The Annals of Probability
1984, Vol. 12, No. 1, 230–242

ON THE BELL-SHAPE OF STABLE DENSITIES

BY WOLFGANG GAWRONSKI

Universität Ulm

The central result of this paper consists in proving that all stable densities are bell-shaped (i.e. its kth derivative has exactly k zeros and they are simple) thereby generalizing the well-known property of the normal distribution and the associated Hermite polynomials.

1. Introduction and summary. The notion of bell-shape is intimately connected with the graph of the density of a normal distribution. As is well known, the Cauchy density has a similar geometrical form. More generally and to be more precise, following the concept of bell-shaped kernels in the theory of games [e.g. 6, Chapter 6, Section 11.C] throughout we call a probability density function p *bell-shaped* iff p is infinitely many times differentiable on the real line and its kth derivative $p^{(k)}$ possesses exactly k zeros on its support and they are simple. Now the bell-shape of the normal density amounts to the well-known property that the zeros of the Hermite polynomials are real and simple [e.g. 9, volume II, problem 58, page 48; or 11, Theorem 3.3.1, page 44]. The same property is readily verified for the Cauchy density too (see [6, Chapter 6, Section 11.C] or, more precisely, Section 3 of this paper). Both functions belong to the class of stable densities for which in general a representation by elementary functions is not known [cf. 7, page 143]. Thus an investigation of finer properties of general stable densities seems to be difficult. On the other hand, Ibragimov and Chernin [3] proved unimodality for *all* stable densities. Sharpening their theorem and extending the bell-shape of the two prominent representatives of stable laws mentioned above, we obtain the central result of this paper by proving that (see Theorem 1):

all stable densities are bell-shaped.

Moreover, for the kth derivative we determine the exact numbers of zeros located on the positive and on the negative axis respectively provided the exponent of stability $\alpha \neq 1$ (Theorem 2). The proofs essentially are based on suitable integral representations and certain variation diminishing transformations [see 6, Chapter 5].

2. Preliminary results. In this section we collect some auxiliary results and in particular we derive suitable integral representations for stable densities and its derivatives which are basic for the technical treatment of the proofs of our theorems.

After a certain normalization we can define stable distributions via the loga-

Received October 1982.

AMS 1980 *subject classifications*. 60E07, 60E10, 42A38, 30C15.

Key words and phrases. Stable densities, zeros of Fourier integrals, bell-shaped kernels.

rithm of their characteristic functions given by

$$\log \phi(t; \alpha, \gamma) := \begin{cases} -|t|^{\alpha}\exp\left(i\frac{\pi}{2}\gamma \operatorname{sign} t\right), & \alpha \neq 1 \\ -|t|\left(\frac{\pi}{2} + i\gamma(\log|t|) \operatorname{sign} t\right), & \alpha = 1 \end{cases} \tag{2.1}$$

[cf. 4, page 581 ff; 5, page 164; 13] where we have for the exponent of stability $\alpha \in (0, 2]$ and for the skewness parameter $\gamma \in \mathbb{R}$

$$|\gamma| \leq \begin{cases} \alpha, & 0 < \alpha \leq 1 \\ 2 - \alpha, & 1 < \alpha \leq 2. \end{cases} \tag{2.2}$$

In view of the integrability properties of ϕ (see (2.1)) we see that the corresponding stable law has an infinitely differentiable density $p(x; \alpha, \gamma)$, say, [cf. 7, Chapter 5.8] which we represent by Fourier's inversion theorem [e.g. 2, Proposition 5.1.10., page 192] as

$$p(x; \alpha, \gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\phi(t; \alpha, \gamma)\, dt = \frac{1}{\pi}\operatorname{Re}\int_{0}^{\infty} e^{-ixt}\phi(t; \alpha, \gamma)\, dt. \tag{2.3}$$

In the first lemma we list some properties of $p(x; \alpha, \gamma)$ known in the literature. In particular concerning concepts such as total positivity sign regularity and related ones we refer to the monograph of Karlin [6, Chapter 2].

