ON THE BEHAVIOUR OF THE CHARACTERISTIC FUNCTION OF A PROBABILITY DISTRIBUTION IN THE NEIGHBOURHOOD OF THE ORIGIN

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1. Introduction

Let X be a real valued random variable with probability measure P and distribution function F. It will be convenient to take F as the *intermediate* distribution function defined by

$$F(x) = \tfrac{1}{2}[P\{X < x\} + P\{X \leqq x\}].$$

In mathematical analysis it is a little more convenient to use this function rather than

$$F_1(x) = P\{X < x\} \text{ or } F_2(x) = P\{X \leqq x\},$$

which arise more naturally in probability theory. In all cases we shall consider

$$F_1(x) \sim F_2(x) \sim F(x), \qquad x \to -\infty,$$

$$1 - F_1(x) \sim 1 - F_2(x) \sim 1 - F(x), \qquad x \to \infty.$$

With this definition, if the distribution function of X is $F(x)$, then the distribution function of $-X$ is $1 - F(-x)$. The distribution of X is symmetrical about 0 if $F(x) = 1 - F(-x)$.

The characteristic function of X, or of F, is ϕ, defined for all real t by

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x).$$

This paper deals with the relation between the value of $F(x)$ for large x and the value of $\phi(t)$ for small t. We are interested in the behaviour of $\phi(t)$ in the neighbourhood of $t = 0$ because upon this depend all limit theorems on sums of random variables. Most of the theorems proved here were stated without proofs in [1].

For $x \geqq 0$, put

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$$H(x) = 1-F(x)+F(-x), \qquad \text{the tail sum,}$$
$$K(x) = 1-F(x)-F(-x), \qquad \text{the tail difference.}$$

If the distribution is symmetrical about 0, then $K(x)$ is identically zero. If X is a non-negative random variable, $F(x) = 0$ when $x < 0$, and $K(x) = H(x)$ for $x > 0$.

We may write

$$\phi(t) = \int_{-\infty}^{0} e^{itx} dF(x) + \int_{0}^{\infty} e^{itx} d[F(x)-1].$$

Integrating by parts, and putting

$$\phi(t) = U(t)+iV(t),$$

where

$$U(t) = \int_{-\infty}^{\infty} \cos tx\, dF(x), \qquad V(t) = \int_{-\infty}^{\infty} \sin tx\, dF(x),$$

we finally obtain

$$\frac{1-U(t)}{t} = \int_0^\infty H(x) \sin tx\, dx,$$
$$\frac{V(t)}{t} = \int_0^\infty K(x) \cos tx\, dx.$$

We have the inversion formulae,

$$H(x) = \frac{2}{\pi} \int_0^\infty \frac{1-U(t)}{t} \sin xt\, dt,$$
$$K(x) = \frac{2}{\pi} \int_0^\infty \frac{V(t)}{t} \cos xt\, dt.$$

$U(t)$ depends only on H, and $V(t)$ depends only on K, and H and K are not closely connected. The only connections between H and K are the relations, $H(x) \geqq |K(x)|$, and $H(x) \pm K(x)$ both non-increasing functions of x. In investigating the behaviour of $\phi(t)$ in the neighbourhood of $t = 0$, it is therefore advisable to consider $U(t)$ and $V(t)$ separately.

Consider $H(x)$ and $U(t)$. The sort of result we are interested in is

(1) $$1-U(t) \sim cH(1/t), \qquad t \downarrow 0,$$

where c is a constant depending on the distribution. If the distribution has finite second moment μ_2, then

$$U(t) = 1-\tfrac{1}{2}\mu_2 t^2 + o(t^2), \qquad t \to 0.$$

Hence

$$1-U(t) \sim \tfrac{1}{2}\mu_2 t^2, \qquad t \to 0.$$

In order to get a result of the type of (1) we must have a distribution of infinite variance.

2. Functions of regular growth

It is shown in [1] that we can expect a result of type (1) only when the tail sum $H(x)$ has the property that for every $\lambda > 0$,

$$\frac{H(\lambda x)}{H(x)} \to \lambda^k \qquad \text{as } x \to \infty.$$

We shall express this property of H by saying that $H(x)$ is of index k as $x \to \infty$. Functions with this property were studied by Karamata [2], [3], who showed that if $G(x)$ is integrable over any finite interval with lower endpoint b, a necessary and sufficient condition for $G(x)$ to be of index k, > -1, as $x \to \infty$ is

$$\frac{\int_b^x G(u)du}{xG(x)} \to \frac{1}{k+1} \qquad \text{as } x \to \infty. \tag{2}$$

He also showed, what can easily be deduced from (2), that if $G(x)$ is of index k, $\neq 0$, as $x \to \infty$, then

$$G(x) \sim G_1(x), \qquad x \to \infty, \tag{3}$$

where $G_1(x)$ is a monotonic function of x, clearly non-decreasing if k is positive, and non-increasing if k is negative.

A function $L(x)$ of index 0 is sometimes called a function of slow growth. It has the property that $L(\lambda x)/L(x) \to 1$ as $x \to \infty$ for every $\lambda > 0$. The functions $\log x$, $\log \log x$ are of index 0, and so is any function with a finite non-zero limit as $x \to \infty$. Clearly, if $G(x)$ is of index k, then $G(x)/x^k$ is of index 0, and so

$$G(x) = x^k L(x),$$

where $L(x)$ is of index 0.

We say that a function $G(x)$ is of index k as $x \downarrow 0$ if for every $\lambda > 0$,

$$\frac{G(\lambda x)}{G(x)} \to \lambda^k \qquad \text{as } x \downarrow 0.$$

Lemma 1. *Let $G(w)$ be positive for $w > b$, bounded in any finite positive interval, and of index k when $w \to \infty$. Let c be greater than 0. If $k > 0$, then B exists such that*

$$\frac{G(\lambda w)}{G(w)} \quad \textit{is bounded for } w \geqq B,\ 0 < \lambda \leqq c.$$

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If $k < 0$, then B exists such that

$$\frac{G(\lambda w)}{G(w)} \quad \text{is bounded for } w \geqq B,\ \lambda \geqq c.$$

PROOF. If $k > 0$, then as stated in (3), $G(w) \sim G_1(w)$, $w \to \infty$, where $G_1(w)$ is a non-decreasing function of w. We may choose $G_1(w)$ so that it is bounded in any finite positive interval, and so that $G_1(w) > 1$.

