P. HÄSTÖ AND A. NEKVINDA: "Open problems in variable exponent Lebesgue and Sobolev spaces", in: FSDONA04 Proceedings, Drabek and Rakosnik (eds.), pp. 38–58, Milovy, Czech Republic, 2005.

[12] L.C. Evans: "The 1–Laplacian, the ∞–Laplacian and Differential Games", *Perspectives in nonlinear partial differential equations*, 245–254, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, 2007.

[13] X. Fan and D. Zhao: "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$", *J. Math. Anal. Appl.* **263** (2001), no. 2, 424–446.

[14] P. Harjulehto, P. Hästö, Út Van Lê and M. Nuortio, "Overview of differential equations with non-standard growth", *Nonlinear Anal.* **72** (2010), no. 12, 4551–4574.

[15] B. Kawohl: "On a family of torsional creep problems", *J. Reine Angew. Math.* **410** (1990), 1–22.

[16] B. Kawohl: "From p-Laplace to mean curvature operator and related questions", Progress in partial differential equations: the Metz surveys, 40–56, Pitman Res. Notes Math. Ser., 249, Longman Sci. Tech., Harlow, 1991.

[17] O. Kováčik and J. Rákosník: "On spaces $L^{p(x)}$ and $W^{k,p(x)}$", *Czechoslovak Math. J.* **41(116)** (1991), no. 4, 592–618.

[18] A. Mercaldo, J.D. Rossi, S. Segura de Leon and C. Trombetti: "Anisotropic p,q-Laplacian equations when p goes to 1", *Nonlinear Anal.*, **73**, (2010), 3546–3560.

[19] A. Mercaldo, S. Segura de Leon and C. Trombetti: "On the behaviour of the solutions to p-Laplacian equations as p goes to 1", *Publ. Mat.*, **52** (2008), 377–411.

[20] A. Mercaldo, S. Segura de Leon and C. Trombetti: "On the solutions to 1–Laplacian equation with L^1 data", *J. Func. Anal.* **256** (2009) 2387–2416

[21] E.Zeidler:*Nonlinear functional analysis and its applications*, Vol. II/A, Springer-Verlag, New York, 1990.

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