LEMMA 1. i) *For all* $x \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}$ *satisfying* (2.2) *we have*

$$p(x; \alpha, \gamma) = p(-x; \alpha, -\gamma), \quad \alpha \in (0, 2]. \tag{2.4}$$

ii) *If* $0 < \alpha < 1$, *then*

$$\begin{aligned} p(x; \alpha, -\alpha) &= 0 \quad \textit{for all} \quad x < 0 \\ p(x; \alpha, \alpha) &= 0 \quad \textit{for all} \quad x > 0. \end{aligned} \tag{2.5}$$

iii) *If* $0 < \alpha < 1$, *then the following representation as Laplace transform holds*

$$e^{-\lambda^{\alpha}} = \int_{0}^{\infty} e^{-\lambda x} p(x; \alpha, -\alpha)\, dx, \quad \operatorname{Re}\lambda \geq 0. \tag{2.6}$$

iv) *If* $0 < \alpha < 1$, *then the kernel*

$$K(x, y) := p(e^{x-y}; \alpha, -\alpha), \quad x, y \in \mathbb{R}, \tag{2.7}$$

is strictly totally positive (STP).

(2.8) v) $$p(x; 1, \gamma) = \frac{1}{\gamma\pi}\int_{-\pi/2}^{\pi/2} \exp(-a(\phi)e^{-x/\gamma})a(\phi)e^{-x/\gamma}\, d\phi$$

for all $x \in \mathbb{R}$ *and* $\gamma \in (0, 1]$ *where*

$$a(\phi) := (\gamma\phi + (\pi/2))\exp\{(\phi + (\pi/2\gamma))\tan\phi\}, \quad |\phi| < \pi/2. \tag{2.9}$$

PROOF. Part i is an immediate consequence of (2.3) and ii follows from a series representation of $p(x; \alpha, \gamma)$ [e.g. 7, Chapter 5.8]. The representations in part iii and v are given in [1] and [13] respectively. The property in part iv is established in [6, page 390].

REMARK. [7, Chapter 5.8]. $p(x; \alpha, \pm\alpha)$, $0 < \alpha < 1$, are the only onesided densities; in all other cases $p(x; \alpha, \gamma)$ is concentrated on the whole real line. If $\gamma = 0$, then $p(x; \alpha, 0)$ is a symmetric density.

Using the abbreviation

$$p_\alpha(x) := p(x; \alpha, -\alpha), \qquad 0 < \alpha < 1, \tag{2.10}$$

and the convention

$$\arg x = \begin{cases} 0 & \text{for } x > 0 \\ \pi & \text{for } x < 0 \end{cases}$$

we derive the following representation formulae for the derivatives in question.

LEMMA. 2. i) *If* $0 < \alpha < 1$, $|\gamma| < \alpha$, $x \neq 0$, *and* $k \in N \cup \{0\}$, *then*

$$p^{(k)}(x; \alpha, \gamma) = \frac{k!}{\pi |x|^k} \int_0^\infty p_\alpha(y|x|)\psi_{\alpha,k}(y)\, dy$$

where

$$\psi_{\alpha,k}(y) := \operatorname{Re} \frac{(-i)^k}{(y \exp(i\pi\gamma/2\alpha) + i \operatorname{sign} x)^{k+1}}, \quad y \geq 0.$$

ii) *If* $1 < \alpha < 2$, $|\gamma| < 2 - \alpha$, $x \neq 0$, *and* $k \in N \cup \{0\}$, *then*

$$p^{(k)}(x; \alpha, \gamma) = \frac{(-1)^{k+1}\Gamma((k+1)/\alpha)}{\pi\alpha |x|^\alpha} \int_0^\infty p_{1/\alpha}(y/|x|^\alpha)\psi_{\alpha,k}(y)\, dy$$

where

$$\psi_{\alpha,k}(y) := \operatorname{Re} \frac{i \exp\{i(k+1)\arg x\}}{(y + \exp\{i\gamma(\pi/2) - i\alpha((\pi/2) - \arg x)\})^{(k+1)/\alpha}} \quad y \geq 0.$$

PROOF. Differentiating (2.3) we obtain, by (2.1),

$$p^{(k)}(x; \alpha, \gamma) = \frac{1}{\pi} \operatorname{Re}(-i)^k \int_0^\infty t^k \exp(-ixt - t^\alpha e^{i\pi\gamma/2})\, dt. \tag{2.11}$$

i) Suppose that $0 < \alpha < 1$, $|\gamma| < \alpha$, and $x \neq 0$. Then, putting

$$a := \exp(i\pi\gamma/2\alpha) \tag{2.12}$$

we get, by Lemma 1, iii, with $\lambda = ta$

$$p^{(k)}(x;\alpha,\gamma) = \frac{1}{\pi}\operatorname{Re}(-i)^k \int_0^\infty dte^{-ixt}t^k \int_0^\infty d\xi e^{-t\xi a}p_\alpha(\xi)$$