If B is sufficiently great, when $w \geqq B$,

$$\tfrac{1}{2}G_1(w) < G(w) < 2G_1(w).$$

When $\lambda w \geqq B$, $w \geqq B$, $\lambda \leqq c$,

$$\frac{G(\lambda w)}{G(w)} < \frac{2G_1(\lambda w)}{\frac{1}{2}G_1(w)} \leqq \frac{4G_1(cw)}{G_1(w)}. \tag{4}$$

The last fraction is bounded for w in any finite positive interval, and $\to 4c^k$ when $w \to \infty$, and so is bounded.

Let $M = \sup\{G(w);\ 0 \leqq w \leqq B\}$. When $\lambda w < B$, $w \geqq B$, $\lambda \leqq c$,

$$\frac{G(\lambda w)}{G(w)} < \frac{M}{\frac{1}{2}G_1(w)} \leqq \frac{2M}{G_1(B)}.$$

If $k < 0$, $G(w) \sim G_1(w)$, $w \to \infty$, where $G_1(w)$ is non-increasing and bounded in any finite positive interval. If $\lambda \geqq c$, when w is great, the relations (4) hold, and as before the last fraction is bounded.

LEMMA 2. *Let $G(w)$ be positive and bounded for $w > b$, and let h and c be greater than 0. If $G(w)$ is of index $-m$ as $w \to \infty$, where $m \geqq 0$, then A, B exist such that*

$$\frac{G(\lambda w)}{G(w)} < \frac{A}{\lambda^{m+h}} \qquad \text{for } w \geqq B,\ 0 < \lambda \leqq c,$$

$$\frac{G(\lambda w)}{G(w)} < \frac{A}{\lambda^{m-h}} \qquad \text{for } w \geqq B,\ \lambda \geqq c.$$

PROOF. $w^{m+h}G(w)$ is of index h, and so by Lemma 1, A, B exist such that

$$\frac{(\lambda w)^{m+h}G(\lambda w)}{w^{m+h}G(w)} < A \qquad \text{for } w \geqq B,\ 0 < \lambda \leqq c,$$

and therefore

$$\frac{G(\lambda w)}{G(w)} < \frac{A}{\lambda^{m+h}} \qquad \text{for } w \geqq B,\ 0 < \lambda \leqq c.$$

The other result is obtained similarly.

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LEMMA 3. *If $G(w)$ is monotonic for $w > a$, and $\int_a^w u^r G(u)du$ is of index k, $w \to \infty$, where $k > 0$, then $w^r G(w)$ is of index $k-1$.*

PROOF. If $b > 0$, $c > 0$, $b \neq c$,

$$\int_{bw}^{cw} u^r G(u)du$$

is of index k, $w \to \infty$. Take $\mu > 1$, $\lambda > 0$. Without loss of generality we may assume that $G(w)$ is positive when $w > a$. We shall give the proof for $G(w)$ non-increasing and $r \geqq 0$. The proofs for the other cases are similar.

When $\lambda w > a$, $w > \mu a$,

$$\frac{\int_{\lambda w}^{\lambda\mu w} u^r G(u)du}{\int_{w/\mu}^{w} u^r G(u)du} \leqq \frac{\lambda\mu(\lambda\mu w)^r G(\lambda w)}{(w/\mu)^r G(w)} = \frac{\lambda^{r+1}\mu^{2r+1}G(\lambda w)}{G(w)}. \tag{5}$$

When $w \to \infty$, the first expression in (5) $\to (\lambda\mu)^k$. Hence

$$\liminf_{w\to\infty} \frac{G(\lambda w)}{G(w)} \geqq \frac{(\lambda\mu)^k}{\lambda^{r+1}\mu^{2r+1}} = \lambda^{k-r-1}\mu^{k-2r-1}.$$

Making $\mu \downarrow 1$, we obtain

$$\liminf_{w\to\infty} \frac{G(\lambda w)}{G(w)} \geqq \lambda^{k-r-1}.$$

By replacing μ in the above argument by $1/\mu$, we obtain similarly

$$\limsup_{w\to\infty} \frac{G(\lambda w)}{G(w)} \leqq \lambda^{k-r-1}.$$

Hence

$$\lim_{w\to\infty} \frac{G(\lambda w)}{G(w)} = \lambda^{k-r-1}.$$

Thus G is of index $k-r-1$ at ∞, and so $w^r G(w)$ is of index $k-1$.

3. Relations between H and U

It is easy to show that if the distribution has infinite second moment, and if $H(x)$ is of index k as $x \to \infty$, then $-2 \leqq k \leqq 0$.

Write

$$S(m) = \frac{\frac{1}{2}\pi}{\Gamma(m)\sin\frac{1}{2}m\pi}, \qquad m > 0,$$

$$= 1, \qquad m = 0,$$

$$C(m) = \frac{\frac{1}{2}\pi}{\Gamma(m)\cos\frac{1}{2}m\pi}, \qquad m > 0.$$

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$S(m)$ is finite for m not an even positive integer, and, for $0 < m < 2$, [2]

$$S(m) = \int_0^\infty \frac{\sin x}{x^m}\, dx.$$

If $2n < m < 2n+2$, where n is a positive integer,

$$S(m) = \int_0^\infty \left\{\sin x - \sum_1^n (-1)^{r-1} \frac{x^{2r-1}}{(2r-1)!}\right\} x^{-m}\, dx.$$