$$= \frac{1}{\pi|x|^k}\operatorname{Re}(-i)^k \int_0^\infty dyp_\alpha(y|x|) \int_0^\infty \tau^k \exp\{-\tau(ay + i \operatorname{sign} x)\}\, d\tau$$

$$= \frac{k!}{\pi|x|^k}\int_0^\infty p_\alpha(y|x|)\psi_{\alpha,k}(y)\, dy.$$

ii) Suppose that $1 < \alpha < 2$, $|\gamma| < 2 - \alpha$, and $x \neq 0$. Then we treat (2.11) by contour integration as in [1] to obtain

$$p^{(k)}(x;\alpha,\gamma) = \frac{1}{\pi}\operatorname{Re}(-i)^k e^{i\phi_0(k+1)} \int_0^\infty \tau^k \exp(-\tau|x|e^{i\theta_2} - \tau^\alpha e^{i\theta_1})\, d\tau$$

where ϕ_0, θ_1, θ_2 are chosen subject to

$$-\frac{\pi}{2} \le \phi_0 \le \frac{\pi}{2}, \qquad \begin{aligned}\theta_1 &= \gamma\frac{\pi}{2} + \alpha\phi_0\\ \theta_2 &= \frac{\pi}{2} - \arg x + \phi_0\end{aligned}$$

such that

$$\theta_i \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right), \quad i = 1, 2, \quad \text{for some} \quad \delta \in (0, \pi/2)$$

and

$$\phi_0 \operatorname{sign} x < 0.$$

Next, we put $\tau^\alpha = \xi$ and get

$$p^{(k)}(x;\alpha,\gamma) = \frac{1}{\pi\alpha}\operatorname{Re}(-i)^k e^{i(k+1)\phi_0} \int_0^\infty \xi^{((k+1)/\alpha)-1}\exp(-\xi e^{i\theta_1})$$

$$\times \exp\{-(\xi|x|^\alpha e^{i\alpha\theta_2})^{1/\alpha}\}\, d\xi.$$

In order to apply Lemma 1, iii to the second exponential we have to make sure that ϕ_0 can be chosen such that $|\theta_1| < \pi/2$ and $|\alpha\theta_2| \le \pi/2$. We suppose that arg $x = 0$. The reasoning for arg $x = \pi$ is very similar. Then we have

$$\theta_1 = \gamma\frac{\pi}{2} + \alpha\phi_0, \quad \alpha\theta_2 = \alpha\left(\frac{\pi}{2} + \phi_0\right)$$

and consequently

$$\theta_1 = (\gamma - \alpha)\frac{\pi}{2} + \alpha\left(\frac{\pi}{2} + \phi_0\right) = (\gamma - \alpha)\frac{\pi}{2} + \alpha\theta_2$$

which gives in view of $|\gamma| < 2 - \alpha$

$$-\pi + \alpha\theta_2 < \theta_1 < \alpha\theta_2.$$

Choosing now $\phi_0 = (1 - \alpha)\pi/2\alpha \in (-\pi/2, 0)$ we get

$$\alpha\theta_2 = \pi/2, \quad -\pi/2 < \theta_1 < \pi/2.$$

(For $\arg x = \pi$ choose $\phi_0 = (\alpha - 1)\pi/2\alpha$).

Now Lemma 1, iii implies ($\lambda = \xi|x|^\alpha \exp(i\alpha\theta_2)$)

$$p^{(k)}(x; \alpha, \gamma)$$

$$= \frac{1}{\pi\alpha} \operatorname{Re}(-i)^k e^{i(k+1)\phi_0} \int_0^\infty dy p_{1/\alpha}(y) \int_0^\infty \xi^{((k+1)/\alpha)-1} \exp\{-\xi(e^{i\theta_1} + y|x|^\alpha e^{i\alpha\theta_2})\}\, d\xi$$

$$= \frac{\Gamma((k+1)/\alpha)}{\pi\alpha} \int_0^\infty p_{1/\alpha}(y) \operatorname{Re} \frac{(-i)^k e^{i(k+1)\phi_0}}{(e^{i\theta_1} + y|x|^\alpha e^{i\alpha\theta_2})^{(k+1)/\alpha}}\, dy$$

$$= \frac{(-1)^{k+1}\Gamma((k+1)/\alpha)}{\pi\alpha|x|^\alpha} \int_0^\infty p_{1/\alpha}(y/|x|^\alpha)\psi_{\alpha,k}(y)\, dy.$$

This completes the proof.

3. Main results. First we prove that $p^{(k)}(x; \alpha, \gamma)$, $k \in \mathbb{N}$, has at least k changes of sign on the real axis by showing that its moments up to order $k - 1$ vanish [cf. 8]. It should be mentioned that this part of the proof for Theorem 1 requires integrability properties of the stable characteristic functions only rather than its particular structure. Putting

$$\chi_k(t) := t^k \phi(t; \alpha, \gamma), \quad t \in \mathbb{R}, \quad k \in \mathbb{N} \tag{3.1}$$

we have, by (2.1), that

$$\chi_k^{(\nu)} \in L_1(\mathbb{R}), \quad \nu = 0, \cdots, k-1, \tag{3.2}$$

and

$$\chi_k^{(\nu)}(0) = 0, \quad \nu = 0, \cdots, k-1. \tag{3.3}$$

Further, from (2.3) we get

$$p^{(k)}(x; \alpha, \gamma) = \frac{(-i)^k}{2\pi} \int_{-\infty}^\infty e^{-ixt} \chi_k(t)\, dt$$

and thus by (3.2) and [2, Proposition 5.1.14, page 194]

$$(ix)^\nu p^{(k)}(x; \alpha, \gamma) = \frac{(-i)^k}{2\pi} \int_{-\infty}^\infty e^{-ixt} \chi_k^{(\nu)}(t)\, dt, \quad \nu = 0, \cdots, k-1. \tag{3.4}$$

Now by Fourier's inversion in the form of Jordan's theorem [2, Problem 5, iii, page 205] and since the left hand side of (3.4) is continuous we get

$$\chi_k^{(\nu)}(t) = C \int_{-\infty}^{\infty} e^{ixt} x^\nu p^{(k)}(x; \alpha, \gamma)\, dx, \quad \nu = 0, \cdots, k-1, \tag{3.5}$$

where the integral exists at least as a principal value. Actually we could apply the classical inversion theorem [2, Proposition 5.1.10, page 192] since by extending the asymptotic results for $p(x; \alpha, \gamma)$ in [10] to its derivatives the left hand side of (3.4) is seen to be in $L_1(\mathbb{R})$. Putting $t = 0$ in (3.5), by (3.3), we obtain

$$\int_{-\infty}^{\infty} x^\nu p^{(k)}(x; \alpha, \gamma)\, dx = 0, \quad \nu = 0, \cdots, k-1.$$

Because $p^{(k)}(x; \alpha, \gamma)$ is continuous on the real line this shows that $p^{(k)}(x; \alpha, \gamma)$ changes its sign at least k times [9, vol. I, problem 140, page 65]. Hereby we have proved part of

THEOREM 1. *All stable densities are bell-shaped.*

To complete the proof of Theorem 1 in case $\alpha \neq 1$ we estimate

$N_k^+(\alpha, \gamma)$:= number of zeros of $p^{(k)}(x; \alpha, \gamma)$ on the positive real axis

and

$N_k^-(\alpha, \gamma)$:= number of zeros of $p^{(k)}(x; \alpha, \gamma)$ on the negative real axis,

from above (both counting multiplicities). First, we observe, by Lemma 1, i, that

$$N_k^+(\alpha, \gamma) = N_k^-(\alpha, -\gamma) \tag{3.6}$$

for all admissable γ in (2.2). Then we prove

THEOREM 2. *Suppose that $\alpha \neq 1$ and $k \in N$, then we have*

i) $p^{(k)}(0; \alpha, \gamma) = \dfrac{\Gamma((k+1)/\alpha)}{\pi\alpha} \cos\left(\dfrac{\pi}{2}\left(k + \dfrac{\gamma}{\alpha}(k+1)\right)\right)$ *that is $x = 0$ is a zero of $p^{(k)}(x; \alpha, \gamma)$ iff $k + (k+1)\gamma/\alpha$ is an odd integer. This zero is simple unless $|\gamma| = \alpha < 1$.*

ii) *If $x = 0$ is a zero of $p^{(k)}(x; \alpha, \gamma)$ so it is a zero of $p^{(k)}(x; \alpha, -\gamma)$.*

iii) *If $|\gamma| < \alpha$ or $|\gamma| < 2 - \alpha$ in cases $\alpha < 1$ or $\alpha > 1$ respectively, then*

$$N_k^\pm(\alpha, \gamma) = \begin{cases} \left[\dfrac{k+1}{2} \mp \dfrac{k+1}{2}\dfrac{\gamma}{\alpha}\right], & \text{when } k + (k+1)\gamma/\alpha \text{ is not an odd integer} \\[2ex] \dfrac{k-1}{2} \mp \dfrac{k+1}{2}\dfrac{\gamma}{\alpha}, & \text{when } k + (k+1)\gamma/\alpha \text{ is an odd integer.} \end{cases}$$