$C(m)$ is finite for m not an odd positive integer, and for $0 < m < 1$,

$$C(m) = \int_0^\infty \frac{\cos x}{x^m}\, dx.$$

If $2n-1 < m < 2n+1$, where n is a positive integer,

$$C(m) = \int_0^\infty \left\{\cos x - \sum_0^{n-1} (-1)^r \frac{x^{2r}}{(2r)!}\right\} x^{-m}\, dx.$$

THEOREM 1. *If $H(x)$ is of index $-m$ when $x \to \infty$, and $0 < m < 2$, then*

$$1-U(t) \sim S(m)H(1/t), \qquad t \downarrow 0.$$

PROOF.

$$\frac{1-U(t)}{t} = \int_0^\infty H(x) \sin tx\, dx \qquad (t > 0),$$

(6)
$$\frac{1-U(t)}{H(1/t)} = \int_0^\infty \frac{H(x/t)}{H(1/t)} \sin x\, dx \qquad (t > 0).$$

If $0 < x \leqq p$, it follows from Lemma 2, that if $h > 0$, when t is sufficiently small,

$$\frac{H(x/t)}{H(1/t)} |\sin x| \leqq \frac{A|\sin x|}{x^{m+h}},$$

where A is a finite constant. We can choose h so that $m+h < 2$. The last function is then integrable over the finite interval $(0, p)$. When $t \downarrow 0$, $H(x/t)/H(1/t) \to x^{-m}$ for $x > 0$. Therefore

$$\int_0^p \frac{H(x/t)}{H(1/t)} \sin x\, dx \to \int_0^p \frac{\sin x}{x^m}\, dx.$$

By the Second Mean Value Theorem

$$\int_p^\infty \frac{H(x/t)}{H(1/t)} \sin x\, dx = \frac{H(p/t)}{H(1/t)} \int_p^q \sin x\, dx,$$

[2] See [4] page 260, Ex. 12 for this and for the first result below for $C(m)$. The results for higher values of m are derived from those for lower values by integration by parts.

which has a modulus $\leqq 2H(p/t)/H(1/t)$. This $\to 2/p^m$ when $t \downarrow 0$, and so can be made arbitrarily small by making p sufficiently great and t sufficiently small. It follows that when $t \downarrow 0$, the integral in (6) tends to

$$\int_0^\infty \frac{\sin x}{x^m}\, dx.$$

This proves Theorem 1.

THEOREM 2. *If $H(x)$ is of index 0 as $x \to \infty$, and*

$$H(x+h) \leqq \tfrac{1}{2}[H(x)+H(x+2h)] \tag{7}$$

when x and h are sufficiently great, then

$$1-U(t) \sim H(1/t), \qquad t \downarrow 0.$$

This is the extension of Theorem 1 to the case $m = 0$. It appears from the counter-example in Section 6 at the end of this paper that some additional condition such as (7) is required for $m = 0$. The condition will be satisfied if $H(x)$ is convex when x is sufficiently great.

PROOF. Suppose that (7) is true when $x, h \geqq B$. Let C be a real number such that $H(C) \leqq \frac{1}{2}H(B)$. (7) is true when $x \geqq B$, $h \geqq C$. When $x < B$, $h \geqq C$,

$$2H(x+h) \leqq 2H(C) \leqq H(B) \leqq H(x)+H(x+2h).$$

Thus (7) is true for all $x \geqq 0$ and all $h \geqq C$.

$$\frac{1-U(t)}{H(1/t)} = \int_0^\infty \frac{H(x/t)\sin x}{H(1/t)}\, dx = \sum_0^\infty \int_{n\pi}^{(n+1)\pi}$$
$$= \int_0^\pi \frac{\sin x}{H(1/t)} \left[\sum_{n=0}^\infty (-1)^n H\left\{\frac{x+n\pi}{t}\right\}\right] dx.$$

If

$$U = u_1 - u_2 + u_3 - \cdots$$

where

$$u_n - u_{n+1} \geqq u_{n+1} - u_{n+2} \geqq 0, \qquad (n = 1, 2, \cdots)$$

and $u_n \to 0$ as $n \to \infty$, then

$$U \geqq u_2 - u_3 + u_4 - \cdots = u_1 - U.$$

Therefore

$$U \geqq \tfrac{1}{2}u_1.$$

Again

$$U = u_1 - (u_2 - u_3 + u_4 - \cdots)$$
$$\leqq u_1 - \tfrac{1}{2}u_2.$$

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Thus
$$\tfrac{1}{2}u_1 \leqq U \leqq u_1 - \tfrac{1}{2}u_2.$$

We have shown that when h is sufficiently great
$$2H(x+h) \leqq H(x)+H(x+2h),$$
i.e.
$$H(x)-H(x+h) \geqq H(x+h)-H(x+2h).$$
Therefore, when t is sufficiently small,
$$H\left\{\frac{x+n\pi}{t}\right\} - H\left\{\frac{x+(n+1)\pi}{t}\right\} \geqq H\left\{\frac{x+(n+1)\pi}{t}\right\} - H\left\{\frac{x+(n+2)\pi}{t}\right\}.$$
Hence when t is sufficiently small,
$$\tfrac{1}{2}H(x/t) \leqq \sum_{n=0}^{\infty}(-1)^n H\left\{\frac{x+n\pi}{t}\right\} \leqq H(x/t) - \tfrac{1}{2}H\left\{\frac{x+\pi}{t}\right\},$$
and therefore

(8)
$$\tfrac{1}{2}I_1 \leqq \frac{1-U(t)}{H(1/t)} \leqq I_1 - \tfrac{1}{2}I_2,$$
where
$$I_1 = \int_0^{\pi} \frac{H(x/t)}{H(1/t)} \sin x dx, \qquad I_2 = \int_0^{\pi} \frac{H\{(x+\pi)/t\}}{H(1/t)} \sin x dx.$$
Both integrands $\to \sin x$ when $t \downarrow 0$. The second integrand is dominated by $\sin x$. From Lemma 2 we see that the first integrand is dominated by $A \sin x/x^{\frac{1}{2}}$. Hence I_1, I_2 both tend to
$$\int_0^{\pi} \sin x dx, = 2.$$
Theorem 2 then follows from (8).