(for real x as customary $[x]$ denotes the largest integer not exceeding x) and all zeros are simple.

iv) *In the extreme cases we have*

$$N_k^{\pm}(\alpha, \mp\alpha) = k \qquad \textit{for } \alpha < 1$$

and for $\alpha > 1$

$$N_k^{+}(\alpha, \alpha - 2) = \begin{cases} \left[\dfrac{k+1}{\alpha}\right], & \textit{when} \quad (k+1)/\alpha \notin \mathbb{N} \\ \dfrac{k+1}{\alpha} - 1, & \textit{when} \quad (k+1)/\alpha \in \mathbb{N} \end{cases}$$

$$N_k^{-}(\alpha, \alpha - 2) = \begin{cases} k - \left[\dfrac{k+1}{\alpha}\right], & \textit{when} \quad \dfrac{k+1}{\alpha} \notin \mathbb{N} \\ k - \dfrac{k+1}{\alpha}, & \textit{when} \quad \dfrac{k+1}{\alpha} \in \mathbb{N}, \end{cases}$$

$$N_k^{\pm}(\alpha, 2 - \alpha) = N_k^{\mp}(\alpha, \alpha - 2),$$

and all zeros on the support of $p(x; \alpha, \gamma)$ *are simple.*

REMARKS.

i) Observe that $k + (k+1)\gamma/\alpha$ is an odd integer iff $(k-1)/2 \mp ((k+1)/2)(\gamma/\alpha)$ is an integer in part iii or $(k+1)/\alpha$ is an integer in part iv when $|\gamma| = 2 - \alpha$.
ii) Theorem 2 in particular shows that $p^{(k)}(x; \alpha, \gamma)$ has exactly k zeros on its support and they are simple ($\alpha \neq 1$).
iii) The numbers $N_k^{\pm}(\alpha, \gamma)$ reflect the skewness of the underlying distribution; that is the more probability mass is concentrated on a half-axis the more zeros of the derivatives are located on this half-axis.
iv) If $\alpha = 2$, then ($\gamma = 0$, by (2.2))

$$N_k^{\pm}(2, 0) = \begin{cases} k/2, & \text{if } k \text{ is even} \\ (k-1)/2, & \text{if } k \text{ is odd} \end{cases}$$

and $p^{(k)}(0; 2, 0) = 0$ in the latter case. This is the well-known property of bell-shape for the normal density following e.g. from [11, Theorem 3.3.1, page 44 and Section 5.5. page 105].

PROOF OF THEOREM 2.

i) From (2.11) we have

$$p^{(k)}(0; \alpha, \gamma) = \frac{1}{\pi} \operatorname{Re}(-i)^k \int_0^\infty t^k \exp(-t^\alpha e^{i\pi\gamma/2})\, dt$$

and, by Cauchy's theorem, we obtain

$$p^{(k)}(0; \alpha, \gamma) = \frac{1}{\pi\alpha} \operatorname{Re}(-i)^k \exp(-(k+1)i\pi\gamma/2\alpha) \int_0^\infty e^{-\xi}\xi^{(k+1)/\alpha-1}\, d\xi$$

$$= \frac{\Gamma((k+1)/\alpha)}{\pi\alpha} \cos\left(\frac{\pi}{2}\left(k + \frac{\gamma}{\alpha}(k+1)\right)\right).$$

ii) Part ii is an immediate consequence of Lemma 1, i.

iii) First, suppose that $0 < \alpha < 1$, $|\gamma| < \alpha$, and $x \neq 0$. Then putting

$$|x| = e^{\xi}, \quad y = e^{-\eta}, \quad \xi, \eta \in \mathbb{R}$$

we rewrite the representation of Lemma 2, i as

$$p^{(k)}(x; \alpha, \gamma) = \frac{k!}{\pi e^{k\xi}} \int_{-\infty}^{\infty} p_\alpha(e^{\xi-\eta}) \psi_{\alpha,k}(e^{-\eta}) e^{-\eta}\, d\eta.$$

By Lemma 1, iv, $K(\xi, \eta) = p_\alpha(e^{\xi-\eta})$ is strictly totally positive (STP).

Using in [6, Chapter 2] Corollary 5.2 (page 66), Theorem 5.1 (page 62), and the remark following definition 1.5 (page 49) successively we conclude that $K(\xi, \eta)$ is extended sign regular of order r in both variables (ESR_r), then extended sign regular of order r with respect to ξ ($ESR_r(\xi)$), and finally extended sign regular of order r and degree s in the variable ξ ($ESR_r(\xi_{(s)})$), for all $r, s \in \mathbb{N}$. To obtain upper estimates for $N_k^{\pm}(\alpha, \gamma)$ by Theorem 3.2 in [6, page 239] we simply have to count the number of sign changes of $\psi_{\alpha,k}(y)$ as y runs through the positive reals (the validity of condition (3.21) in [6] is readily verified). To this end by Lemma 2, i and (2.12) we write

$$\psi_{\alpha,k}(y) = \operatorname{Re} \frac{(-i)^k}{(ya + i \operatorname{sign} x)^{k+1}}$$

from which its zeros are easily recognized to be

$$y_\nu = \frac{\sin \dfrac{\pi\nu}{k+1}}{\sin\left(\dfrac{\pi\nu}{k+1} + \dfrac{\pi}{2}\left(1 + \dfrac{\gamma}{\alpha}\right)\right)} \operatorname{sign} x \tag{3.7}$$

where $\nu = 0, \cdots, k$, but $\nu \neq ((k+1)/2)(1 - (\gamma/\alpha))$.

Case I. Suppose that $k + (k+1)\gamma/\alpha$ is not an odd integer (which is equivalent to the fact that $(k+1)(1-\gamma/\alpha)/2$ is not an integer). Then a simple counting in (3.7) gives

$$N_k^+(\alpha, \gamma) \leq \left[\frac{k+1}{2}\left(1 - \frac{\gamma}{\alpha}\right)\right] \tag{3.8}$$

(use Theorem 3.2 in [6, page 239]) and further (see (3.6))

$$N_k^-(\alpha, \gamma) = N_k^+(\alpha, -\gamma) \le \left[\frac{k+1}{2}\left(1+\frac{\gamma}{\alpha}\right)\right]. \tag{3.9}$$

Since x is not a zero in this case the considerations before Theorem 1 imply that

$$\begin{aligned} k \le N_k^+(\alpha, \gamma) + N_k^-(\alpha, \gamma) &\le \left[\frac{k+1}{2}\left(1-\frac{\gamma}{\alpha}\right)\right] + \left[\frac{k+1}{2}\left(1+\frac{\gamma}{\alpha}\right)\right] \\ &= k+1+\left[\frac{k+1}{2}\left(1-\frac{\gamma}{\alpha}\right)\right] + \left[\frac{k+1}{2}\left(\frac{\gamma}{\alpha}-1\right)\right] = k \end{aligned}$$

thereby showing that (3.8) and (3.9) hold with equality.

Case II. Suppose now that $k + (k+1)\gamma/\alpha$ is an odd integer which implies that $x = 0$ is a simple zero and $(k+1)(1-\gamma/\alpha)/2$ is a positive integer. Now the conclusion from (3.7) is

$$N_k^+(\alpha, \gamma) \le \frac{k+1}{2}\left(1-\frac{\gamma}{\alpha}\right) - 1 \quad \text{and} \quad N_k^-(\alpha, \gamma) = N_k^+(\alpha, -\gamma) \le \frac{k+1}{2}\left(1+\frac{\gamma}{\alpha}\right) - 1.$$

The same reasoning as above completes the proof in case $\alpha < 1$.

Now suppose that $1 < \alpha < 2$, $|\gamma| < 2 - \alpha$, and $x \ne 0$. Then by the same arguments as above we have to determine the number of positive zeros of

$$\psi_{\alpha,k}(y) = \operatorname{Re} \frac{i \exp\{i(k+1)\arg x\}}{(y + \exp\{i\gamma(\pi/2) - i\alpha((\pi/2) - \arg x)\})^{(k+1)/\alpha}}.$$

By (3.6), we may confine ourselves to the case $x > 0$. Writing

$$b := \exp(i\pi(\gamma - \alpha)/2)$$

the zeros of $\psi_{\alpha,k}$ are the finite solutions of

$$\left(\frac{y+\bar{b}}{y+b}\right)^{(k+1)/\alpha} = 1.$$

Putting $r := (k+1)/\alpha$ the latter are given by

$$y_\nu = \frac{\sin\left(\dfrac{\pi\nu}{r} + \dfrac{\pi}{2}(\gamma - \alpha + 2)\right)}{\sin\dfrac{\pi\nu}{r}} \tag{3.10}$$

where

$$\begin{aligned} \nu &= 1, \cdots, r-1, && \text{if } r \in \mathbb{N} \\ \nu &= 1, \cdots, [r], && \text{if } r \notin \mathbb{N}. \end{aligned} \tag{3.11}$$

Further, observe that

$$0 < \frac{\nu}{r} + \frac{\gamma - \alpha + 2}{2} < 2 \tag{3.12}$$

for ν satisfying (3.11).