THEOREM 3.[3] *If $H(x)$ is of index -2 as $x \to \infty$, then*
$$1-U(t) \sim t^2 \int_0^{1/t} xH(x)dx, \qquad t \downarrow 0.$$

PROOF.
$$\frac{1-U(t)}{H(1/t)} = \int_0^{\infty} \frac{H(x/t)}{H(1/t)} \sin x dx.$$
Here H is of index -2, and so for $x > 0$,
$$\frac{H(x/t)}{H(1/t)} \to \frac{1}{x^2} \qquad \text{as } t \downarrow 0.$$
Also, if $c > 0$, and t positive and sufficiently small, we see from Lemma 2 with $h = \frac{1}{2}$, that

[3] A stronger result than this can be obtained by putting $n = 0$ in Theorem 6 (iii).

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$$\frac{H(x/t)}{H(1/t)} \leqq \frac{A}{x^{\frac{3}{2}}} \qquad \text{when } x \geqq c.$$

Therefore

$$\int_c^\infty \frac{H(x/t)}{H(1/t)} \sin x\,dx \to \int_c^\infty \frac{\sin x}{x^2}\,dx \qquad \text{as } t \downarrow 0.$$

The last integral $\to \infty$ when $c \downarrow 0$, and therefore

$$\int_0^\infty \frac{H(x/t)}{H(1/t)} \sin x\,dx \to \infty \qquad \text{as } t \downarrow 0.$$

Hence

$$\int_0^\infty \frac{H(x/t)}{H(1/t)} \sin x\,dx \sim \int_0^c \frac{H(x/t)}{H(1/t)} \sin x\,dx, \qquad t \downarrow 0,$$

and so

$$\int_0^\infty H(x/t) \sin x\,dx \sim \int_0^c H(x/t) \sin x\,dx, \qquad t \downarrow 0.$$

Since c may be arbitrarily small, this must be

$$\sim \int_0^c H(x/t)x\,dx \sim \int_0^1 H(x/t)x\,dx, \qquad t \downarrow 0.$$

Thus

$$1-U(t) = \int_0^\infty H(x/t) \sin x\,dx \sim \int_0^1 H(x/t)x\,dx \sim t^2 \int_0^{1/t} H(x)x\,dx, \qquad t \downarrow 0.$$

THEOREM 4. *Let $U_1(t)$ be the real part at t of the characteristic function corresponding to the tail function H_1. If H satisfies the conditions of Theorem* 1, 2 *or* 3 *then*

$$H_1(x) = O\{H(x)\},\ x \to \infty \ \Rightarrow\ 1-U_1(t) = O\{1-U(t)\},\ t \downarrow 0,$$
$$H_1(x) = o\{H(x)\},\ x \to \infty \ \Rightarrow\ 1-U_1(t) = o\{1-U(t)\},\ t \downarrow 0.$$

PROOF. Consider the relation $H_1(x) = O\{H(x)\}$, and suppose $H_1(x) \leqq kH(x)$.

$$1-U_1(t) = \int_0^\infty H_1(x/t) \sin x\,dx \leqq \int_0^\pi H_1(x/t) \sin x\,dx,$$

because $H_1(x)$ is a non-increasing function of x. Hence

$$1-U_1(t) \leqq k \int_0^\pi H(x/t) \sin x\,dx = kH(1/t) \int_0^\pi \frac{H(x/t)}{H(1/t)} \sin x\,dx$$
$$\sim kH(1/t) \int_0^\pi \frac{\sin x}{x^m}\,dx \sim c(1-U(t)), \qquad t \downarrow 0,$$

if $m < 2$, where c is a constant. When $m = 2$

$$1-U_1(t) \leqq k\int_0^{\pi} H(x/t)\sin x\,dx = kt\int_0^{\pi/t} H(x)\sin tx\,dx$$
$$\leqq kt^2\int_0^{\pi/t} H(x)x\,dx \sim kt^2\int_0^{1/t} H(x)x\,dx \sim k\{1-U(t)\}, \qquad t\downarrow 0.$$

This proves the first part of the theorem, and the second part is proved similarly.

THEOREM 5. *If* $1-U(t)$ *is of index* m *as* $t\downarrow 0$ *and* $0 \leqq m < 2$, *then*

$$H(x) \sim \frac{1-U(1/x)}{S(m)}, \qquad x\to\infty,$$

and if $m = 2$, *then*

$$\int_0^x uH(u)du \sim x^2\{1-U(1/x)\} \qquad x\to\infty. \tag{9}$$

PROOF. For $x \geqq 0$ define

$$H_1(x) = \int_0^x uH(u)du, \qquad H_2(x) = \int_0^x H_1(u)du.$$
$$H(x) = \frac{2}{\pi}\int_0^\infty \frac{1-U(t)}{t}\sin xt\,dt,$$
$$H_1(x) = \frac{2}{\pi}\int_0^x\int_0^\infty \frac{1-U(t)}{t}u\sin ut\,dt\,du.$$

The integrand is bounded, and

$$\int_0^T \frac{1-U(t)}{t}u\sin ut\,dt$$

is bounded in $0 \leqq u \leqq x$, $T \geqq 0$. See (a) below. We may therefore reverse the order of integration, and so

$$H_1(x) = \frac{2}{\pi}\int_0^\infty\int_0^x \frac{1-U(t)}{t}u\sin ut\,du\,dt$$
$$= \frac{2}{\pi}\int_0^\infty \{1-U(t)\}\frac{\sin xt - xt\cos xt}{t^3}dt.$$

We now have an absolutely convergent integral.

$$H_2(x) = \frac{2}{\pi}\int_0^x\int_0^\infty \{1-U(t)\}\frac{\sin ut - ut\cos ut}{t^3}dt\,du.$$

By Fubini's theorem we may reverse the order of integration, and so

$$H_2(x) = \frac{2}{\pi}\int_0^\infty \{1-U(t)\}\frac{2(1-\cos xt) - xt\sin xt}{t^4}dt.$$

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Hence

$$\frac{H_2(x)}{x^3\{1-U(1/x)\}} = \frac{2}{\pi}\int_0^\infty \frac{1-U(t/x)}{1-U(1/x)}\,\frac{2(1-\cos t)-t\sin t}{t^4}\,dt.$$

$1-U(1/x)$ is of index $-m$ as $x \to \infty$. Hence by Lemma 2, with

$$w = x,\ G(x) = 1-U(1/x),\ \lambda = 1/t,\ c = 1$$

when x is sufficiently great,

$$\frac{1-U(t/x)}{1-U(1/x)} < At^{m+h} \qquad \text{when } t \geqq 1, \text{ and}$$

$$< At^{m-h} \qquad \text{when } t < 1,$$

where $h > 0$. If $0 \leqq m < 2$, we can choose h so that $h < 1$, $m+h < 2$. The integrand will then be dominated by

$$A\,(t^{m+h}+t^{m-h})\,\frac{2(1-\cos t)-t\sin t}{t^4},$$

which is integrable over $(0, \infty)$. When $x \to \infty$

$$\frac{1-U(t/x)}{1-U(1/x)} \to t^m.$$

Therefore

$$\frac{H_2(x)}{x^3\{1-U(1/x)\}} \to \frac{2}{\pi}\int_0^\infty \frac{2(1-\cos t)-t\sin t}{t^{4-m}}\,dt, = g(m).$$

Thus

(10) $$H_2(x) \sim g(m)x^3\{1-U(1/x)\}, \qquad x \to \infty.$$

It can be shown that

(11) $$g(m) = \frac{1}{(3-m)(2-m)S(m)};$$

but we do not need this evaluation to prove the theorem.

The relation (10) shows that $H_2(x)$ is of index $3-m$ as $x \to \infty$. Hence by Lemma 3, $H_1(x)$ is of index $2-m$, and $xH(x)$ is of index $1-m$. Therefore $H(x)$ is of index $-m$, and the stated result will follow from Theorem 1 if $0 < m < 2$. We cannot argue in this way when $m = 0$ because of the additional convexity condition in Theorem 2. We may get over the difficulty by using the result obtained by Karamata in [2], that if $G(x)$ is integrable over any finite positive interval, and is of index n, $x \to \infty$, where $n > -1$, then

(12) $$\int_0^x G(u)du \sim \frac{xG(x)}{n+1}, \qquad x \to \infty.$$

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When $m = 0$, $H(x)$ is of index 0. Therefore $xH(x)$ is of index 1, and

$$H_1(x) \sim x^2 H(x)/2,$$
$$H_2(x) \sim xH_1(x)/3 \sim x^3 H(x)/6,$$

and therefore it follows from (10) that

$$H(x) \sim 6g(0)\{1-U(1/x)\} = 1-U(1/x).$$

That $6g(0) = 1$ follows from Theorem 2 or from (11).

When $m = 2$, we consider H_3, defined by

$$H_3(x) = \int_0^x H_2(u)du.$$

We can show that

$$H_3(x) \sim kx^4\{1-U(1/x)\}, \qquad x \to \infty,$$

where k is a constant. Thus H_3 is of index 2 at ∞. Hence, by Lemma 3, H_2 is of index 1, and H_1 of index 0 at ∞. Using Karamata's result (12), we obtain

$$H_2(x) \sim xH_1(x), \qquad x \to \infty$$
$$H_3(x) \sim \tfrac{1}{2}xH_2(x) \sim \tfrac{1}{2}x^2 H_1(x), \qquad x \to \infty$$

and so

$$H_1(x) \sim 2kx^2\{1-U(1/x)\}, \qquad x \to \infty.$$

Theorem 3 shows that $2k = 1$, and so (9) is proved.

(a)
$$\int_0^T \frac{1-U(t)}{t} \sin ut dt = \int_0^T \left\{\int_0^\infty H(y) \sin yt \sin ut dy\right\} dt$$
$$= \int_0^\infty \left\{\int_0^T H(y) \sin yt \sin ut dt\right\} dy,$$

because $|\int_0^Y H(y) \sin yt \sin ut dy| \leqq 4u$ for $Y > 0$. See (b). Thus

$$\int_0^T \frac{1-U(t)}{t} \sin ut dt = \int_0^\infty \left\{\int_0^T \tfrac{1}{2}H(y)\{\cos (y-u)t - \cos (y+u)t\}dt\right\} dy$$
$$= \int_0^\infty \tfrac{1}{2}H(y) \left\{\frac{\sin (y-u)T}{y-u} - \frac{\sin (y+u)T}{y+u}\right\} dy,$$

which has a modulus $\leqq 2A$. See (c). Thus

$$\left|\int_0^T \frac{1-U(t)}{t} u \sin ut dt\right| \leqq 2Ax \text{ for } 0 \leqq u \leqq x, \qquad T > 0.$$

(b) $\int_0^Y H(y)\sin yt \sin ut\, dy = \sin ut\, H(0)\int_0^\xi \sin yt\, dt + \sin ut\, H(Y)\int_\xi^Y \sin yt\, dt,$

where $0 \leqq \xi \leqq Y$. The modulus of this is

$$\leqq |\sin ut|(2/t+2/t) \leqq 4u.$$

(c)
$$\int_a^b \frac{\sin k(y+c)}{y+c}\, dy$$

is bounded for all real a, b, c, k, because it is equal to

$$\int_{k(a+c)}^{k(b+c)} \frac{\sin v}{v}\, dv.$$

This is bounded because

$$\int_{-\infty}^{\infty} \frac{\sin v}{v}\, dv$$

exists as a finite (semi-convergent) integral. Thus

$$\left|\int_a^b \frac{\sin k(y+c)}{y+c}\, du\right| \leqq A.$$

$$\int_0^Y H(y)\frac{\sin k(y+c)}{y+c}\, dy$$
$$= H(0)\int_0^\xi \frac{\sin k(y+c)}{y+c}\, dy + H(Y)\int_\xi^Y \frac{\sin k(y+c)}{y+c}\, dy,$$

where $0 \leqq \xi \leqq Y$, and so has a modulus $\leqq H(0)A + H(Y)A \leqq 2A$.