CASE I. Suppose that $k + (k+1)\gamma/\alpha = k + r\gamma$ is not an odd integer; that is $x = 0$ is not a zero by part i. Using the considerations preceeding Theorem 1, (3.6) and Theorem 3.2 in [6] again (observe Remark i following Theorem 2) by (3.10), (3.11), and (3.12) we get

$$N_k^{\dagger}(\alpha, \pm\gamma) \le \left[r \frac{\alpha \mp \gamma}{2} \right]$$

and

$$\begin{aligned} k &\le N_k^{+}(\alpha, \gamma) + N_k^{-}(\alpha, \gamma) = N_k^{\dagger}(\alpha, \gamma) + N_k^{\dagger}(\alpha, -\gamma) \\ &\le \left[\frac{k+1}{2} - (k+1)\frac{\gamma}{2\alpha} \right] + \left[\frac{k+1}{2} + (k+1)\frac{\gamma}{2\alpha} \right] \\ &= \left[\frac{k+1}{2} - (k+1)\frac{\gamma}{2\alpha} \right] + \left[-\frac{k+1}{2} + (k+1)\frac{\gamma}{2\alpha} \right] + k + 1 = k. \end{aligned}$$

Hence the truth of the assertion follows in this case.

CASE II. If $k + (k+1)\gamma/\alpha$ is an odd integer the proof runs along the same lines as the previous one.

iv) Finally we mention that stable laws are continuous functions of the parameters provided $\alpha \neq 1$ [12, Section 2.4, p. 447]. Thus parts i, iii imply iv when $0 < \alpha < 1$ or $1 < \alpha < 2$. If $\alpha = 2$, then see Remark iv following Theorem 2. Now the proof of Theorem 2 is finished.

Finally we complete the:

PROOF OF THEOREM 1. By Theorem 2 and the result preceeding Theorem 1, it suffices to prove that $p^{(k)}(x; 1, \gamma)$ has at most k real zeros (counting multiplicities). Since stable laws are not continuous with respect to α near $\alpha = 1$ [12, Section 2.4] we cannot pass to the limit in Theorem 2. To this end we use representation (2.8) to get for $x \in \mathbb{R}$, $\gamma \in (0, 1]$,

$$p^{(k)}(x; 1, \gamma) = \frac{1}{\pi\gamma^{k+1}} \int_{-\pi/2}^{\pi/2} \exp(-a(\phi)e^{-x/\gamma}) P_{k+1}(a(\phi)e^{-x/\gamma})\, d\phi$$

where P_k is a real polynomial of degree k being recursively defined by

$$P_1(\xi) := \xi, \quad P_{k+1}(\xi) := \xi(P_k(\xi) - P_k'(\xi)), \quad k \in \mathbb{N}.$$

Since

$$\lim_{\phi\to\pi/2-0} a(\phi) = \infty, \quad \lim_{\phi\to-\pi/2+0} a(\phi) = 0$$

and $a(\phi)$ is strictly increasing on $(-\pi/2, \pi/2)$ (note that $a'(\phi) > 0$ if $|\phi| < \pi/2$) by substituting

$$\xi := e^{-x/\gamma}, \quad t := a(\phi) \tag{3.13}$$

we obtain

$$p^{(k)}(x; 1, \gamma) = \frac{1}{\pi\gamma^{k+1}} \int_0^\infty e^{-\xi t} P_{k+1}(\xi t)\psi(t)\, dt$$

where

$$\psi(t) := \frac{1}{a'(a^{-1}(t))} > 0, \quad \text{for} \quad 0 < t < \infty.$$

Further, by the formula

$$\frac{1}{z^j} = \frac{1}{(j-1)!} \int_0^\infty e^{-zs} s^{j-1}\, ds, \quad z > 0, j \in N$$

we conclude

$$\begin{aligned} p^{(k)}(x; 1, \gamma) &= \xi^{k+2} \int_0^\infty ds\, Q_k(s) \int_0^\infty e^{-\xi t(1+s)} t^{k+2} \psi(t)\, dt \\ &=: \xi^{k+2} \int_1^\infty R_k(y) \int_0^\infty e^{-\xi y t} \phi(t)\, dt\, dy \\ &=: \xi^{k+2} \int_1^\infty R_k(y) L(\xi, y)\, dy. \end{aligned}$$