4. Relations between *H* and *U* for distributions of finite variance

If n is a positive integer,

$$\mu_{2n} = \int_{-\infty}^{\infty} x^{2n}\, dF(x) = -\int_0^\infty x^{2n}\, dH(x) = \int_0^\infty 2nx^{2n-1}H(x)\, dx.$$

THEOREM 6. *Let n be a non-negative integer. Suppose that μ_{2n} is finite. Put*

$$U(t) = \sum_0^n (-1)^r \frac{\mu_{2r}t^{2r}}{(2r)!} - U_{2n}(t).$$

Let H be of index $-m$ at ∞.

(i) *If $2n < m < 2n+2$, then*

$$U_{2n}(t) \sim S(m)H(1/t), \qquad t \downarrow 0.$$

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(ii) *If $m = 2n$, where $n > 0$,*

$$U_{2n}(t)-(-1)^n t^{2n}\int_{1/t}^{\infty}\frac{x^{2n-1}}{(2n-1)!}H(x)dx \sim (-1)^{n-1}\frac{\Gamma'(2n)}{\Gamma(2n)^2}H(1/t), \quad t\downarrow 0.$$

This implies the weaker statement,

$$U_{2n}(t) \sim (-1)^n t^{2n}\int_{1/t}^{\infty}\frac{x^{2n-1}}{(2n-1)!}H(x)dx, \qquad t\downarrow 0.$$

(iii) *If $m = 2n+2$*

$$U_{2n}(t)-(-1)^n t^{2n+2}\int_{0}^{1/t}\frac{x^{2n+1}}{(2n+1)!}H(x)dx \sim (-1)^{n}\frac{\Gamma'(2n+2)}{\Gamma(2n+2)^2}H(1/t), \quad t\downarrow 0,$$

which implies

$$U_{2n}(t) \sim (-1)^n t^{2n+2}\int_{0}^{1/t}\frac{x^{2n+1}}{(2n+1)!}H(x)dx, \qquad t\downarrow 0.$$

PROOF.

$$\begin{aligned} U_{2n}(t) &= 1-U(t)-\sum_1^n(-1)^{r-1}\frac{\mu_{2r}t^{2r}}{(2r)!} \\ &= t\int_0^\infty H(x)\left\{\sin tx-\sum_1^n(-1)^{r-1}\frac{x^{2r-1}t^{2r-1}}{(2r-1)!}\right\}dx \\ &= \int_0^\infty H(x/t)G_n(x)dx, \end{aligned}$$

where

$$G_n(x) = \sin x-\sum_1^n(-1)^{r-1}\frac{x^{2r-1}}{(2r-1)!}.$$

Note that $(-1)^n G_n(x)$ is positive when x is positive.[4] Also

$$(-1)^n G_n(x) \sim \frac{x^{2n-1}}{(2n-1)!}, \qquad x\to\infty$$

$$(-1)^n G_n(x) \sim \frac{x^{2n+1}}{(2n+1)!}, \qquad x\to 0.$$

$$\frac{U_{2n}(t)}{H(1/t)} = \int_0^c\frac{H(x/t)}{H(1/t)}G_n(x)dx+\int_c^\infty\frac{H(x/t)}{H(1/t)}G_n(x)dx,$$

where $c > 0$.

When $0 < x \leqq c$, $h > 0$, and t sufficiently small,

$$\left|\frac{H(x/t)}{H(1/t)}G_n(x)\right| \leqq \frac{A|G_n(x)|}{x^{m+h}}$$

[4] See [5] pages 80, 81.

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by Lemma 2. If $m < 2n+2$, we can choose h so that $m+h < 2n+2$. $AG_n(x)/x^{m+h}$ is then integrable over $[0, c]$. When $t \downarrow 0$, $H(x/t)/H(1/t) \to x^{-m}$, and therefore

$$\int_0^c \frac{H(x/t)}{H(1/t)} G_n(x)dx \to \int_0^c \frac{G_n(x)}{x^m} dx.$$

Similarly we can show that if $m > 2n$,

$$\int_c^\infty \frac{H(x/t)}{H(1/t)} G_n(x)dx \to \int_c^\infty \frac{G(x)}{x^m} dx \qquad \text{as } t \downarrow 0.$$

Hence, when $2n < m < 2n+2$,

$$\lim_{t\downarrow 0} \frac{U_{2n}(t)}{H(1/t)} = \int_0^c \frac{G_n(x)}{x^m} dx + \int_c^\infty \frac{G_n(x)}{x^m} dx = \int_0^\infty \frac{G_n(x)}{x^m} dx = S(m).$$

This proves (i).

$$\frac{U_{2n}(t)-(-1)^n t^{2n+2} \int_0^{1/t} \frac{x^{2n+1}}{(2n+1)!} H(x)dx}{H(1/t)}$$
$$= \int_0^\infty \frac{H(x/t)}{H(1/t)} G_n(x)dx - (-1)^n \int_0^1 \frac{x^{2n+1}}{(2n+1)!} \frac{H(x/t)}{H(1/t)} dx$$
$$= \int_0^1 \frac{H(x/t)}{H(1/t)} G_{n+1}(x)dx + \int_1^\infty \frac{H(x/t)}{H(1/t)} G_n(x)dx.$$

If $m = 2n+2$, when $t \downarrow 0$, the limit of this is

$$\int_0^1 \frac{G_{n+1}(x)}{x^{2n+2}} dx + \int_1^\infty \frac{G_n(x)}{x^{2n+2}} dx. \tag{13}$$

The sum of the two integrals in (13) is the limit as $h \downarrow 0$ of

$$\int_0^1 \frac{G_{n+1}(x)}{x^{2n+2+h}} dx + \int_1^\infty \frac{G_n(x)}{x^{2n+2+h}} dx,$$
$$= \int_0^\infty \frac{G_{n+1}(x)}{x^{2n+2+h}} dx + (-1)^n \int_1^\infty \frac{x^{2n+1}}{(2n+1)!\, x^{2n+2+h}} dx$$
$$= S(2n+2+h) + \frac{(-1)^n}{h(2n+1)!}.$$

The limit of this, as $h \downarrow 0$, is easily shown to be

$$\frac{(-1)^n \Gamma'(2n+2)}{\Gamma(2n+2)^2},$$

and so (iii) is proved.

We prove (ii) simply by replacing n in (iii) by $n-1$.

5. Relations between K and V

If K is of index $-m$ at ∞, where $m \geqq 0$, $K(\lambda x)/K(x) \to \lambda^{-m}$ as $x \to \infty$, for $\lambda > 0$. Hence $K(x)$ must be either always positive or always negative when x is great. We shall give the proofs always for the case $K(x) > 0$ when x is great.

THEOREM 7. *Put*

$$K(x) = K_1(x) - K_2(x),$$

where

$$K_1(x) = 1 - F(x), \quad K_2(x) = F(-x).$$

Suppose that K is of index $-m$ at ∞, and that, when x is great, either $K(x)$ is monotonic, or $K_2(x) < \lambda K_1(x)$, where $0 < \lambda < 1$.

(i) *If $0 < m < 1$, then*

$$V(t) \sim C(m)K(1/t), \qquad t \downarrow 0.$$

(ii) *If $m = 0$, then*

$$\int_0^t \frac{V(u)}{u}\,du \sim \tfrac{1}{2}\pi K(1/t), \qquad t \downarrow 0.$$

(ii) *If $m = 1$, then*

$$V(t) - t\int_0^{1/t} K(x)dx \sim -\gamma K(1/t), \qquad t \downarrow 0,$$

where γ is Euler's constant.

The proof of (i) when $K(x)$ is ultimately monotonic is similar to the proof of Theorem 1. When we assume that $K_2(x) < \lambda K_1(x)$ when x is great, a slight modification is necessary. In the course of the proof we need to show that the integral

$$\int_p^\infty \frac{K(x/t)}{K(1/t)} \cos x\,dx$$

can be made arbitrarily small by making p sufficiently great and t sufficiently small.

It follows from $K_2(x) < \lambda K_1(x)$ that

$$K_1(x) + K_2(x) < \frac{(1+\lambda)K(x)}{1-\lambda}.$$

Using the Second Mean Value Theorem, we obtain

$$\int_p^\infty \frac{K(x/t)}{K(1/t)} \cos x\,dx = \frac{K_1(p/t)}{K(1/t)} \int_p^{q_1} \cos x\,dx - \frac{K_2(p/t)}{K(1/t)} \int_p^{q_2} \cos x\,dx.$$

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The modulus of the right hand side is not greater than

$$\frac{2K_1(p/t)+2K_2(p/t)}{K(1/t)},$$

which is less than

$$\frac{2(1+\lambda)K(p/t)}{(1-\lambda)K(1/t)}$$

when p/t is great. Since $K(p/t)/K(1/t) \to 1/p$ as $t \to 0$, the required result follows.

We now consider the proof of (ii).

$$\frac{V(u)}{u} = \int_0^\infty K(x) \cos ux\, dx.$$

If $t_0 > 0$, the integral is uniformly convergent with respect to $u > t_0$.

Therefore, if $t > t_0$,

$$\int_{t_0}^t \frac{V(u)}{u}\, du = \int_0^\infty K(x) \frac{\sin tx}{x}\, dx - \int_0^\infty K(x) \frac{\sin t_0 x}{x}\, dx$$
$$= \int_0^\infty K(x/t) \frac{\sin x}{x}\, dx - \int_0^\infty K(x/t_0) \frac{\sin x}{x}\, dx.$$

The second integral on the right is uniformly convergent with respect to $t_0 > 0$, and therefore $\to 0$ when $t_0 \downarrow 0$. Hence

$$\int_0^t \frac{V(u)}{u}\, du = \int_0^\infty K(x/t) \frac{\sin x}{x}\, dx,$$

and

$$\frac{\int_0^t \frac{V(u)}{u}\, du}{K(1/t)} = \int_0^\infty \frac{K(x/t)}{K(1/t)} \frac{\sin x}{x}\, dx. \tag{14}$$

If $m = 0$, when $t \downarrow 0$, $K(x/t)/K(1/t) \to 1$. Using the property that when x is great, either $K(x)$ is monotonic or $K_2(x) < \lambda K_1(x)$, we can as in the proof of (i), and Theorem 1, show that when $t \downarrow 0$, the right hand side of (14) tends to

$$\int_0^\infty \frac{\sin x}{x}\, dx = \tfrac{1}{2}\pi.$$

This proves (ii).

$$V(t) = \int_0^\infty K(x/t) \cos x\, dx.$$

$$\frac{V(t)-t\int_0^{1/t}K(x)dx}{K(1/t)} = \frac{V(t)-\int_0^1 K(x/t)dx}{K(1/t)}$$
$$= \int_0^1 \frac{K(x/t)}{K(1/t)}(\cos x-1)dx+\int_1^\infty \frac{K(x/t)}{K(1/t)}\cos x\,dx,$$

which tends to

$$\int_0^1 \frac{\cos x-1}{x}dx+\int_1^\infty \frac{\cos x}{x}dx, \tag{15}$$

when $t \downarrow 0$, if $m = 1$, so that $K(x/t)/K(1/t) \to 1/x$.

The sum of the two integrals in (15) is the limit, when $\theta \to 0$, of

$$\int_0^1 \frac{\cos x-1}{x^{1+\theta}}dx+\int_1^\infty \frac{\cos x}{x^{1+\theta}}dx$$
$$= \int_0^\infty \frac{\cos x-1}{x^{1+\theta}}dx+\int_1^\infty \frac{dx}{x^{1+\theta}} = C(1+\theta)+1/\theta$$
$$= \frac{1}{\theta}-\frac{\frac{1}{2}\pi}{\Gamma(1+\theta)\sin\frac{1}{2}\pi\theta}.$$

This limit of this, when $\theta \to 0$, is easily shown to be

$$\Gamma'(1), = -\gamma.$$

This proves (iii).