Q_k and R_k being polynomials of degree k. Next, we have

$$\det\left(\frac{\partial^{i+j-2}}{\partial\xi^{i-1}\partial y^{j-1}} L(\xi, y)\right)_{i,j=1}^r = (\xi y)^{r(r-1)/2} \det\left(\int_0^\infty t^{i-1} t^{j-1} e^{-\xi y t} \phi(t)\, dt\right)$$

which is positive for all $\xi, y > 0$ and all $r \in N$, because the latter determinant is a gram determinant of the linearly independent monomials $1, t, \cdots, t^{r-1}$. Thus the kernel $L(\xi, y)$ is extended totally positive (ETP) and hence $\mathrm{ESR}_r(\xi_{(s)})$ for all $r, s \in N$ [6, page 49]. Now another application of Theorem 3.2 in [6, page 239] completes the proof of Theorem 1 (observe also (2.4)).

Finally we mention two simple properties of two special cases in the following:

REMARK.

i) Writing the kth derivative of Cauchy's density ($\alpha = 1, \gamma = 0$) in the form

$$p^{(k)}(x; 1, 0) = \frac{1}{\pi} \operatorname{Im} \frac{d^k}{dx^k}\left(\frac{1}{x-i}\right) = \frac{(-1)^k k!}{\pi} \operatorname{Im} \frac{1}{(x-i)^{k+1}}$$

it is readily verified that its zeros x_ν can be given explicitly by

$$x_\nu = \operatorname{cotan} \frac{\pi\nu}{k+1}, \quad \nu = 1, \cdots, k.$$

[see also 9, volume II, problem 57, page 46].

ii) If $\alpha = -\gamma = ½$, then $p(x; ½, -½)$ can be expressed by elementary functions [5, page 171; 7, page 143] and hence we obtain inductively ($k \in \mathbb{N} \cup \{0\}$)

$$p^{(k)}\left(x; \frac{1}{2}, -\frac{1}{2}\right) = \begin{cases} \dfrac{P_k(x)}{2^{2k+1}\pi^{1/2}x^{(4k+3)/2}} \exp\left(-\dfrac{1}{4x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where the polynomials P_k (of degree k) are recursively computed by

$$P_0(x) :\equiv 1, \quad P_k(x) := (1 - (8k-2)x)P_{k-1}(x) + 4x^2 P'_{k-1}(x), \quad k \in \mathbb{N}.$$

REFERENCES

[1] BERGSTRÖM, H. (1952). On some expansions of stable distributions. *Arkiv för Matematik* **2** 375–378.

[2] BUTZER, P. L., and NESSEL, R. J. (1971). *Fourier Analysis and Approximation*, vol. I. Birkhäuser, Basel.

[3] CHERNIN, K. E. and IBRAGIMOV, I. A. (1959). On the unimodality of stable laws. *Theory Probab. Appl.* **4** 417–419.

[4] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, vol. II. Wiley, New York.

[5] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1968). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge, Mass.

[6] KARLIN, S. (1968). *Total Positivity*, vol. I. Stanford University Press, Stanford, California.

[7] LUKACS, E. (1970). *Characteristic Functions*. Griffin, London.

[8] PÓLYA, G. (1923). On the zeros of an integral function represented by Fourier's integral. *Mess. of Mathematics* **3** 185–188.

[9] PÓLYA, G. and SZEGÖ, G. (1971). *Aufgaben und Lehrsätze aus der Analysis*, vol. I, II. Heidelberger Taschenbücher, Springer, Berlin.

[10] SKOROHOD, A. V. (1956). Asymptotic formulas for stable distribution laws. In *Selected Trans. Math. Statist. Probab.* **1** 157–161. Amer. Math. Soc., Providence, R. I.

[11] SZEGÖ, G. (1939). Orthogonal polynomials. *Amer. Math. Soc., Colloqu. Publ.* **23**. Amer. Math. Soc., Providence, R. I.

[12] ZOLOTAREV, V. M. (1957). Mellin-Stieltjes transforms in probability theory. *Theory Probab. Appl.* **2** 433–460.

[13] ZOLOTAREV, V. M. (1966). On representation of stable laws by integrals. In *Selected Trans. Math. Statist. Probab.* **6** 84–88. Amer. Math. Soc., Providence, R. I.

ABTEILUNG FÜR MATHEMATIK
UNIVERSITÄT ULM
OBERER ESELSBERG
D-7900 ULM
GERMANY