We shall write

$$\mu^*_{2n-1} = \int_0^\infty (2n-1)x^{2n-2}K(x)dx, \tag{16}$$

when this integral exists. In all the cases we shall discuss, $K(x)$ is asymptotically monotonic, and therefore, if the integral is finite, $x^{2n-2}K(x)$ must $\to 0$ as $x \to \infty$. When this is so, integration by parts shows that

$$\begin{aligned}\mu^*_{2n-1} &= -\int_0^\infty x^{2n-1}dK(x) = -\lim_{T\to\infty}\int_0^T x^{2n-1}dK(x)\\ &= \lim_{T\to\infty}\int_{-T}^T x^{2n-1}dF(x).\end{aligned} \tag{17}$$

It is possible for the integral in (16) to be finite and the limit in (17) not to exist; but not in the cases we are investigating.

It follows from Lemma 2 that if K is of index $-m$ at ∞, μ^*_{2n-1} is finite if $m > 2n-1$, and infinite if $m < 2n-1$.

Theorem 8. *Suppose that μ^*_{2n-1} exists and is finite, where n is a positive integer. Put*

$$V(t) = \sum_1^n (-1)^{r-1}\frac{\mu^*_{2r-1}t^{2r-1}}{(2r-1)!}+V_{2n-1}(t).$$

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Let K be of index $-m$ at ∞

(i) *If $2n < m < 2n+1$, then*

$$V_{2n-1}(t) \sim C(m)K(1/t), \qquad t \downarrow 0.$$

(ii) *If $m = 2n-1$, then*

$$V_{2n-1}(t) - (-1)^n t^{2n-1} \int_{1/t}^{\infty} \frac{x^{2n-2}}{(2n-2)!} K(x)dx$$

$$\sim (-1)^{n-1} \frac{\Gamma'(2n-1)}{\Gamma(2n-1)^2} K(1/t), \qquad t \downarrow 0.$$

(iii) *If $m = 2n+1$,*

$$V_{2n-1}(t) - (-1)^n t^{2n+1} \int_{0}^{1/t} \frac{x^{2n}}{(2n)!} K(x)dx$$

$$\sim (-1)^{n} \frac{\Gamma'(2n+1)}{\Gamma(2n+1)^2} K(1/t) \qquad t \downarrow 0.$$

The proofs are similar to those of Theorem 6. Here

$$V_{2n-1}(t) = \int_0^{\infty} K(x/t) J_n(x)dx,$$

where

$$J_n(x) = \cos x - \sum_{0}^{n-1} (-1)^r \frac{x^{2r}}{(2r)!}.$$

6. Counter-example

In this section we give an example where H is of index 0 at ∞ but the conclusion of Theorem 2 is not true, i.e. $1-U(t)$ is not asymptotically equal to $H(1/t)$ as $t \downarrow 0$.

Let $\{T_n\}$ be an increasing sequence of integers > 1 such that T_n/T_{n-1} is integral. For

$$2T_r \leqq x \leqq 2T_{r+1}$$

define

$$f(x) = \frac{1}{\log(2mT_r)} - \frac{1}{\log((2m+2)T_r)}$$

if

$$x = (2m+1)T_r, \quad m = 1, 2, \cdots, T_{r+1}/T_r - 1,$$
$$= 0 \text{ otherwise.}$$

For

$$0 \leqq x \leqq 2T_1$$

define

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$$f(x) = 1 - \frac{1}{\log (2T_1)},$$

if $$x = T_1,$$

$$= 0 \text{ otherwise.}$$

Now define $$H(x) = \tfrac{1}{2}f(x) + \sum_{y>x} f(y).$$

Clearly H is a tail-sum function. We easily obtain

$$H(2mT_r) = \frac{1}{\log (2mT_r)}, \qquad m = 1, 2, \cdots, T_{r+1}/T_r.$$

If $$2mT_r \leqq x \leqq 2(m+1)T_r,$$

where m is a positive integer, then

$$\tfrac{1}{2}x \leqq 2mT_r \leqq x \leqq 2(m+1)T_r \leqq 2x,$$

and so

$$\frac{1}{\log (\frac{1}{2}x)} \geqq H(2mT_r) \geqq H(x) \geqq H(2(m+1)T_r) \geqq \frac{1}{\log (2x)}.$$

Thus for all $x > 2T_1$,

$$\frac{1}{\log(\frac{1}{2}x)} \geqq H(x) \geqq \frac{1}{\log (2x)}.$$

It follows that $H(x) \sim 1/\log x$, $x \to \infty$. Thus H is of index 0 at ∞

$$1-U(\pi/T_r) = -\int_0^\infty (1-\cos (\pi x/T_r))dH(x)$$

$$\geqq -\int_{2T_r}^{2T_{r+1}} (1-\cos (\pi x/T_r))dH(x) = -\int_{2T_r}^{2T_{r+1}} 2dH(x).$$

The last equality is true because at a point of increase of H in the interval $[2T_r, 2T_{r+1}]$, $\cos (\pi x/T_r) = -1$. Hence

$$1-U(\pi/T_r) \geqq 2[H(2T_r)-H(2T_{r+1})]. \tag{18}$$

Now $H(T_r) \sim 1/\log T_r$, and so if we choose the sequence $\{T_n\}$ so that $\log T_{r+1}/\log T_r \to \infty$ as $r \to \infty$, the right side of (18) will be asymptotically equal to

$$2H(2T_r) \sim 2H(T_r/\pi), \qquad r \to \infty.$$

Hence $1-U(\pi/T_r)$ will not be asymptotically equal to $H(T_r/\pi)$ as $r \to \infty$, and so $1-U(t)$ will not be asymptotically equal to $H(1/t)$ as $t \downarrow 0$.

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